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On the Jacobson radical of strongly group graded rings

A.V. Kelarev

Abstract. For any non-torsion group $G$ with identity $e$, we construct a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R_e)$ is locally nilpotent, but $J(R)$ is not locally nilpotent. This answers a question posed by Puczyłowski.

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Several interesting results of ring theory establish the local nilpotency of the Jacobson radical of some ring constructions (cf. [9]). In this paper we consider an analogous question for strongly group graded rings. Let $G$ be a group. An associative ring $R = \bigoplus_{g \in G} R_g$ is said to be strongly $G$-graded if $R_gR_h = R_{gh}$ for all $g, h \in G$. Strongly group graded rings have been intensively investigated for several years (cf., for example, [12],[15],[20]). In [18] the following question was posed: is it true that for every free group $G$ of rank $\geq 2$ the Jacobson radical of each strongly $G$-graded ring is locally nilpotent? (As it is noted in [18], the question is also connected with [14], Problem 24, and with a problem on the local nilpotency of the Jacobson radical of a skew polynomial ring, cf. [19].) It follows from the results of [6] that the answer is positive in the case when $R_e$ satisfies the ascending chain condition for left annihilators, where $e$ is the identity of $G$. It is also known that the answer is positive for group rings of free groups of rank $\geq 2$ (cf. [18]). The answer to the analogous question for the rings of polynomials in at least two non-commuting variables is also positive (cf. [18]).

We shall show that in general the answer is negative. Namely, for an arbitrary group $G$, we construct a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R)$ is not nil. On the other hand, we shall prove that, for the positive answer to the question above, it suffices to assume that $J(R_e)$ is left $T$-nilpotent. It will also be shown that the weaker condition that $J(R_e)$ is equal to the Baer radical $B(R_e)$, is not sufficient for the local nilpotency of $J(R)$.

Our proofs are based on the previous results of [7], [10] and [21].

Theorem 1. For each group $G$, there exists a strongly $G$-graded ring $R$ such that the Jacobson radical $J(R)$ is not nil.
Lemma 1. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring, and let $h \in G$. Then there exists a $G$-graded ring $Q = \bigoplus_{g \in G} Q_g$ such that $Q \supseteq R$, $J(Q) \supseteq J(R)$, $Q_g \supseteq R_g$ and $Q_gQ_h \supseteq R_{gh}$ for each $g \in G$.

Proof: Let $R$ be the ring of integers, $R^1$ the ring $R$ with identity 1 adjoined, $\mathbb{Z}[x, y]$ the ring of polynomials with non-commuting variables $x, y$. Denote by $W$ the free product of $R$ and $\mathbb{Z}[x, y]$. For $w \in W$, let $\langle w \rangle$ be the subring generated in $W$ by $w$. Put $M = R + Ry + xR + xRy$, $S = M + \langle xy \rangle + \langle x \rangle x + y \langle xy \rangle + \langle y \rangle x$. To simplify the notation, we shall denote by the same letters elements and their images in the quotient rings which will be introduced. If we factor out the ideal generated in $W$ by $x^2, y^2, yR, Rx$ and all $r - xy, r - ry, x, y$ runs over $R$, then the resulting quotient ring $Q$ is equal to $Z + \langle x \rangle + \langle y \rangle + S$. Clearly, $S$ and $M$ are ideals of $Q$. It is routine to verify that

$$M = \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix}$$

and

$$S/M = \begin{bmatrix} \langle xy \rangle & \langle xy \rangle x \\ y\langle xy \rangle & \langle y \rangle x \end{bmatrix}$$

are Morita contexts (cf. [1]). Further, $R, xRy \cong R$ and $Ry, xR \cong R^0$, where $R^0$ stands for the ring with zero multiplication defined on the additive group of $R$. Since $\langle xy \rangle$ and $\langle yx \rangle$ are semiprime rings and $S/M$ satisfies the left annihilator condition in the sense of [21], then [21], Lemma 2.6, implies that $S/M$ is semiprime. Therefore $J(S) = J(M)$.

