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## M-mappings make their images less cellular

MICHAEL G. TKAČENKO

*Abstract.* We consider  $M$ -mappings which include continuous mappings of spaces onto topological groups and continuous mappings of topological groups elsewhere. It is proved that if a space  $X$  is an image of a product of Lindelöf  $\Sigma$ -spaces under an  $M$ -mapping then every regular uncountable cardinal is a weak precaliber for  $X$ , and hence  $X$  has the Souslin property. An image  $X$  of a Lindelöf space under an  $M$ -mapping satisfies  $cel_\omega X \leq 2^\omega$ . Every  $M$ -mapping takes a  $\Sigma(\aleph_0)$ -space to an  $\aleph_0$ -cellular space. In each of these results, the cellularity of the domain of an  $M$ -mapping can be arbitrarily large.

*Keywords:*  $M$ -mapping, topological group, Maltsev space,  $\aleph_0$ -cellularity

*Classification:* 54A25

### 1. Introduction

We define the notion of an  $M$ -mapping that takes its origin from topological groups and Maltsev spaces (see [7], [18] for the discussion of Maltsev spaces). Then we prove the results mentioned above thus generalizing [11, Theorem 1], [16, Theorem 1] and [9, Theorem 1.1].

Let  $G$  be a topological group. The mapping  $F : G^3 \rightarrow G$  defined by  $F(x, y, z) = x \cdot y^{-1} \cdot z$  is continuous and satisfies the condition

$$(*) \quad F(x, y, y) = F(y, y, x) = x \text{ for all } x, y \in G.$$

This reason gives rise for the notion of Maltsev spaces [7], [18], or  $M$ -spaces for short: we say that  $X$  is an  $M$ -space, if there exists a continuous mapping  $F : X^3 \rightarrow X$  satisfying (\*). The mapping  $F$  is called a Maltsev operation on  $X$ . It is clear that every topological group is an  $M$ -space. In fact, every retract of a topological group is an  $M$ -space (if  $r$  is a continuous retraction of a topological group  $G$  onto its subspace  $X$ , define  $F : X^3 \rightarrow X$  by  $F(x, y, z) = r(x \cdot y^{-1} \cdot z)$  for all  $x, y, z$  of  $X$ ). It is an open problem whether every  $M$ -space is a retract of some topological group.

Every  $\sigma$ -compact topological group has the Souslin property [11]. Uspenskii [17] generalized this result by proving that any Maltsev Lindelöf  $\Sigma$ -space has the same property (and even is  $\aleph_0$ -cellular). In fact, a little bit more is proved in [11]: if  $\gamma$  is a family of open sets in a  $\sigma$ -compact topological group  $G$  and the cardinality of  $\gamma$  is uncountable and regular, then there exists a subfamily  $\mu$  of  $\gamma$  such that  $|\mu| = |\gamma|$  and  $U \cap V \neq \emptyset$  for all  $U, V$  of  $\mu$ . In other words, every uncountable

regular cardinal is a weak precaliber for  $G$ . The same conclusion can be proved for a Lindelöf  $\Sigma$ -group or more generally, for a Maltsev Lindelöf  $\Sigma$ -space.

Our aim is to show that both the Souslin property and the weak precaliber property arise when the structures of an  $M$ -space and of a Lindelöf  $\Sigma$ -space are 'separated' by a continuous mapping of a special kind. In fact, the structure of an  $M$ -space will be completely replaced by the so-called  $M$ -mapping.

**1.1 Definition.** We call  $f : X \rightarrow Y$  an  $M$ -mapping provided that there exists a continuous mapping  $F : X^3 \rightarrow Y$  such that  $F(x, y, y) = F(y, y, x) = f(x)$  for all  $x, y$  of  $X$ .

It is clear that every  $M$ -mapping is continuous. If  $f : X \rightarrow Y$  is continuous and  $X$  (or  $Y$ ) is an  $M$ -space, then  $f$  is an  $M$ -mapping. One can easily see that if every continuous mapping of a space  $X$  to elsewhere is an  $M$ -mapping, then  $X$  is an  $M$ -space. Furthermore,  $X$  is an  $M$ -space iff the identity mapping  $id_X$  is an  $M$ -mapping. Also note that  $f : X \rightarrow Y$  is an  $M$ -mapping if there exist an 'intermediate'  $M$ -space  $Z$  and continuous mappings  $g : Z \rightarrow Y$  and  $h : X \rightarrow Z$  such that  $f = g \circ h$ . The class of  $M$ -mappings does not consist only of mappings generated by  $M$ -spaces: it is not necessary to require that  $g$  be defined on the whole space  $Z$ ; it suffices to assume that  $g$  is defined on a subspace  $Z_0$  of  $Z$  containing the set  $F_0(h(X)^3)$ , where  $F_0$  is a Maltsev operation for  $Z$ . Indeed, define a continuous mapping  $F : X^3 \rightarrow Y$  by  $F(x, y, z) = gF_0(h(x), h(y), h(z))$  for all  $x, y, z$  of  $X$ . Obviously,  $F$  is as in Definition 1.1. Note that neither  $h(X)$  nor  $Z_0$  have to be  $M$ -subspaces of  $Z$ . This is the reason why the notion of an  $M$ -mapping seems to be flexible. Some additional information on  $M$ -mappings is contained in [13].

We denote by  $w(X)$ ,  $l(X)$  and  $c(X)$  the weight, Lindelöf number and cellularity of a space  $X$  respectively. A  $G_\tau$ -set in  $X$  is an intersection of at most  $\tau$  many open subsets of  $X$ . The  $\tau$ -cellularity of  $X$ , denoted by  $cel_\tau X$ , is the least cardinal  $\lambda$  such that every family  $\gamma$  of  $G_\tau$ -sets in  $X$  contains a subfamily  $\mu$  of cardinality less than or equal to  $\lambda$  satisfying  $cl(\cup\mu) = cl(\cup\lambda)$ . A space  $X$  is said to be  $\tau$ -cellular if  $cel_\tau X \leq \tau$ .

All considered spaces are assumed to be completely regular.

The notion of a  $\Sigma$ -space (see [8]) is one of the most important in the paper. For the reader's convenience we give some definitions related to this concept.

