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Two cardinal inequalities for functionally Hausdorff spaces

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Abstract. In this paper, two cardinal inequalities for functionally Hausdorff spaces are established. A bound on the cardinality of the $\tau\theta$-closed hull of a subset of a functionally Hausdorff space is given. Moreover, the following theorem is proved: if $X$ is a functionally Hausdorff space, then $|X| \leq 2^{\chi(X)wcd(X)}$.

Keywords: cardinal functions, $\tau\theta$-closed sets, $w$-compactness degree

Classification: 54A25, 54D20

A space $X$ is said to be functionally Hausdorff if whenever $x \neq y$ in $X$ there is a continuous real valued function $f$ defined on $X$ such that $f(x) = 0$ and $f(y) = 1$. A well-known Arkhangel’skii’s theorem states that if $X$ is a Hausdorff space, then $|X| \leq 2^{\chi(X)L(X)}$ ([1], [6]). Bella and Cammaroto [2] established some cardinal inequalities for Urysohn spaces that improve, for non regular spaces, the Arkhangel’skii’s formula. In this paper, a bound on the cardinality of the $\tau\theta$-closed hull of a subset of a functionally Hausdorff space and a bound on the cardinality of a functionally Hausdorff space are given. We refer the reader to [3] and [4] for notations and definitions not explicitly given. All topological spaces considered here are assumed to be infinite. Let $E$ be a set; the cardinality of $E$ is denoted by $|E|$, $P_k(E)$ is the collection of all subsets of $E$ of cardinality $\leq k$. $\chi(X)$ and $L(X)$ denote respectively the character and the Lindelöf degree of a space $X$.

Definition 1 [5]. Let $A$ be a subset of a space $X$. $A$ is called $\tau$-open if $A$ is a union of cozere-sets of $X$. The $\tau$-closure of $A$, denoted by $\text{cl}_\tau(A)$, is the set of all points $x \in X$ such that any cozere-set neighbourhood of $x$ intersects $A$. The $\tau$-interior of $A$, denoted by $\text{int}_\tau(A)$, is the set of all $x$ such that there is a cozere-set neighbourhood of $x$ contained in $A$.

Definition 2. Let $X$ be a topological space and $A$ a subset of $X$. The $\tau\theta$-closure of $A$, denoted by $\text{cl}_{\tau\theta}(A)$, is the set of all points $x \in X$ such that $\text{cl}_\tau(V) \cap A \neq \emptyset$ for every open neighbourhood $V$ of $x$. $A$ is said to be $\tau\theta$-closed if $A = \text{cl}_{\tau\theta}(A)$.

As pointed to me by S. Watson, the $\tau\theta$-closure is not in general idempotent.

Definition 3. Let $X$ be a topological space and $A$ a subset of $X$. The $\tau\theta$-closed hull of $A$, denoted by $[A]_{\tau\theta}$, is the smallest $\tau\theta$-closed subset of $X$ containing $A$. 
Clearly, $[A]_{\tau \theta} = \bigcap \{ F : A \subset F \text{ and } \text{cl}_{\tau \theta}(F) = F \}$. For every space $X$ and every $A \subset X$ we have $A \subset \text{cl}_{\tau \theta}(A) \subset [A]_{\tau \theta} \subset \text{cl}_{\tau}(A)$. It is obvious that if $X$ is a Tychonoff space, then $A = \text{cl}_{\tau \theta}(A) = [A]_{\tau \theta} = \text{cl}_{\tau}(A)$ for any $A \subset X$.

The next result gives some conditions on a functionally Hausdorff space which are equivalent to $\text{cl}_{\tau \theta} = \text{cl}_{\tau}$.

**Proposition 4.** For a functionally Hausdorff space $X$ the following conditions are equivalent:

1. For each $\tau$-open set $V$ of $X$, $\overline{V} = \text{cl}_{\tau}(V)$.
2. For each open set $G$ of $X$, $G \subset \text{int}_{\tau}(\text{cl}_{\tau}(G))$.
3. For each subset $A$ of $X$, $\text{cl}_{\tau \theta}(A) = \text{cl}_{\tau}(A)$.
4. For each $\tau$-open subset $V$ of $X$, $\text{cl}_{\tau \theta}(V) = \text{cl}_{\tau}(V)$.

