Jaroslav Ivančo; Stanislav Jendroľ; Michal Tkáč

Note on Petrie and Hamiltonian cycles in cubic polyhedral graphs

Commentationes Mathematicae Universitatis Carolinae, Vol. 35 (1994), No. 2, 413--417

Persistent URL: http://dml.cz/dmlcz/118681

Note on Petrie and Hamiltonian cycles in cubic polyhedral graphs

J. Ivančo, S. Jendroľ, M. Tkáč

Abstract. In this note we show that deciding the existence of a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of the existence of an associated cubic plane multi-3-gonal graph with a Hamiltonian cycle which takes alternately left and right edges at each successive vertex, i.e. it is also a Petrie cycle. The Petrie Hamiltonian cycle in an n-vertex plane cubic graph can be recognized by an $O(n)$-algorithm.

Keywords: Hamiltonian cycles, Petrie cycles, cubic polyhedral graphs
Classification: 05C45, 05C38

Throughout this note we shall consider cubic polyhedral graphs, i.e. 3-valent plane 3-connected graphs (see Grünbaum [4], Malkevitch [6]).

Many papers are devoted to the study of the existence of Hamiltonian cycles in cubic plane graphs, see e.g. Holton and McKay [5] or Malkevitch [6] for recent surveys. In Fleischner [2, Chapter VI] there is proved that the problem of finding a Hamiltonian cycle in a cubic plane graph is equivalent to the problem of finding an *A-trail*, that is an Eulerian trail whose consecutive edges (including the last and the first) lie on a common face, in an associated Eulerian plane graph.

In this note we show that the cubic hamiltonian problem is equivalent to the problem of finding a cubic multi-3-gonal plane graph M (i.e. having sizes of all faces $\equiv 0 \pmod 3$) with a Petrie cycle which passes through all vertices of M. A cycle C in a cubic graph is said to be a *Petrie cycle* if every two, but no three, consecutive edges of C (including the last and the first) lie on a common face. A path with this property is known to be a Petrie path (a Petrie arc), cf. Coxeter [1], Grünbaum [4, p. 258].

Petrie cycles do not always exist in cubic plane graphs. For example, a graph of a k-side prisma, $k \geq 3$, has a Petrie cycle if and only if $k \equiv 0 \pmod 4$. Because every Petrie cycle is uniquely determined by arbitrary two of its consecutive edges, the existence of an $O(n)$-algorithm which decides if there is a Petrie cycle crossing all vertices of an n-vertex cubic plane graph is easily seen. Such cycle is called a Petrie Hamiltonian cycle (a *PH-cycle* in the sequel).

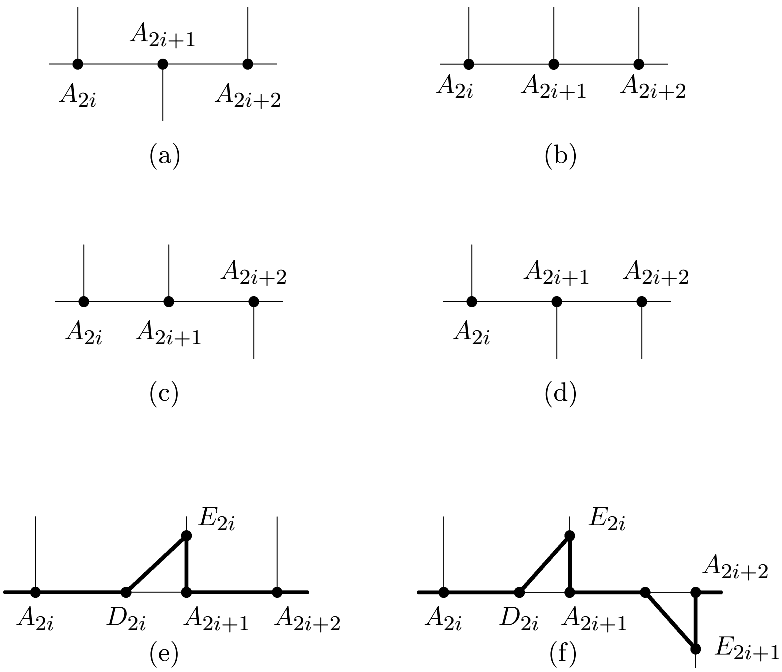
Let G be a cubic plane graph and A be its vertex adjacent to the vertices B_1, B_2, B_3 and incident with faces α_1, α_2, α_3. By a *cutting off* the vertex A of G we mean the placing new vertices A_1 and A_2 on the edges AB_1 and AB_2 of G, respectively, and joining them by a new edge A_1A_2 (i.e. a replacing of the vertex