Take any $q \in J(Q)$, say $q = a + bx + cy + s$, where $a, b, c \in Z$, $s \in S$. If $a \neq 0$, then $qxy \notin M$ and so $0 \neq qxy \in J(S/M)$, a contradiction. If $a = 0, b \neq 0$, then $0 \neq qy \in J(S/M)$ gives a contradiction. Finally, if $a = b = 0, c \neq 0$, then $0 \neq qx \in J(S/M)$, a contradiction again. Therefore $a = b = c = 0$, that is $q \in J(M)$. Thus $J(Q) = J(M)$.

Denote by $I$ the ideal generated in $Q$ by $J(R)$. Then

$$I = \begin{bmatrix} J(R) & J(R)y \\ xJ(R) & xJ(R)y \end{bmatrix}.$$  

Clearly, $I$ is the largest ideal of $M$ satisfying the property that $I \cap R \subseteq J(R)$ and $I \cap xRy \subseteq J(xRy) = xJ(R)y$. In view of [10], Corollary 1, and [11], Corollary 6, we conclude $I = J(M)$. Hence $I = J(Q)$. In particular, $J(Q) \supseteq J(R)$.

To make $Q$ a $G$-graded ring, we put $x \in Q_h$, $y \in Q_{h^{-1}}$, $Z \subseteq Q_e$, and then the grading naturally comes from $R$. For example, $xR_gy \subseteq Q_{gh^{-1}} \subseteq Q_{gh}$. Since $R_{gh}y \subseteq Q_g$ and $x \in Q_h$, we get $Q_gQ_h \supseteq (R_{gh}y)x = R_{gh}$, as required.  \qed
Lemma 2. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. Then there exists a $G$-graded ring $Q = \bigoplus_{g \in G} Q_g$ such that $Q \supseteq R$, $J(Q) \supseteq J(R)$, $Q_g \supseteq R_g$ and $Q_gQ_h \supseteq R_{gh}$ for all $g, h \in G$.

Proof: Denote by $R^{(h)}$ the ring constructed by $R$ and $h$ in Lemma 1. We may order the set $G$, identify the elements of $G$ with ordinal numbers and define an ascending chain of $G$-graded rings $T_\alpha$ by putting $T_1 = R^{(1)}$, $T_\alpha = (\bigcup_{\beta < \alpha} T_\beta)^{(\alpha)}$.

The transfinite induction shows that $J(T_\alpha) \supseteq J(R)$ in view of Lemma 1. However, $G = \{\alpha| \alpha \leq \tau\}$ for some $\tau$. Hence a straightforward verification shows that $Q = \bigcup_{\alpha \leq \tau} T_\alpha$ is the desired ring.

Lemma 3. Let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded ring. Then there exists a strongly $G$-graded ring $Q = \bigoplus_{g \in G} Q_g$ such that $Q \supseteq R$, $J(Q) \supseteq J(R)$, and $Q_g \supseteq R_g$ for all $g \in G$.

Proof: Denote by $R'$ the ring constructed in Lemma 2, and put $R^{[1]} = R'$, $R^{[n+1]} = (R^{[n]})'$. Then it is routine to verify that $Q = \bigcup_{n=1}^{\infty} R^{[n]}$ is the required example.

Proof of Theorem 1 easily follows from Lemma 3 if we take any quasi-regular but not nil ring $R$ and make it $G$-graded with $R_e = R$.