**Proof:** (i) $\leftrightarrow$ (ii) Lemma 28 in [9]. (ii) $\Rightarrow$ (iii) Let $A \subset X$ and $x \notin \text{cl}_{\tau \theta}(A)$, then there is an open neighbourhood $G$ of $x$ such that $\text{cl}_{\tau}(G) \cap A = \emptyset$. By hypothesis $G \subset \text{int}_{\tau}(\text{cl}_{\tau}(G))$, then there is a cozero set $V$ such that $x \in V \subset \text{cl}_{\tau}(G)$, so $V \cap A = \emptyset$ and $x \notin \text{cl}_{\tau}(A)$. Hence, $\text{cl}_{\tau \theta}(A) = \text{cl}_{\tau}(A)$. (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (i) Let $V$ be a $\tau$-open subset of $X$, by hypothesis $\text{cl}_{\tau \theta}(V) = \text{cl}_{\tau}(V)$. Now let $x \notin V$, then there is an open set $G$ such that $x \in G$ and $G \cap V = \text{cl}_{\tau}(V)$. Since $V$ is $\tau$-open, we have $\text{cl}_{\tau}(G) \cap V = \emptyset$, hence $x \notin \text{cl}_{\tau \theta}(V)$. Therefore, $\overline{V} = \text{cl}_{\tau \theta}(V) = \text{cl}_{\tau}(V)$. \qed

**Remark 5.** A functionally Hausdorff space $X$ is called weakly absolutely closed [8] provided that every $\tau$-open filter base on $X$ has an adherent point. An SW space is a functionally Hausdorff space $X$ such that every point-separating subalgebra of $C^*(X)$ which contains the constants is uniformly dense in $C^*(X)$ [8]. It is worth noting that by Lemma 25 in [9] and Proposition 4, a functionally Hausdorff space $X$ is weakly absolutely closed iff it is an SW space and $\text{cl}_{\tau \theta}(A) = \text{cl}_{\tau}(A)$ for every $A \subset X$.

The following result gives an upper bound on the $\tau \theta$-closed hull.

**Theorem 6.** Let $X$ be a functionally Hausdorff space. If $A$ is a subset of $X$, then $|[A]_{\tau \theta}| \leq |A|^\chi(X)$.

**Proof:** Let $m = \chi(X)$ and $k = |A|$. For each $x \in X$ let $\mathcal{B}(x)$ be a base for $X$ at the point $x$ such that $|\mathcal{B}(x)| \leq m$. If $x \in \text{cl}_{\tau \theta}(A)$, choose a point in $\text{cl}_{\tau}(U) \cap A$ for every $U \in \mathcal{B}(x)$ and let $B_x$ be the set so obtained. Clearly, $x \in \text{cl}_{\tau \theta}(B_x)$ and $|B_x| \leq m$. Let $\mathcal{G}_x = \{ \text{cl}_{\tau}(U) \cap B_x : U \in \mathcal{B}(x) \}$. For every $U \in B_x$ we have $x \in \text{cl}_{\tau \theta}(\text{cl}_{\tau}(U) \cap B_x)$, in fact, if $V \in \mathcal{B}(x)$ let $W \in \mathcal{B}(x)$ such that $W \subset V \cap U$, then

$$\emptyset \neq \text{cl}_{\tau}(W) \cap B_x \subset \text{cl}_{\tau}(V \cap U) \cap B_x \subset \text{cl}_{\tau}(V) \cap (\text{cl}_{\tau}(U) \cap B_x).$$

Since $X$ is functionally Hausdorff, then $\bigcap \{ \text{cl}_{\tau \theta}(\text{cl}_{\tau}(U) \cap B_x) : U \in \mathcal{B}(x) \} = \{ x \}$, in fact let $y \neq x$, then there exist open sets $G$ and $H$ such that $x \in G$, $y \in H$ and $\text{cl}_{\tau}(G) \cap \text{cl}_{\tau}(H) = \emptyset$, now let $U \in \mathcal{B}(x)$ such that $U \subset G$, then $\text{cl}_{\tau}(H) \cap \text{cl}_{\tau}(U) = \emptyset$, so $y \notin \bigcap \{ \text{cl}_{\tau \theta}(\text{cl}_{\tau}(U) : U \in \mathcal{B}(x)) \}$, and, a fortiori, $y \notin \bigcap \{ \text{cl}_{\tau \theta}(\text{cl}_{\tau}(U) \cap B_x) : U \in \mathcal{B}(x)) \}$. 