**1.2 Definition.** We call  $X$  a  $\Sigma$ -space, if there are two covers  $\mathcal{K}$  and  $\mathcal{C}$  of  $X$  such that  $\mathcal{K}$  is  $\sigma$ -locally finite and consists of closed sets,  $\mathcal{C}$  consists of countably compact sets and for every  $C \in \mathcal{C}$  and every neighborhood  $U$  of  $C$  there exists  $K \in \mathcal{K}$  satisfying  $C \subseteq K \subseteq U$ . If in addition  $|\mathcal{K}| \leq \tau$ , we call  $X$  a  $\Sigma(\tau)$ -space.

Note that every countably compact space is obviously a  $\Sigma(\aleph_0)$ -space. The classes of  $\Sigma$ - and  $\Sigma(\tau)$ -spaces are closed with respect to countable unions and passing to perfect (even quasiperfect) images [8].

The following cardinal invariant was defined by Arhangel'skiĭ [1] in a slightly different form. Let  $\beta X$  be the Čech-Stone compactification of  $X$  and  $\mathcal{F}$  the family

of all closed subsets of  $\beta X$ .

**1.3 Definition.** The Nagami number of a space  $X$  is the cardinal number

$$Nag(X) = \min\{|\gamma| : (\gamma \subseteq \mathcal{F}) \& (\forall x \in X \forall y \in \beta X \setminus X \exists F \in \gamma \text{ with } x \in F \not\supseteq y)\}.$$

In other words, the Nagami number of  $X$  is the least cardinality of a family  $\gamma \subseteq \mathcal{F}$  separating the points of  $X$  from the points of  $\beta X \setminus X$ . Note that a  $\Sigma$ -space  $X$  is Lindelöf iff  $Nag(X) \leq \aleph_0$ . Lindelöf  $\Sigma$ -spaces can also be characterized as continuous images of spaces that admit perfect mappings onto separable metric ones [1]. Compact and  $\sigma$ -compact spaces constitute proper subclasses of the class of Lindelöf  $\Sigma$ -spaces, and the latter is countably productive. Clearly,  $l(X) \leq Nag(X)$  for every space  $X$ . It needs mentioning that the Čech-Stone compactification  $\beta X$  in Definition 1.3 can be replaced by any compactification of  $X$ .

## 2. Main results

Let  $\Pi = \prod_{i \in J} P_i$  be a product space and  $p^* \in \Pi$ . Denote by  $\sigma(p^*)$  the subspace of  $\Pi$  consisting of all points which differ from  $p^*$  on at most finitely many coordinates. The following theorem presents the main result of the paper.

**2.1 Theorem.** Let  $\Pi = \prod_{i \in J} P_i$  be a product of spaces satisfying  $Nag(P_i) \leq \tau$  for each  $i \in J$ , and  $Z$  be a subspace of  $\Pi$  such that  $\sigma(p^*) \subseteq Z$ . If  $X$  is an image of  $Z$  under an  $M$ -mapping, then every regular cardinal  $\lambda > \tau$  is a weak precaliber for  $X$ .

We will prove the theorem by making use of two auxiliary set-theoretic lemmas. The first of them reminds of the  $\Delta$ -lemma [5, Appendix 2] and has a similar proof.

**2.2 Lemma.** Suppose that  $\lambda$  and  $\tau$  are infinite cardinals such that  $cf(\lambda) > \tau^+$  and  $k^\tau < \lambda$  for each  $k < \lambda$ . If  $\xi$  is a family of sets of cardinality less than or equal to  $\tau$  and  $|\xi| = \lambda$ , then  $\xi$  contains a quasidisjoint subfamily  $\eta$  of the same cardinality  $\lambda$ , i.e. there exists a set  $R$  such that  $M \cap N = R$  whenever  $M, N \in \eta$  and  $M \neq N$ .

PROOF: For an arbitrary set  $A$  put  $\xi_A = \{M \setminus A : M \in \xi\}$ . It suffices to consider the case when the family  $\xi_A$  does not contain disjoint subfamilies of cardinality  $\lambda$  whenever  $|A| < \lambda$ . If  $\alpha < \tau^+$  and a subfamily  $\eta(\beta)$  of  $\xi$  with  $|\eta(\beta)| < \lambda$  has already been defined for each  $\beta < \alpha$ , we put  $B(\beta) = \cup \eta(\beta)$ ,  $A(\alpha) = \bigcup_{\beta < \alpha} B(\beta)$  and define  $\eta(\alpha)$  as a maximal disjoint subfamily of  $\xi_{A(\alpha)}$ . Obviously  $|A(\alpha)| < \lambda$ , and hence  $|\eta(\alpha)| < \lambda$  as well. If now  $M \in \xi \setminus \bigcup_{\alpha < \tau^+} \eta(\alpha)$ , then  $M \cap B(\alpha) \neq \emptyset$  for each  $\alpha < \tau^+$ . The latter implies  $|M| > \tau$ , for  $B(\alpha) \cap B(\beta) = \emptyset$  whenever  $\alpha \neq \beta$ , which contradicts the assumption on the family  $\xi$ . □

If  $\gamma$  is a cover of a set  $X$  and  $x \in X$ , we denote  $St(x, \gamma) = \cup\{U \in \gamma : x \in U\}$ .

**2.3 Lemma** (see [11]). *Let  $\{\gamma_\alpha : \alpha < \lambda\}$  be a family of finite covers of a set  $T$ , where  $\lambda$  is a regular uncountable cardinal. Then for every sequence  $\{x_\alpha : \alpha < \lambda\} \subseteq T$  there exists a subset  $A \subseteq \lambda$  such that  $|A| = \lambda$  and  $St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset$  for all  $\alpha, \beta \in A$ .*

**Proof of Theorem 2.1.** Let  $f$  be an  $M$ -mapping of  $Z$  onto  $X$  and suppose that a continuous mapping  $F : Z^3 \rightarrow X$  witnesses that. Suppose also that  $\{O_\alpha : \alpha < \lambda\}$  is a family of non-void open sets in  $X$ , where  $\lambda$  is a regular cardinal,  $\lambda > \tau$ . Since  $\sigma = \sigma(p^*)$  is dense in  $\Pi$  (and in  $Z$ ), for each  $\alpha < \lambda$  one can find a point  $x_\alpha \in \sigma$  with  $f(x_\alpha) \in O_\alpha$ . For all  $x \in \sigma$  and  $\alpha < \lambda$  define  $U_\alpha(x) \ni x$  as a maximal open set in  $\sigma$  such that

$$(1) F(x_\alpha, y, z) \in O_\alpha \text{ and } F(z, y, x_\alpha) \in O_\alpha \text{ for all } y, z \in U_\alpha(x).$$

Then define  $\gamma_\alpha = \{U_\alpha(x) : x \in \sigma\}$ ;  $\alpha < \lambda$ . It is clear that  $\gamma_\alpha$  is an open cover of  $\sigma$  for each  $\alpha < \lambda$ . Our aim is to show that the conclusion of Lemma 2.3 holds for the family  $\{\gamma_\alpha : \alpha < \lambda\}$  and the sequence  $\{x_\alpha : \alpha < \lambda\}$ . The further reasoning is divided into three parts.