Now we shall give a new condition sufficient for the Jacobson radical of a ring strongly graded by a free group to be locally nilpotent. In fact, our condition is applicable not only to free groups, but also to all u.p.-groups. A group $G$ is called a unique product (u.p.)-group if, for any non-empty subsets $X, Y$ of $S$, there exists at least one element uniquely presented in the form $xy$, where $x \in X$, $y \in Y$ (cf. [16], §13.1). The radicals of rings graded by u.p.-groups were considered, in particular, in [6] and [7]. A ring $R$ is said to be left $T$-nilpotent if, for every sequence $x_1, x_2, \ldots \in R$, there exists $n$ such that $x_1 \ldots x_n = 0$. The class of left $T$-nilpotent rings lies strictly between the class of nilpotent rings and the Baer radical class (cf. [5]).

Theorem 2. Let $G$ be a u.p.-group, $R = \bigoplus_{g \in G} R_g$ a strongly $G$-graded ring. If $J(R_{e})$ is left $T$-nilpotent, then $J(R)$ is locally nilpotent.

Proof: Given that $G$ is a u.p.-group, it follows from [7], Theorem 2.2, that the Levitzky radical $L(R)$ is homogeneous, i.e. $L(R) = \bigoplus_{g \in G} L(R) \cap R_{g}$. Since $R/L(R)$
is strongly $G$-graded, we may assume that from the very beginning $L(R) = 0$.

Suppose to the contrary that $J(R) \neq 0$. For $r \in R$, $g \in G$, denote by $r_g$ the projection of $r$ on $R_g$, and put supp $(r) = \{g \in G | r_g \neq 0\}$. Let $l(r) = |\text{supp} (r)|$. Choose a non-zero element $s$ in $J(R)$ with minimal length $l(s)$, and take any $h \in \text{supp} (s)$. If $s_h R_{h^{-1}} = 0$, then $s_h R_{h^{-1}} R = 0$, and so $s_h \in A = \{r \in R | r R = 0\}$. However, $A \subseteq L(R) = 0$, because $A^2 = 0$. Thus $s_h R_{h^{-1}} \neq 0$. Therefore the set $P = \{r_e | r \in J(R), \ l(r) = l(s)\}$ is non-zero. Given that $G$ is a u.p.-group, Theorem 3.2 of [7] tells us that $P \subseteq J(R_e)$. Denote by $I$ the ideal generated in $R$ by $P$. We claim that $I$ is left $T$-nilpotent.

Suppose that there exists a sequence of elements $x_1, x_2, \ldots$ of $I$ such that $x_1 \ldots x_n \neq 0$ for all $n$. Each $x_i$ is a finite sum of elements of the form $ar_e b$, where $r \in J(R)$, $l(r) = l(s)$, $a$ and $b$ are homogeneous elements of $R^1$. We may assume that all $b$ belong to $R$. (Indeed, if $b \in Z$, then we can replace $x_i$ by $x_i x_{i+1}$, and consider the sequence $x_1, \ldots, x_i x_{i+1}, x_{i+2}, \ldots$.) Denote by $S(x_i)$ the set of such summands of $x_i$. For arbitrarily large $n$ we can pick $y_1 \in S(x_1), \ldots, y_n \in S(x_n)$ such that $y_1 \ldots y_n \neq 0$. Since all the $S(x_i)$ are finite, a standard argument shows that there exists an infinite sequence $y_1, y_2, \ldots$ where $y_i \in S(x_i)$ and $y_1 \ldots y_n \neq 0$ for all $n$. Then $y_i = a^{(i)} r_e^{(i)} b^{(i)}$ where $r^{(i)} \in J(R)$, $l(r^{(i)}) = l(s)$, $a^{(i)}$ and $b^{(i)}$ are homogeneous elements of $R^1$, and $b^{(i)} \in R$. Given that $R$ is strongly graded, $b^{(2)} \in R$, and $G$ is a group, it follows that $b^{(2)} = c^{(2)} d^{(2)}$ for some homogeneous elements $c^{(2)}, d^{(2)}$ such that $b^{(1)} a^{(2)} r_e^{(2)} c^{(2)} \in R_e$. Similarly, for any $i \geq 3$, there exist homogeneous elements $c^{(i)}, d^{(i)}$ such that $b^{(i)} = c^{(i)} d^{(i)}$ and $d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)} \in R_e$. Let $z_1 = r_e^{(1)}, z_2 = b^{(1)} a^{(2)} r_e^{(2)} c^{(2)}$, and $z_i = d^{(i-1)} a^{(i)} r_e^{(i)} c^{(i)}$ for $i \geq 3$. Then, $z_1, z_2, z_3, \ldots \in P$. Since $J(R_e)$ is left $T$-nilpotent and contains $P$, we get $z_1 \ldots z_n = 0$ for some $n > 1$. Hence $y_1 \ldots y_n = a^{(1)} z_1 \ldots z_n d^{(n)} = 0$.