U ∈ B(x). So the map ψ : cl_{τθ}(A) → \mathcal{P}_m(\mathcal{P}_m(A)) defined by ψ(x) = G_x for every x ∈ cl_{τθ}(A), is one to one. Since |\mathcal{P}_m(\mathcal{P}_m(A))| ≤ (k^m)^m = k^m, then |cl_{τθ}(A)| ≤ k^m = |A|\chi(X). Let A_0 = A and, by transfinite induction, define for every \alpha < m^+ sets A_\alpha such that A_\alpha = cl_{τθ}(\bigcup\{A_\beta : \beta < \alpha\}). Clearly \bigcup\{A_\alpha : \alpha < m^+\} ∈ [A]_{τθ}. Now let x ∈ cl_{τθ}(\bigcup\{A_\alpha : \alpha < m^+\}), for each V ∈ B(x) choose a point in cl_{τ}(V) ∩ (\bigcup\{A_\alpha : \alpha < m^+\}) and let B be the set so obtained, obviously B ∈ \mathcal{P}_m(\bigcup\{A_\alpha : \alpha < m^+\}) and x ∈ cl_{τθ}(B). Since m^+ is regular, there is an ordinal \alpha < m^+ such that B ⊆ A_\alpha, so

\[ x ∈ cl_{τθ}(B) ⊆ cl_{τθ}(A_\alpha) ⊆ A_{\alpha + 1} ∪ \{A_\alpha : \alpha < m^+\}, \]

therefore \bigcup\{A_\alpha : \alpha < m^+\} is \tauθ-closed. Hence [A]_{τθ} = \bigcup\{A_\alpha : \alpha < m^+\}. It remains to show that |A_\alpha| ≤ k^m for each \alpha < m^+ (this is equivalent to |\bigcup\{A_\alpha : \alpha < m^+\}| ≤ k^m). Suppose there is an ordinal \alpha < m^+ such that |A_\alpha| > k^m and let γ = min\{\alpha : |A_\alpha| > k^m\}. Since |A_\alpha| ≤ k^m for every \beta < γ, we have |\bigcup\{A_\beta : \beta < γ\}| ≤ k^m. Now A_γ = cl_{τθ}(\bigcup\{A_\beta : \beta < γ\}), hence

\[ |A_γ| = |cl_{τθ}(\bigcup\{A_\beta : \beta < γ\})| ≤ |\bigcup\{A_\beta : \beta < γ\}|\chi(X) ≤ (k^m)^m = k^m, \]

a contradiction. □

**Definition 7.** Let X be a topological space. The w-compactness degree of X, denoted by wcd(X), is defined as the smallest infinite cardinal number k with the property that for every open cover \mathcal{U} of X there is a subcollection \mathcal{V} ∈ \mathcal{P}_k(\mathcal{U}) for which X = \bigcup\{cl_τ(V) : V ∈ \mathcal{V}\}.

For every space X we have wcd(X) ≤ L(X) and this inequality can be proper.

**Example 8.** Let X be any infinite T₃-space such that every continuous real valued function defined on X is constant. Clearly wcd(X) = \aleph_0 < L(X).

**Example 9.** For each \alpha < \omega_1 let I(\alpha) = \{\alpha\} × an open interval in the real line. Set X = \omega_1 ∪ \bigcup\{I(\alpha) : \alpha < \omega_1\} and for x, y ∈ X define x < y if (i) x, y ∈ \omega_1 and x < y in \omega_1, or (ii) x ∈ \omega_1, y ∈ I(\beta) and x ≤ β in \omega_1, or (iii) x ∈ I(γ), y ∈ \omega_1 and γ < y in \omega_1, or (iv) x ∈ I(\alpha), y ∈ I(\beta) and \alpha < \beta in \omega_1, or (v) x, y ∈ I(\alpha) and x < y in I(\alpha). Let σ be the order topology on X. Let Y = X ∪ \{\omega_1\}, define x < \omega_1 for every x ∈ X and let \rho be the order topology on Y. If τ is the topology on Y generated by \rho ∪ \{Y − L : L is the set of limit ordinals in Y − \{\omega_1\}\}, then (Y, τ) is a functionally Hausdorff H-closed space which fails to be Lindelöf [7], so wcd(Y) = \aleph_0 < L(Y).