*Case 1.* Suppose that  $\lambda \leq 2^\tau$ . For each  $x \in \sigma$  denote by  $supp(x)$  the set of all indices of  $J$  on which  $x$  differs from  $p^*$ . Put  $B = \bigcup_{\alpha < \lambda} supp(x_\alpha)$ . Then  $|B| \leq \lambda$ , because  $|supp(x)| < \aleph_0$  whenever  $x \in \sigma$ . Denote by  $\pi_B$  the projection of  $\Pi$  onto  $\Pi_B = \prod_{i \in B} P_i$ . The subspace  $\sigma_B = \{x \in \sigma : supp(x) \subseteq B\}$  of  $\Pi$  is homeomorphic to  $\pi_B(\sigma)$ , and the latter is the  $\sigma$ -product  $\sigma(\pi_B(p^*)) \subseteq \Pi_B$  with the base point  $\pi_B(p^*)$ . We have  $|B| \leq \lambda \leq 2^\tau$  and  $Nag(P_i) \leq \tau$  for all  $i \in B$ . Therefore  $Nag(\sigma_B) = Nag(\pi_B(\sigma)) \leq \tau$  by a theorem of Ranchin [2]. Let  $\beta\Pi$  be the Čech-Stone compactification of  $\Pi$  and  $\mathcal{F}$  a family of closed sets in  $\beta\Pi$  separating points of  $\sigma_B$  from the points of  $\beta\Pi \setminus \sigma_B$ ,  $|\mathcal{F}| \leq \tau$ . We can assume that the family  $\mathcal{F}$  is closed under finite intersections. For every  $\alpha < \lambda$  denote by  $\tilde{\gamma}_\alpha$  a family of open sets in  $\beta\Pi$  whose traces on  $\sigma$  constitute the cover  $\gamma_\alpha$  of  $\sigma$  and find an element  $L \in \mathcal{F}$  such that  $x_\alpha \in L \subseteq \cup \tilde{\gamma}_\alpha$ . It does exist, for otherwise  $\cap \mathcal{F}_\alpha \setminus \cup \tilde{\gamma}_\alpha \neq \emptyset$ , where  $\mathcal{F}_\alpha = \{K \in \mathcal{F} : x_\alpha \in K\}$ . The latter, however, contradicts the separating property of  $\mathcal{F}$ . Since  $\lambda$  is a regular cardinal and  $|\mathcal{F}| \leq \tau < \lambda$ , one can find a set  $A_0 \subseteq \lambda$ ,  $|A_0| = \lambda$ , and an element  $K^* \in \mathcal{F}$  such that  $x_\alpha \in K^* \subseteq \cup \tilde{\gamma}_\alpha$  for all  $\alpha \in A_0$ . Using compactness of  $K^*$ , we can assume that  $|\tilde{\gamma}_\alpha| < \aleph_0$  for each  $\alpha \in A_0$ . Thus  $\gamma_\alpha$  is a finite cover of the set  $T = \{x_\beta : \beta \in A_0\}$  for each  $\alpha \in A_0$ , and Lemma 2.3 implies the existence of a subset  $A \subseteq A_0$  with  $|A| = \lambda$  such that  $St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset$  for all  $\alpha, \beta \in A$ .

*Case 2.* There exists a cardinal  $k$  such that  $\tau < k < \lambda \leq 2^k$ .

The argument here is just the same as in Case 1 with  $k$  playing the rôle of  $\tau$ .

*Case 3.*  $2^k < \lambda$  for each  $k < \lambda$  (i.e.  $\lambda$  is a strong limit cardinal). This is the only case to care about thoroughly. Since  $X$  is completely regular, we can assume that for each  $\alpha < \lambda$  there exists a continuous function  $\varphi_\alpha : X \rightarrow [0, 1]$  such that  $O_\alpha = \varphi_\alpha^{-1}(0, 1]$ . Consider continuous mappings  $\psi_\alpha = \varphi_\alpha \circ F$  and  $\tilde{\psi}_\alpha = \psi_\alpha|_{\sigma^3}$ ;  $\alpha < \lambda$ . Note that for every finite  $B \subseteq J$  the space  $\Pi_B$  satisfies the inequality  $l(\Pi_B) \leq Nag(\Pi_B) \leq \tau$ , and hence  $l(\sigma) \leq \tau$  by a theorem of [6]. By the same

reason we have  $l(\sigma^3) \leq \tau$ . Since  $\tilde{\psi}_\alpha$  is a continuous mapping of  $\sigma^3$  to a second-countable space,  $\tilde{\psi}_\alpha$  depends on at most  $\tau$  coordinates by Corollary 1 of [10] (another way is to apply a theorem of Engelking [3]). Let  $M_\alpha$  be the set of all the indices of  $J$  the function  $\tilde{\psi}_\alpha$  depends on;  $|M_\alpha| \leq \tau$ . Note that in fact we should speak about coordinates of the index set  $J^3$ , but ‘spilling’ the triples of  $J^3$  we get the sets  $M_\alpha$  lying in  $J$ . Obviously,  $\psi_\alpha$  does not depend on  $J \setminus M_\alpha$  either, because  $\sigma^3$  is dense in  $Z^3$ . For every  $\alpha < \lambda$  denote  $N_\alpha = M_\alpha \cup \text{supp}(x_\alpha)$  and consider the family  $\xi = \{N_\alpha : \alpha < \lambda\}$ . If  $|\xi| < \lambda$ , by the regularity of  $\lambda$  there exists a set  $E \subseteq \lambda$  of cardinality  $\lambda$  such that  $N_\alpha = N_\beta = B$  for all  $\alpha, \beta \in E$ . Then  $|B| \leq \tau$  and we will get Case 1 if we consider the subspace  $\sigma_B$  of  $Z$  and the family  $\{\gamma_\alpha : \alpha \in E\}$  of covers of  $\sigma_B$ . So it remains to assume that  $|\xi| = \lambda$ . Then by Lemma 2.2 the family  $\xi$  contains a quasidisjoint subfamily  $\eta$  with  $|\eta| = \lambda$ . Let  $\eta = \{N_\alpha : \alpha \in C\}$ , where  $C \subseteq \lambda$ , and suppose that  $R$  is a root of  $\eta$ . Diminishing  $\eta$ , we can also assume that  $N_\alpha \neq N_\beta$  for all distinct  $\alpha, \beta$  of  $C$ . For each  $\alpha \in C$  put  $N_\alpha^* = N_\alpha \setminus R$ . Then  $\{N_\alpha^* : \alpha \in C\}$  is a disjoint family of cardinality  $\lambda$ .