This contradiction shows that $I$ is left $T$-nilpotent, and so $I \subseteq L(R) = 0$. Therefore $J(R) = 0$, which completes the proof.

Let $P$ denote the set of positive integers. The well-known Golod’s example of a nil but not locally nilpotent ring $R$ is $\mathbb{P}$-graded (cf. [16]). Therefore one cannot replace strongly graded rings by arbitrary graded rings in Theorem 2. Now we shall show that the left $T$-nilpotence cannot be weakened to Baer radicalness, either.

**Theorem 3.** Let $G$ be a non-periodic group with identity $e$. Then there exists a strongly $G$-graded ring $Q$ such that $J(Q_e) = B(Q_e)$ but $J(Q) \neq L(Q)$.

**Proof:** Let $R$ be the Golod ring. Since $R$ is $\mathbb{P}$-graded, $R$ can easily be made $G$-graded with $R_e = 0$. Take any $h \in G$ and denote by $Q, S, M$ the rings constructed by $R$ as in the proof of Lemma 1. It has been proved that $J(Q) = J(M)$. The same reasoning shows that $B(Q) = B(M)$. Further,

$$M \cong \begin{bmatrix} R & Ry \\ xR & xRy \end{bmatrix} ,$$
whence
\[ M_e = \begin{bmatrix} R_e & R_h y \\ x R_h^{-1} & x R_y \end{bmatrix}. \]

Evidently \( R_e \) and \( x R_e y \) are isomorphic to \( R_e \) which satisfies \( J(R_e) = B(R_e) \), because it is zero. It follows from [10], Corollary 1, and [11], Corollary 6, that \( J(M_e) \) is equal to the largest ideal \( I \) of \( M_e \) with the property that \( I \cap R_e \subseteq J(R_e) \) and \( I \cap x R_e y \subseteq J(x R_e y) \). Besides, [10], Corollary 3, and [11], Corollary 6, imply that \( B(M_e) \) is the largest ideal of \( M_e \) with the property that \( I \cap R_e \subseteq B(R_e) \) and \( I \cap x R_e y \subseteq B(x R_e y) \). Therefore \( J(M_e) = B(M_e) \). Further, \( Q_e/S_e \cong Z \) and \( S_e/M_e = \langle xy \rangle + \langle yx \rangle \) imply \( J(Q_e) = B(Q_e) \). Thus \( J(Q_e) = B(Q_e) \). It follows from [21], Lemma 2.3, that \( J(Q_e) \supseteq J(R_e) \). This and transfinite induction show that all rings \( Q \) obtained from \( R \) in Lemmas 2 and 3 satisfy \( J(Q_e) = B(Q_e) \). However \( J(Q) \) is not locally nilpotent, because \( J(Q) \supseteq J(R) \). Thus \( Q \) is the required example. \( \square \)

Note that in the opposite case, where \( G \) is locally finite, it follows from the results of [2] and [3] that \( J(R_e) = L(R_e) \) implies \( J(R) = L(R) \) (cf. [13], Lemma 1.1). Analogous results were obtained in [2] for the more general case of rings graded by locally finite semigroups. A sufficient condition for the Jacobson radical of an algebra graded by a finite group to be nilpotent follows from the main theorem of [17].

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