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Note on special arithmetic and geometric means

HORST ALZER

Abstract. We prove: If $A(n)$ and $G(n)$ denote the arithmetic and geometric means of the first $n$ positive integers, then the sequence $n \mapsto nA(n)/G(n) - (n-1)A(n-1)/G(n-1)$ ($n \geq 2$) is strictly increasing and converges to $e/2$, as $n$ tends to $\infty$.

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In this paper we denote by $A(n)$ and $G(n)$ the arithmetic and geometric means of the first $n$ positive integers, that is,

$$A(n) = \frac{1}{n} \sum_{i=1}^{n} i = \frac{n+1}{2} \quad \text{and} \quad G(n) = \prod_{i=1}^{n} i^{1/n} = (n!)^{1/n}.$$ 

In 1964 H. Minc and L. Sathre [2] published several remarkable inequalities involving $G(n)$. “Probably the most interesting of them, and certainly the hardest to prove” [2, p. 41], is

$$1 < n \frac{G(n+1)}{G(n)} - (n-1) \frac{G(n)}{G(n-1)} \quad (n \geq 2).$$

It is the aim of this paper to present a closely related result. We prove the following counterpart of inequality (1):

$$\frac{3}{\sqrt{2}} - 1 \leq n \frac{A(n)}{G(n)} - (n-1) \frac{A(n-1)}{G(n-1)} < \frac{e}{2} \quad (n \geq 2).$$

Both bounds are best possible. The double-inequality (2) is an immediate consequence of the following

Theorem. The sequence

$$n \mapsto n \frac{A(n)}{G(n)} - (n-1) \frac{A(n-1)}{G(n-1)} \quad (n \geq 2)$$

is strictly increasing and converges to $e/2$, as $n$ tends to $\infty$.