We claim that  $U = \sigma \cap \pi_{N_\alpha}^{-1} \pi_{N_\alpha}(U)$  for all  $U \in \gamma_\alpha$  and  $\alpha < \lambda$ . First, note that the restriction of  $\pi_{N_\alpha}$  to  $\sigma$  is an open mapping onto  $\pi_{N_\alpha}(\sigma)$ ; therefore by the maximality of a set  $U (= U_\alpha(x)$  for some point  $x \in \sigma$ ) it suffices to verify that the open set  $\sigma \cap \pi_{N_\alpha}^{-1} \pi_{N_\alpha}(U)$  satisfies the above condition (1). Indeed, take  $y, z, y', z' \in \sigma$  such that  $y, z \in U_\alpha(x)$ ,  $\pi_{N_\alpha}(y') = \pi_{N_\alpha}(y)$  and  $\pi_{N_\alpha}(z') = \pi_{N_\alpha}(z)$ . Then  $\pi_{N_\alpha}^3(x_\alpha, y', z') = \pi_{N_\alpha}^3(x_\alpha, y, z)$ , and the definition of the subset  $M_\alpha$  of  $N_\alpha$  implies

$$\psi_\alpha(x_\alpha, y', z') = \varphi_\alpha F(x_\alpha, y', z') = \varphi_\alpha F(x_\alpha, y, z) = \psi_\alpha(x_\alpha, y, z).$$

By the definition of  $U_\alpha(x)$  we have

$$y, z \in U_\alpha(x) \implies \varphi_\alpha F(x_\alpha, y, z) > 0 \implies \varphi_\alpha F(x_\alpha, y', z') > 0.$$

Analogously,

$$y, z \in U_\alpha(x) \implies \varphi_\alpha F(z, y, x_\alpha) > 0 \implies \varphi_\alpha F(z', y', x_\alpha) > 0.$$

Since  $O_\alpha = \varphi_\alpha^{-1}(0, 1]$ , the inequalities above imply that the set  $\tilde{U}_\alpha(x) = \sigma \cap \pi_{N_\alpha}^{-1} \pi_{N_\alpha}(U_\alpha(x))$  satisfies (1), and hence  $\tilde{U}_\alpha(x) = U_\alpha(x)$ .

Denote  $\Pi_R^* = \Pi_R \times \{\pi_{J \setminus R}(p^*)\}$  and  $\Phi = cl_{\beta \Pi} \Pi_R^*$ . Since  $|R| \leq \tau$ , the subspace  $\sigma_R = \sigma \cap \Pi_R^*$  of  $\Pi$  satisfies  $Nag(\sigma_R) \leq \tau$ . The set  $\sigma_R$  is dense in  $\Phi$ , and hence there exists a family  $\mathcal{F}$  of closed sets in  $\Phi$  separating points of  $\sigma_R$  from the points of  $\Phi \setminus \sigma_R$ ,  $|\mathcal{F}| \leq \tau$ . For each  $\alpha \in C$ , define a point  $x_\alpha^* \in \sigma_R$  by  $\pi_R(x_\alpha^*) = \pi_R(x_\alpha)$  and  $\pi_{J \setminus R}(x_\alpha^*) = \pi_{J \setminus R}(p^*)$ . The same argument as in Case 1 gives us an element  $K^* \in \mathcal{F}$  and a subset  $D$  of  $C$  such that  $|D| = \lambda$  and  $x_\alpha^* \in K^* \subseteq \cup \tilde{\gamma}_\alpha$  for all  $\alpha \in D$ .

**Claim.** If  $\mu \subseteq \gamma_\alpha$  for some  $\alpha \in D$  and  $K^* \subseteq \cup\mu$ , then  $\{x_\beta : \beta \in D\} \setminus \{x_\alpha\} \subseteq \cup\mu$ .

Indeed, let  $\alpha \in D$  and  $\mu \subseteq \gamma_\alpha$ ,  $F \subseteq \cup\mu$ . For an arbitrary  $\beta \in D \setminus \{\alpha\}$  choose  $U \in \mu$  with  $x_\beta^* \in U$ . The points  $x_\beta$  and  $x_\beta^*$  coincide on all coordinates of  $J \setminus N_\beta^*$ , and  $N_\beta^* \cap N_\alpha = \emptyset$ . Since  $x_\beta^* \in U$  and  $U = \pi_{N_\alpha}^{-1} \pi_{N_\alpha}(U)$ , we conclude that  $x_\beta \in U$ . This proves the claim.