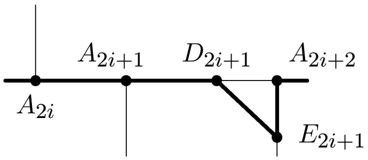
A by a triangle AA_1A_2). This changes the graph G into a cubic plane graph G^* with a new triangle AA_1A_2 and new faces α_1', α_2', α_3' instead of the faces α_1, α_2, α_3 of G. If the face α_i, $i = 1, 2, 3$ is an r_i-gon, the face α_i' is an $(r_i + 1)$-gon. The change G into G^* is denoted by $G^* = G\nabla A$. Let $\mathcal{S} = \{A_i | 1 \le i \le t\}$ be a set of vertices of G. Let $G_0 = G$, $G_i = G_{i-1}\nabla A_i$ for all $i = 1, 2, \ldots, t$. We put

$$G\nabla\mathcal{S} := G_t.$$

Lemma 1. *Let C be a cycle of the length $2k$ in a cubic plane graph G. Then there is a set $\mathcal{S}$ of, say t, vertices of C such that $G^* = G\nabla\mathcal{S}$ has a Petrie cycle C^* of the length $2(k + t)$.*

Proof: Denote the vertices of cycle C successively $A_0, A_1, \ldots, A_{2k-1}$. Let h be an edge incident with the vertex A_0 lying outside of C. We will construct G^* together with its Petrie cycle C^*. Let $G_0 = G$. Suppose we have a graph G_i, $i = 0, 1, \ldots, k-2$ with a Petrie path P_i starting in A_0 and ending in A_{2i} and such that for continuation of P_i the right edge in the vertex A_{2i} must be chosen. In the graph G_i one of the four situations (a), (b), (c), (d) depicted in Fig. 1 appears.





(g)

Figure 1

In the situation (a) of Fig. 1 we put $G_{i+1} := G_i$ and $P_{i=1} := P_i \cup A_{2i+1}A_{2i+2}$. In the situation (b) of Fig. 1 we cut off the vertex A_{2i+1} as it is shown in Fig. 1 (e) and put $G_{i+1} = G_i \nabla A_{2i+1}$ and $P_{i+1} := P_i \cup D_{2i}E_{2i}A_{2i+1}A_{2i+2}$. Changes for the situation (c) and (d) are depicted in Fig. 1 (f) and (g), respectively.

In the graph G_{k-1} we have the Petrie path from A_0 to A_{2k-2} and, because of h, only the situation of Figure (a) or (b) appears. In the first case we put $G^* := G_{k-1}$ and $C^* := P_{k-1} \cup A_{2k-1}A_0$. In the second case $G^* := G_{k-1}\nabla A_{2k-1}$ and $C^* := P_{k-1} \cup D_{2k-2}E_{2k-2}A_{2k-1}A_0$.

The proposition concerning the length of C^* is clear from the above. □

Corollary 2. *If C is a Hamiltonian cycle in a cubic plane graph, then there is a set $\mathcal{S}$ of vertices of G such that $G\nabla\mathcal{S}$ has a Hamiltonian cycle C^* which is also a Petrie cycle.* □

Theorem 3. *A cubic plane graph G is Hamiltonian if and only if there exists a set $\mathcal{S}$ of vertices of G such that the graph $G\nabla\mathcal{S}$ has a PH-cycle.*

Proof: Since G is cubic it has even number of vertices and the necessity follows from Lemma 1 and Corollary 2.