**Theorem 10.** If X is a functionally Hausdorff space, then |X| ≤ 2\chi(X)wcd(X).

**Proof:** Let m = \chi(X)wcd(X) and for every x ∈ X let B(x) be a base for X at the point x such that |B(x)| ≤ m. Construct a family \{C_\alpha : \alpha < m^+\} of subsets of X such that
(1) for any \( \alpha < m^+ \), \( C_\alpha \) is \( \tau_\theta \)-closed;
(2) for any \( \alpha < m^+ \), \( |C_\alpha| \leq 2^m \);
(3) if \( \alpha < \beta < m^+ \), then \( C_\alpha \subset C_\beta \);
(4) for any \( \alpha < m^+ \), if \( U \subset \bigcup \{ B(x) : x \in \bigcup \{ C_\beta : \beta < \alpha \} \} \), \( |U| \leq m \) and 
\( X - \bigcup \{ \text{cl}_U(U) : U \in U \} \neq \emptyset \), then \( C_\alpha - \bigcup \{ \text{cl}_U(U) : U \in U \} \neq \emptyset \).

The construction is done by transfinite induction. Let \( p \in X \) and \( C_0 = \{ p \} \). Let \( 0 < \alpha < m^+ \) and assume that \( C_\beta \) has been constructed for every \( \beta < \alpha \). Let \( B_\alpha = \bigcup \{ B(x) : x \in \bigcup \{ C_\beta : \beta < \alpha \} \} \), clearly \( |B_\alpha| \leq 2^m \). For any \( U \subset B_\alpha \) such that \( |U| \leq m \) and \( X - \bigcup \{ \text{cl}_U(U) : U \in U \} \neq \emptyset \), choose a point in \( X - \bigcup \{ \text{cl}_U(U) : U \in U \} \) and let \( A \) be the set so obtained, obviously \( |A| \leq 2^m \).

Let \( C_\alpha = \bigcup \{ B_\alpha : \beta < \alpha \} \). \( C_\alpha \) satisfies (1), (3), (4) and, by Theorem 6, also (2). The set \( C = \bigcup \{ C_\alpha : \alpha < m^+ \} \) is \( \tau_\theta \)-closed, in fact let \( x \in \text{cl}_\tau(C) \), for every \( V \in B(x) \) choose a point in \( \text{cl}_\tau(V) \cap C \) and let \( K \) be the set so obtained, clearly \( |K| \leq m \), therefore there exists an \( \alpha < m^+ \) such that \( K \subset C_\alpha \), then \( x \in \text{cl}_\tau(K) \subset \text{cl}_\tau(C_\alpha) = C_\alpha \subset C \). Obviously \( |C| \leq 2^m \), so to complete the proof it suffices to show that \( C = X \). Let us suppose that \( y \in X - C \), since \( X \) is functionally Hausdorff, then for any \( x \in C \) there is a \( U_x \in B(x) \) such that \( y \notin \text{cl}_\tau(U_x) \); for every \( x \in X - C \) let \( U_x \in B(x) \) such that \( \text{cl}_\tau(U_x) \cap C = \emptyset \) (\( C \) is \( \tau_\theta \)-closed). \( \{ U_x \} \subset X \) is an open cover of \( X \), since \( \text{wcd}(X) \leq m \) there is a \( B \subset X \) such that \( |B| \leq m \) and \( X = \bigcup \{ \text{cl}_\tau(U_x) : x \in B \} \), clearly \( C \subset \bigcup \{ \text{cl}_\tau(U_x) : x \in B \cap C \} \). Since \( |B \cap C| \leq m \), there is an \( \alpha < m^+ \) such that \( B \cap C \subset C_\alpha \). Let \( U = \{ U_x : x \in B \cap C \} \), \( U \subset \bigcup \{ B(x) : x \in \bigcup \{ C_\beta : \beta < \alpha + 1 \} \} \), \( |U| \leq m \), \( y \in X - \bigcup \{ \text{cl}_\tau(U_x) : U_x \in U \} \) and \( C_{\alpha+1} - \bigcup \{ \text{cl}_\tau(U_x) : U_x \in U \} = \emptyset \), a contradiction. Hence \( C = X \) and the proof is complete.

**Remark 11.** Let \( X \) be a functionally Hausdorff space and let \( wX \) be the completely regular space which has the same points and continuous real valued functions as those of \( X \). Clearly \( L(wX) \leq \text{wcd}(X) \) for every functionally Hausdorff space \( X \). On the other hand, there exist functionally Hausdorff spaces \( X \) such that \( \chi(X) < \chi(wX) \) (see e.g. [9, Example 36]). I do not know if \( \chi(wX)L(wX) \leq \chi(X) \text{wcd}(X) \) for every functionally Hausdorff space \( X \); if this is the case, then Theorem 10 is a consequence of the Arkhangel’skii’s inequality quoted at the beginning.

**References**

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