For each  $\alpha \in D$ , find a finite subfamily  $\mu_\alpha$  of  $\gamma_\alpha$  that covers the compact set  $K^* \cup \{x_\alpha\}$  and apply Lemma 2.4 to the family  $\{\mu_\alpha : \alpha \in D\}$  of covers of the sequence  $\{x_\alpha : \alpha \in D\}$  (Claim works here). This gives us a subset  $A \subseteq D$  of cardinality  $\lambda$  such that  $St(x_\alpha, \mu_\beta) \cap St(x_\beta, \mu_\alpha) \neq \emptyset$  for all  $\alpha, \beta \in A$ . Since  $\mu_\alpha \subseteq \gamma_\alpha$ , we conclude that

$$(*) \quad St(x_\alpha, \gamma_\beta) \cap St(x_\beta, \gamma_\alpha) \neq \emptyset \text{ for all } \alpha, \beta \in A.$$

*Final step.* In each of Cases 1–3 we have found a subset  $A \subseteq \lambda$  of cardinality  $\lambda$  with the above property (\*). It remains to show that  $O_\alpha \cap O_\beta \neq \emptyset$  for all  $\alpha, \beta \in A$ . To this end apply an argument similar to the proof of Proposition 1 of [16]. Let  $\alpha, \beta \in A$  and  $\alpha \neq \beta$ . By (\*) one can find  $U \in \gamma_\alpha$ ,  $V \in \gamma_\beta$  and a point  $z \in \sigma$  such that  $\{x_\alpha, z\} \subseteq V$  and  $\{x_\beta, z\} \subseteq U$ . Then by the definition of  $\gamma_\alpha$  and  $\gamma_\beta$ , we have  $F(x_\alpha, z, x_\beta) \in O_\alpha \cap O_\beta \neq \emptyset$ . □

The following result was earlier announced by the author in [12] for  $\tau = \aleph_0$ .

**2.4 Theorem.** Let  $\Pi = \prod_{i \in J} P_i$  be a product of spaces satisfying  $Nag(P_i) \leq \tau$  for all  $i \in J$ , and suppose that  $f : \Pi \rightarrow X$  is an  $M$ -mapping onto a Tikhonov space  $X$ . Then every regular cardinal  $\lambda > \tau$  is a weak precaliber for  $X$  and  $cel_\tau X \leq \tau$ .

PROOF: It suffices to put  $Z = \Pi$  in Theorem 2.1. The inequality  $cel_\tau \leq \tau$  follows from [13, Theorem 2.2]. □

**2.5 Remark.** The subspace  $Z$  of the product  $\Pi$  in Theorem 2.1 could not be an arbitrary dense set in  $\Pi$  even if the index set  $J$  is one-point, i.e.  $Nag(\Pi) \leq \tau$ . In fact, every space is an image of some dense subspace of a compact space under an  $M$ -mapping. Indeed, for a given space  $X$  let  $X_d$  be a discrete group with the underlying set  $X$ . Then the identity mapping  $id_X : X_d \rightarrow X$  is obviously an  $M$ -mapping and  $X_d$  is dense in the compact space  $\beta X_d$ .

**2.6 Remark.** A proof of the ‘weak precaliber’ part of Theorem 2.4 can be essentially simplified in comparison with the proof of Theorem 2.1. Indeed, let us have defined the points  $x_\alpha \in \Pi$  with  $f(x_\alpha) \in O_\alpha$  for all  $\alpha < \lambda$ . For every  $\alpha < \lambda$  there exists a standard open set  $V_\alpha \ni x_\alpha$  in  $\Pi$  such that  $f(V_\alpha) \subseteq O_\alpha$ . Then  $V_\alpha = \pi_{D(\alpha)}^{-1} \pi_{D(\alpha)} V_\alpha$  for some finite subset  $D(\alpha)$  of  $J$ . By the usual  $\Delta$ -lemma, the family  $\xi = \{D(\alpha) : \alpha < \lambda\}$  contains a quasisdisjoint subfamily of the same cardinality  $\lambda$ , so we can assume that  $\xi$  is quasisdisjoint itself and has a root  $R$ ,  $|R| < \aleph_0$ . Put  $E(\alpha) = D(\alpha) \setminus R$  for each  $\alpha < \lambda$ . The family  $\{E(\alpha) : \alpha < \lambda\}$  is disjoint, and one can find a point  $q \in \Pi_{J \setminus R}$  such that  $q|_{E(\alpha)} = x_\alpha|_{E(\alpha)}$  for all  $\alpha < \lambda$ . Denote  $\Pi_R^* = \Pi_R \times \{q\}$ . Then  $V_\alpha \cap \Pi_R^* \neq \emptyset$ , and hence  $f(\Pi_R^*) \cap O_\alpha \neq \emptyset$

for all  $\alpha < \lambda$ . Since  $Nag(\Pi_R^*) = Nag(\Pi_R) \leq \tau$ , it remains to apply the argument of Case 1 and Final step of the corresponding proof.

**2.7 Corollary.** *If an  $M$ -space  $X$  is a continuous image of a product of Lindelöf  $\Sigma$ -spaces, then every regular uncountable cardinal is a weak precaliber for  $X$  and  $X$  is  $\aleph_0$ -cellular.*

PROOF: Every continuous mapping onto an  $M$ -space is an  $M$ -mapping. The conclusion now follows from Theorem 2.4.  $\square$

**2.8 Corollary.** *If  $X$  is a dense subspace of a Lindelöf  $\Sigma$ -group, then every regular uncountable cardinal is a weak precaliber for  $X$ .*

PROOF: The ‘weak precaliber’ property is hereditary with respect to dense subspaces. It remains to note that a topological group is an  $M$ -space and apply Corollary 2.7.  $\square$

It is known that the cellularity can be raised by multiplying of two spaces [14]. Recently Todorčević [15] constructed (in ZFC only) an example of a topological group  $H$  with  $c(H \times H) > c(H)$ . However, the latter is impossible in the class of subgroups of Lindelöf  $\Sigma$ -groups.

**2.9 Proposition.** *Let  $G$  be a subgroup of a Lindelöf  $\Sigma$ -group. Then  $c(G \times H) = c(H)$  for every infinite topological group  $H$ .*

PROOF: There exists a Lindelöf  $\Sigma$ -group  $\hat{G}$  containing  $G$  as a subgroup. Then  $K = cl_{\hat{G}}G$  is a closed subgroup of  $\hat{G}$ , and hence is a Lindelöf  $\Sigma$ -group. Since  $G$  is dense in  $K$ , every uncountable regular cardinal is a weak precaliber for  $G$  by Corollary 2.8. In particular, this is the case for  $\tau^+$  where  $\tau = c(H)$ . For the following argument, topological group structures of  $G$  and  $H$  are unessential. Let  $\gamma$  be a family of open sets in  $G \times H$ ,  $|\gamma| = \tau^+$ . It suffices to find two distinct elements of  $\gamma$  with a non-empty intersection. We can assume that each element of  $\gamma$  has the form  $U \times V$  for some open sets  $U \subseteq X$  and  $V \subseteq Y$ . For every subfamily  $\mu$  of  $\gamma$  denote  $\tilde{\mu} = \{p_1(O) : O \in \mu\}$ , where  $p_1$  is the projection of  $G \times H$  onto  $G$ . Since  $\tau^+$  is a weak precaliber for  $G$ , one can find a subfamily  $\mu$  of  $\gamma$  such that  $|\mu| = \tau^+$  and  $U \cap U' \neq \emptyset$  for all  $U, U' \in \tilde{\mu}$ . The inequality  $c(H) \leq \tau$  implies that  $p_2(O) \cap p_2(O') \neq \emptyset$  for some distinct  $O, O' \in \mu$  where  $p_2$  is the projection of  $G \times H$  onto  $H$ . Since  $O$  and  $O'$  have rectangular form, we conclude that  $O \cap O' \neq \emptyset$ .  $\square$