Proof: In the first part of the proof we show that the function

$$f(x) = x(x+1)(\Gamma(x+1))^{-1/x} \quad (0 < x \in \mathbb{R})$$

is strictly increasing and converges to $e/2$, as $n$ tends to $\infty$.
is strictly convex on $[4, \infty)$. In what follows we assume $x \geq 4$. Differentiation yields
\begin{equation}
\frac{x^2(x+1)f''(x)}{f(x)} = 2x - 2x\Psi(x+1) + 2\log\Gamma(x+1) + (x+1)(\Psi(x+1))^2
- \frac{2(x+1)}{x}\Psi(x+1)\log\Gamma(x+1) + \frac{x+1}{x^2}(\log\Gamma(x+1))^2
- x(x+1)\Psi'(x+1),
\end{equation}
where $\Psi = \Gamma'/\Gamma$ designates the logarithmic derivative of the gamma function. Using the inequalities
\begin{align*}
0 < (x-1/2)\log(x) - x + \log\sqrt{2\pi} < \log\Gamma(x) < 1/(12x) + (x-1/2)\log(x) - x + \log\sqrt{2\pi},
\end{align*}
and
\begin{align*}
0 < \log(x) - 1/(2x) - 1/(12x^2) < \Psi(x) < \log(x) - 1/(2x),
\end{align*}
and\(\Psi'(x) < 1/x + 1/(2x^2) + 1/(6x^3)\), (see [1, p. 820 ff.]), we get from (3):
\begin{equation}
\frac{x^2(x+1)f''(x)}{f(x)} > 2x - 2x \left[ \log(x+1) - \frac{1}{2(x+1)} \right]
+ 2 \left[ (x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi} \right]
+ (x+1) \left[ \log(x+1) - \frac{1}{2(x+1)} - \frac{1}{12(x+1)^2} \right]^2
- \frac{2(x+1)}{x} \left[ \log(x+1) - \frac{1}{2(x+1)} \right] \times
\left[ \frac{1}{12(x+1)} + (x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi} \right]
+ \frac{x+1}{x^2} \left[ (x+1/2)\log(x+1) - (x+1) + \log\sqrt{2\pi} \right]^2
- x(x+1) \left[ \frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3} \right]
= \frac{1}{2} + \frac{1}{x} \left[ \frac{25}{12} - \frac{3}{2} \log(2\pi) + (\log\sqrt{2\pi})^2 \right] - \frac{1}{2(x+1)}
+ \frac{1}{x^2} \left[ 1 - \log(2\pi) + (\log\sqrt{2\pi})^2 \right] + \frac{1}{4(x+1)^2} + \frac{1}{144(x+1)^3}
+ \frac{x+1}{4x^2} (\log(x+1))^2 + \log(x+1) \left\{ \frac{1}{x} \left[ \frac{1}{2} \log(2\pi) - \frac{5}{3} \right] \right\}
- \frac{1}{6(x+1)} + \frac{1}{x^2} \left[ \frac{1}{2} \log(2\pi) - 1 \right].
\end{equation}
Since
\[ 1 - \log(2\pi) + (\log \sqrt{2\pi})^2 > 0 \]
and
\[ \frac{x + 1}{4x^2} (\log(x + 1))^2 > \frac{1}{2x} \log(x + 1) \]
we conclude from (4):
\[ x^2(x + 1) f''(x) f(x) > \frac{1}{2} + \frac{a}{x} - \frac{1}{2(x + 1)} - \log(x + 1) \left[ \frac{b}{x} + \frac{1}{6(x + 1)} + \frac{c}{x^2} \right], \]
where
\[ a = \frac{25}{12} - \frac{3}{2} \log(2\pi) + (\log \sqrt{2\pi})^2 = 0.170 \ldots, \]
\[ b = \frac{7}{6} - \frac{1}{2} \log(2\pi) = 0.247 \ldots, \]
\[ c = 1 - \frac{1}{2} \log(2\pi) = 0.081 \ldots. \]
Using \( \log(x + 1) < x \), we obtain from (5):
\[ x^2(x + 1) f''(x) f(x) > \frac{1}{3} - b + \left( a - c - \frac{1}{3} \right) \frac{1}{x} \]
\[ \geq \frac{1}{3} - b + \left( a - c - \frac{1}{3} \right) \frac{1}{4} = 0.024 \ldots, \]
valid for all \( x \geq 4 \).

Thus, \( f \) is strictly convex on \([4, \infty)\). From Jensen’s inequality we get
\[ 2f(n) < f(n - 1) + f(n + 1) \]
for all integers \( n \geq 5 \). This implies that the sequence
\[ n \mapsto [f(n) - f(n - 1)] / 2 = nA(n)/G(n) - (n - 1)A(n - 1)/G(n - 1) \]
is strictly increasing for \( n \geq 5 \). The approximate values of \( nA(n)/G(n) - (n - 1)A(n - 1)/G(n - 1) \) for \( n = 2, 3, 4, 5 \), are 1.121, 1.180, 1.216, 1.239, respectively. Hence, \( n \mapsto nA(n)/G(n) - (n - 1)A(n - 1)/G(n - 1) \) is strictly increasing for all \( n \geq 2 \).

Finally we prove that
\[ \lim_{n \to \infty} [nA(n)/G(n) - (n - 1)A(n - 1)/G(n - 1)] = e/2. \]
If we set
\[ \alpha(n) = n/G(n) \quad \text{and} \quad \beta(n) = G(n)/G(n - 1), \]
then we have for \( n \geq 2 \):
\[ \frac{n A(n)}{G(n)} - (n - 1) \frac{A(n - 1)}{G(n - 1)} \]
\[ = \frac{1}{2} \left[ \alpha(n) + \frac{n}{n - 1} \alpha(n - 1) - \frac{n}{n - 1} \alpha(n) \beta(n) - \frac{1}{\log \beta(n)} \log \alpha(n) \right]. \]
Since
\[ \lim_{n \to \infty} \alpha(n) = e \quad \text{and} \quad \lim_{n \to \infty} \beta(n) = 1 \]
we obtain (6). This completes the proof of the Theorem. \( \square \)
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