Sufficiency. Let H^* be a PH-cycle in $G\nabla\mathcal{S}$. It is easy to see that each triangle of $G\nabla\mathcal{S}$ has two of its edges on H^*. Let $\tau_1, \tau_2, \dots, \tau_s$, $s \geq 1$, be triangles obtained by cutting off the vertices from $\mathcal{S}$ in G. If we delete from $G\nabla\mathcal{S}$ the edge of τ_j, for any $1 \leq j \leq s$, not lying on H^* and then forget the vertices of degree two, we get a Hamiltonian cycle H in G. □

The problem of deciding the existence of Hamiltonian cycles in cubic, planar, 3-connected graphs, is an NP-complete problem, see Garey et al [3]. Therefore one could think, to find a Hamiltonian cycle by using Theorem 3, it is necessary to consider as set $\mathcal{S}$ all of 2^n subsets of the vertex set of an n-vertex cubic plane graph. But the following theorem provides some restrictions on $\mathcal{S}$.

Theorem 4. *If a cubic polyhedral n-vertex graph G has a PH-cycle then*

(i) *all faces of G are multi-3-gonal,*
(ii) $4 \leq n \equiv 0 \pmod 4$, $n \neq 8$.

Proof: Suppose C is a PH-cycle in G. Then it is easy to see that each third edge of any face in G is a chord of C. Further there is the same number, say

t, of chords in the interior and in the exterior of C. Every chord makes two non-adjacent vertices of C trivalent. Hence C must have $4t$ vertices.

Let G be a cubic polyhedral graph on 8 vertices and with a PH-cycle C. Let the vertices of C be successively $A_1, A_2, \dots, A_8$. Without loss of generality we can assume that the edges A_1A_3 and A_5A_7 lie inside of C. Because of planarity of G, the edges A_2A_6 and A_4A_8 cannot exist in G. The existence of an edge A_2A_4 or A_2A_8 leads to the contradiction with the 3-connectivity of G. □

Note that for any n, $4 \leq n \equiv 0 \pmod 4$, $n \neq 8$, there exists an n-vertex cubic polyhedral graph with PH-cycle. The proof of this statement is left to the reader.

As the cutting off a vertex A of a graph G leads to the increasing of the number in $G\nabla A$ by two, Theorem 4 yields

Corollary 5. *Let G be an n-vertex plane cubic graph having PH-cycle, then*

$$|\mathcal{S}| \equiv \frac{n}{2} \pmod 2.$$

□

Here and in the sequel, $\mathcal{S}$ is as in Theorem 3.

Many other restrictions on $\mathcal{S}$ are given by (i) of Theorem 4. To obtain a multi-3-gonal face from an m-gonal face α, $m \equiv j \pmod 3$, $j = 1, 2, 3$, $3t - j$ vertices must be cut off for some $t = 1, 2, \dots, \lfloor \frac{m}{3} \rfloor$. By this we have

Corollary 6. *Let $p_k(G)$ denote the number of k-gonal faces of an n-vertex cubic plane graph G having PH-cycle and $K = 2\sum_{k\geq 1} p_{3k+1}(G) + \sum_{k\geq 1} p_{3k+2}(G)$. Then*

$$\frac{K}{3} \leq |\mathcal{S}| \leq n - \frac{K}{3}.$$

□

References

[1] Coxeter H.S.M., *Regular Polytopes*, MacMillan, London, 1948.

[2] Fleischner H., *Eulerian graphs and related topics, Part 1, Vol. 1*, North-Holland, Amsterdam, 1990.

[3] Garey M.R., Johnson D.S., Tarjan R.E., *The plane Hamiltonian problem is NP-complete*, SIAM J. Comput. **5** (1968), 704–714.

[4] Grünbaum B., *Convex Polytopes*, Interscience, New York, 1967.

[5] Holton D.A., McKay B.D., *The smallest non-hamiltonian 3-connected cubic planar graphs have 38 vertices*, J. Comb. Theory B **45** (1988), 305–319.

[6] Malkevitch J., *Polytopal graphs*, Selected topics in graph theory III (L.W. Beineke and R.J. Wilson, eds.), Academic Press, London, 1988, pp. 169–188.

J. Ivančo, S. Jendroľ

Department of Geometry and Algebra, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia

M. Tkáč
Department of Mathematics, Technical University, Letná 9, 040 01 Košice, Slovakia

(*Received* May 26, 1993)