By Theorem 1.1 of [9], every topological group  $G$  with  $l(G) \leq \tau$  is  $2^\tau$ -cellular, i.e.  $cel_\tau(G) \leq 2^\tau$ . Roughly speaking, this result was proved by making use of a ‘good’ lattice of open mappings of  $G$  onto quotient groups of countable pseudocharacter. Again, we show that the existence of an appropriate  $M$ -mapping is responsible for this phenomenon.

**2.10 Theorem.** *Let  $f : \Pi \rightarrow X$  be an  $M$ -mapping of a space  $\Pi$  with  $l(\Pi) \leq \tau$  onto  $X$ . Then  $cel_\tau(X) \leq 2^\tau$ .*

PROOF: Suppose the contrary. Then there exists a sequence  $\{(K_\alpha, O_\alpha) : \alpha < \lambda\}$  such that  $K_\alpha \subseteq O_\alpha \subseteq X$ ,  $K_\alpha$  is a non-empty closed  $G_\tau$ -set in  $X$ ,  $O_\alpha$  is open in



$X$  and  $K_\beta \cap O_\alpha = \emptyset$  whenever  $\beta < \alpha < \lambda$ ;  $\lambda = (2^\tau)^+$ . Diminishing  $K_\alpha$  and  $O_\alpha$  if necessary, we can assume that for each  $\alpha < \lambda$  there exists a continuous mapping  $\varphi_\alpha$  of  $X$  onto a space  $X_\alpha$  of weight at most  $\tau$  such that  $K_\alpha = \varphi_\alpha^{-1}\varphi_\alpha K_\alpha$ .

Let  $F : \Pi^3 \rightarrow X$  be a continuous mapping with the property  $F(x, y, y) = F(y, y, x) = f(x)$  for all  $x, y \in \Pi$ . For every  $\alpha < \lambda$  pick a point  $x_\alpha \in \Pi$  with  $f(x_\alpha) \in K_\alpha$  and define a continuous mapping  $\psi_{\alpha,\beta} : \Pi \rightarrow X_\alpha$  by  $\psi_{\alpha,\beta}(x) = \varphi_\alpha F(x_\alpha, x_\beta, x)$  for all  $\alpha, \beta < \lambda$  and  $x \in \Pi$ . Note that  $\psi_{\beta,\beta} = \varphi_\beta \circ f$  for all  $\beta < \lambda$ . Denote  $T = \{x_\alpha : \alpha < \lambda\}$  and  $T_\alpha = \{x_\beta : \beta < \alpha\}$ ;  $\alpha < \lambda$ . If  $g : \Pi \rightarrow Y$  is an arbitrary mapping to a space  $Y$  with  $w(Y) \leq \tau$ , then  $|Y| \leq 2^\tau$ ; therefore there exists an ordinal  $\delta(g) < \lambda$  such that  $g(T_{\delta(g)}) = g(T)$ .

The following step is a transfinite construction on  $\alpha < \lambda$ . Put  $\nu(0) = 0$ . Suppose that for some  $\alpha < \tau^+$  we have already defined ordinals  $\nu(\beta) < \lambda$  for all  $\beta < \alpha$ . If  $\alpha$  is limit, we put  $\nu(\alpha) = \sup_{\beta < \alpha} \nu(\beta)$ . Otherwise  $\alpha = \beta + 1$  for some  $\beta$  and we put  $\mathcal{G}_\alpha = \{\psi_{\gamma,\gamma'} : \gamma, \gamma' \leq \beta\}$  and denote by  $\mathcal{H}_\alpha$  the family  $\{\Delta\Psi : \Psi \subseteq \mathcal{G}_\alpha, |\Psi| \leq \tau\}$ , where  $\Delta\Psi$  is the diagonal product of mappings of  $\Psi$ . Obviously,  $|\mathcal{H}_\alpha| \leq 2^\tau$  and the weight of the space  $g(\Pi)$  does not exceed  $\tau$  for each  $g \in \mathcal{H}_\alpha$ . Therefore we can define  $\nu(\alpha)$  as the maximum of the ordinals  $\nu(\beta)$  and  $\sup\{\delta(g) : g \in \mathcal{H}_\alpha\}$ . Clearly,  $\nu(\alpha) < \lambda$ . This completes our construction.

Put  $\nu = \sup_{\alpha < \tau^+} \nu(\alpha)$ ,  $\mathcal{G} = \{\psi_{\gamma,\gamma'} : \gamma, \gamma' < \nu\}$  and  $h = \Delta\mathcal{G}$ . The crucial statement is that the set  $C = cl_{\Pi} T_\nu$  has the property  $h(T) \subseteq h(C)$ . Indeed, by the construction, for every subfamily  $\Psi$  of  $\mathcal{G}$  with  $|\Psi| \leq \tau$  we have  $(\Delta\Psi)(T) \subseteq (\Delta\Psi)(T_\nu)$ , and the statement follows from the fact that  $l(C) \leq \tau$ .

Choose a point  $z \in C$  with  $h(z) = h(x_\nu)$ . We have  $F(z, z, x_\nu) = f(x_\nu) \in K_\nu \subseteq O_\nu$ , and the continuity of  $F$  in the first argument implies that there exists an open neighbourhood  $V$  of  $z$  in  $\Pi$  such that  $F(x, z, x_\nu) \in O_\nu$  for all  $x \in V$ . Since  $z$  is in the closure of  $T_\nu$ , one can find  $\alpha < \nu$  with  $x_\alpha \in V$ . Thus we have  $F(x_\alpha, z, x_\nu) \in O_\nu$ . To get a contradiction it remains to show that  $F(x_\alpha, z, x_\nu) \in K_\alpha$ .

Since  $\psi_{\alpha,\beta} \in \mathcal{G}$  for all  $\beta < \nu$ , from the choice of the point  $z$  it follows that  $\varphi_\alpha F(x_\alpha, x_\beta, x_\nu) = \varphi_\alpha F(x_\alpha, x_\beta, z)$  for each  $\beta < \nu$ . Using the continuity of  $F$  in the first argument, we get

$$\varphi_\alpha F(x_\alpha, z, x_\nu) = \varphi_\alpha F(x_\alpha, z, z) = \varphi_\alpha f(x_\alpha) \in \varphi_\alpha(K_\alpha).$$

However,  $K_\alpha = \varphi_\alpha^{-1}\varphi_\alpha(K_\alpha)$ , whence it follows that the point  $p = F(x_\alpha, z, x_\nu)$  belongs to  $K_\alpha$ . Thus  $p \in O_\nu \cap K_\alpha \neq \emptyset$  and  $\alpha < \nu$ , a contradiction with the choice of  $K_\alpha$  and  $O_\nu$ . □

**2.11 Remark.** An easy examination of the above proof shows that we have not used the continuity of the mapping  $F : \Pi^3 \rightarrow X$  in all its power. In fact, just the separate continuity of  $F$  in its arguments is necessary. This enables to consider groups with a separately continuous multiplication (and continuous inverse); let us call them quasitopological. Thus we have the following.

**2.12 Corollary.** *A quasitopological group  $G$  with  $l(G) \leq \tau$  satisfies  $cel_\tau(G) \leq 2^\tau$ .*

This corollary of Theorem 2.10 is a slight generalization of [9, Theorem 1.1]. Note that one cannot improve Corollary 2.12 by showing  $cel_\tau(G) \leq \tau$  even for a topological group  $G$ . Indeed, if  $X$  is a one-point  $\tau$ -Lindelöfication of a discrete set of cardinality greater than  $\tau$ , then the free Abelian group  $G = A(X)$  satisfies  $l(G) \leq \tau$  and  $c(G) > \tau$  (see [11] for details).

**2.13 Problem.** Does Theorem 2.10 remain valid for a weakly  $\tau$ -Lindelöf space  $\Pi$  (i.e. for a space each open cover of which contains a subfamily of cardinality  $\leq \tau$  with a dense union)?

The following problem is closely connected with the previous one.

**2.14 Problem.** Let  $S$  be a weakly Lindelöf subspace of a product  $\prod_{i \in J} P_i$  and suppose that  $f : S \rightarrow X$  is an  $M$ -mapping to a second-countable space  $X$ . Does  $f$  depend on countably many coordinates?

The answer to Problem 2.14 is “yes” if  $S$  is a *subgroup* of a product of *topological groups*  $P_i$ , or even if  $S$  is an  $M$ -*subspace* of a product of  $M$ -*spaces*  $P_i$  [4, Theorem 1.1].

Combining the argument exposed in the proof of Theorems 2.2 and 2.10, we can prove a result generalizing Proposition 6 of [18]. This requires the following notation. Suppose  $f : \Pi \rightarrow Y$  and  $g : \Pi \rightarrow Z$  are continuous mappings and  $f$  is onto. We write  $f \prec g$  if there exists a continuous mapping  $h : Y \rightarrow Z$  such that  $g = h \circ f$ .

**2.15 Theorem.** *An image of a  $\Sigma(\aleph_0)$ -space under an  $M$ -mapping is  $\aleph_0$ -cellular.*

PROOF: Let  $f : \Pi \rightarrow X$  be an  $M$ -mapping of a  $\Sigma(\aleph_0)$ -space  $\Pi$  onto  $X$  and suppose that  $F : \Pi^3 \rightarrow X$  witnesses that. If  $X$  is not  $\aleph_0$ -cellular, there exists a sequence  $\{(K_\alpha, O_\alpha) : \alpha < \omega_1\}$  such that  $K_\alpha \subseteq O_\alpha \subseteq X$ ,  $K_\alpha$  is a non-empty  $G_\delta$ -set in  $X$ ,  $O_\alpha$  is open and  $K_\beta \cap O_\alpha = \emptyset$  whenever  $\beta < \alpha < \omega_1$ . For each  $\alpha < \omega_1$  pick a point  $x_\alpha \in \Pi$  with  $f(x_\alpha) \in K_\alpha$ . We can also assume that for each  $\alpha < \omega_1$  there exists a continuous function  $\varphi_\alpha : X \rightarrow [0, 1]$  such that  $K_\alpha = \varphi_\alpha^{-1}(1)$  and  $X \setminus O_\alpha = \varphi_\alpha^{-1}(0)$ . Define a continuous mapping  $\psi_{\alpha,\beta} : \Pi \rightarrow [0, 1]$  by  $\psi_{\alpha,\beta}(x) = \varphi_\alpha F(x_\alpha, x_\beta, x)$  for all  $\alpha, \beta < \omega_1$  and  $x \in \Pi$ . Let two covers  $\mathcal{K}$  and  $\mathcal{C}$  of  $\Pi$  witness that  $\Pi$  is a  $\Sigma(\aleph_0)$ -space,  $|\mathcal{K}| \leq \aleph_0$ . The family  $\mathcal{K}$  can be chosen closed under finite intersections.

Let  $\alpha_0$  be a countable ordinal. Put  $g_0 = \Delta\{\psi_{\beta,\gamma} : \beta, \gamma \leq \alpha_0\}$ . Then the space  $Y_0 = g_0(\Pi)$  is second-countable. Suppose that for some integer  $n$  we have defined an ordinal  $\alpha_n < \omega_1$  and a continuous mapping  $g_n : \Pi \rightarrow Y_n$  onto a space  $Y_n$  with  $w(Y_n) \leq \aleph_0$ . Since  $Y_n$  is hereditarily separable, for each  $K \in \mathcal{K}$  there exists an ordinal  $\delta = \delta_n(K) < \omega_1$  such that  $g_n(K \cap T_\delta)$  is dense in  $g_n(K \cap T)$ , where  $T = \{x_\alpha : \alpha < \omega_1\}$  and  $T_\delta = \{x_\nu : \nu < \delta\}$ . Put  $\alpha_{n+1} = \max\{\alpha_n, \sup\{\delta_n(K) : K \in \mathcal{K}\}\}$  and  $g_{n+1} = g_n \Delta(\Delta\{\psi_{\beta,\gamma} : \beta, \gamma \leq \alpha_{n+1}\})$ ; the symbol  $\Delta$  stands for the diagonal product of mappings. Thus we have defined an increasing sequence  $\{\alpha_n : n \in \omega\}$  of countable ordinals and can now put  $\alpha = \sup_{n \in \omega} \alpha_n$  and  $g = \Delta\{\psi_{\beta,\gamma} : \beta, \gamma < \alpha\}$ . Then  $\alpha < \omega_1$  and the space  $Y = g(\Pi)$  is second-countable.

From the construction it follows that the following conditions are fulfilled:

- (1)  $g(K \cap T_\alpha)$  is dense in  $g(K \cap T)$  for each  $K \in \mathcal{K}$ ;
- (2)  $g \prec \psi_{\beta,\gamma}$  for all  $\beta, \gamma < \alpha$ .

We claim that  $g(x_\alpha) \in g(cl_\Pi T_\alpha)$ . Indeed, choose  $C^* \in \mathcal{C}$  with  $x_\alpha \in C^*$ . Denote by  $\mathcal{B}$  a countable base of the space  $Y$  at  $g(x_\alpha)$ . By (1), the family  $\mathcal{F} = \{g^{-1}(cl_Y U) \cap K \cap cl_\Pi T_\alpha : U \in \mathcal{B}, C^* \subseteq K \in \mathcal{K}\}$  consists of non-empty closed subsets of  $\Pi$ . The set  $g^{-1}(cl_Y U) \cap cl_\Pi T_\alpha$  meets  $C^*$  for each  $U \in \mathcal{B}$ , otherwise by the choice of  $\mathcal{K}$  there exists  $K^* \in \mathcal{K}$  such that  $C^* \subseteq K^* \subseteq \Pi \setminus (g^{-1}(cl_Y U) \cap cl_\Pi T_\alpha)$ , which contradicts the fact that all elements of  $\mathcal{F}$  are non-empty. Thus, the countable family  $\mathcal{F}^* = \{g^{-1}(cl_Y U) \cap C^* \cap cl_\Pi T_\alpha : U \in \mathcal{B}\}$  consists of non-empty closed subsets of  $C^*$  and by the choice of  $\mathcal{B}$  has the finite intersection property. Since  $C^*$  is countably compact, we have  $\emptyset \neq \bigcap \mathcal{F}^* = g^{-1}g(x_\alpha) \cap C^* \cap cl_\Pi T_\alpha$ . Consequently,  $g(x_\alpha) \in g(cl_\Pi T_\alpha)$ .

Pick a point  $z \in cl_\Pi T_\alpha$  with  $g(z) = g(x_\alpha)$ . Since  $F(z, z, x_\alpha) = f(x_\alpha) \in K_\alpha \subseteq O_\alpha$ , there exists a neighbourhood  $V$  of  $z$  such that  $F(V \times \{z\} \times \{x_\alpha\}) \subseteq O_\alpha$ . From  $z \in cl_\Pi T_\alpha$  it follows that  $x_\beta \in V$  for some  $\beta < \alpha$ , and we have  $F(x_\beta, z, x_\alpha) \in O_\alpha$ . Since  $g(x_\alpha) = g(z)$ , (2) implies that  $\psi_{\beta,\gamma}(x_\alpha) = \psi_{\beta,\gamma}(z)$ , i.e.  $\varphi_\beta F(x_\beta, x_\gamma, x_\alpha) = \varphi_\beta F(x_\beta, x_\gamma, z)$  for all  $\gamma < \alpha$ . Taking into account the continuity of  $F$  in the first argument and the fact that  $z \in cl_\Pi T_\alpha$ , we get the equalities

$$\varphi_\beta F(x_\beta, z, x_\alpha) = \varphi_\beta F(x_\beta, z, z) = \varphi_\beta f(x_\beta) = 1.$$

Since  $K_\beta = \varphi_\beta^{-1}(1)$ , the latter means that  $y = F(x_\beta, z, x_\alpha) \in K_\beta$ . Thus  $y \in K_\beta \cap O_\alpha \neq \emptyset$  and  $\beta < \alpha$ , which contradicts the choice of the sequence  $\{(K_\nu, O_\nu) : \nu < \omega_1\}$ . □

Note again that the separate continuity of the mapping  $F : \Pi^3 \rightarrow X$  was only used in the proof of the above theorem. Thus we have the following corollary.

**2.16 Corollary.** *If a quasitopological group  $G$  is a continuous image of some  $\Sigma(\aleph_0)$ -space, then  $G$  is  $\aleph_0$ -cellular.*

**2.17 Corollary.** *Suppose a space  $\Pi$  admits a quasiperfect (i.e. continuous closed with countably compact fibers) mapping onto a space with a countable network. Then every image of  $\Pi$  under an  $M$ -mapping is  $\aleph_0$ -cellular.*

PROOF: Let  $h : \Pi \rightarrow Z$  be a quasiperfect mapping of  $\Pi$  onto a space  $Z$  with a countable network  $\mathcal{N}$ . Since  $Z$  is regular, we can assume that  $\mathcal{N}$  consists of closed sets. Put  $\mathcal{C} = \{h^{-1}(z) : z \in Z\}$  and  $\mathcal{K} = \{h^{-1}(N) : N \in \mathcal{N}\}$ . Then the covers  $\mathcal{C}$  and  $\mathcal{K}$  of  $\Pi$  witness that  $\Pi$  is a  $\Sigma(\aleph_0)$ -space. It remains to apply Theorem 2.15. □

**2.18 Problem.** *Suppose  $f : \Pi \rightarrow X$  is an  $M$ -mapping of a countably compact space  $\Pi$  onto  $X$ . Is then every regular uncountable cardinal a caliber for  $X$ ?*

**2.19 Problem** (see also [13, Problem 2.4]). Does an image of a pseudocompact space under an  $M$ -mapping have the Souslin property?

It is still unknown whether every pseudocompact  $M$ -space has the Souslin property; see [18] for details.

We conclude with a little bit alien problem to the area.

**2.20 Problem** (see [13, Problem 2.5]). If  $X$  is an image of a compact space under an  $M$ -mapping, must  $X$  be dyadic?

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