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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 35 (1994), No. 4, 645--652

Persistent URL: http://dml.cz/dmlcz/118706

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Analytic functions are $I$-density continuous

Krzysztof Ciesielski, Lee Larson

Abstract. A real function is $I$-density continuous if it is continuous with the $I$-density topology on both the domain and the range. If $f$ is analytic, then $f$ is $I$-density continuous. There exists a function which is both $C^\infty$ and convex which is not $I$-density continuous.

Keywords: analytic function, $I$-density continuous, $I$-density topology
Classification: 26A21

Let $T_N$ stand for the density topology on the real line, $\mathbb{R}$. A function $f: \mathbb{R} \to \mathbb{R}$ is density continuous at the point $x$ if it is continuous at $x$ when $T_N$ is used on both the domain and the range. The class of all everywhere density continuous functions is written as $C_{NN}$. It is known that all locally convex functions are density continuous, and it follows quite easily from this that all analytic functions are in $C_{NN}$. But, there are $C^\infty$ functions which are not in $C_{NN}$ [2].

W. Wilczyński [4] introduced the $I$-density topology on $\mathbb{R}$, which has many properties in common with the density topology, except that it is based upon category instead of measure. (For its definition see [4] or [3].) The $I$-density topology is denoted here by $T_I$. The $I$-density continuous functions, $C_{II}$, are those functions $f: \mathbb{R} \to \mathbb{R}$ which are continuous when the domain and range are both given the topology $T_I$.

It is natural to ask if the known properties of the density continuous functions can be proved in the case of the $I$-density continuous functions. It turns out that some properties can and some cannot be proved. Theorem 7, given below, establishes that analytic functions are $I$-density continuous, but the proof is necessarily different from the case of the density continuous functions because we also exhibit in Example 10, a convex and $C^\infty$ function which is not $I$-density continuous.

The notation used here is fairly standard. The set of subsets of $\mathbb{R}$ with the Baire property is written as $B$. $I$ stands for the ideal of first category subsets of $\mathbb{R}$. $C^\infty$ is the set of all functions $f: \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable at every point and $A$ stands for the collection of all real analytic functions. A set $E$ is a right interval set at a point $a \in \mathbb{R}$, if $E = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ or $E = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ where $a_n \to a$ and $a_n > b_{n+1} > a_{n+1}$ for all $n \in \mathbb{N}$. The definition of a left interval set at $a$ is similar. The set $E$ is an interval set at $a$, if it is the union of a right and left interval set at $a$. Any interval set at 0 is just called an interval set.
An open set $S$ is said to be regular, if $S = \text{int} (\text{cl}(S))$. In particular, it can be shown that for any $B \in \mathcal{B}$, there is a unique regular open set, $\tilde{B}$ such that $B \triangle \tilde{B} \in \mathcal{I}$. This observation is important below because it often enables us to replace an arbitrary $B \in \mathcal{T}_\mathcal{I}$ by $\tilde{B}$ without losing any generality in a proof.

We begin by stating several known results which are needed below. The first is essentially the same as [5, Theorem 2].

**Lemma 1.** Let $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero and, for each $n \in \mathbb{N}$, let $(a_n, b_n)$ be an open interval centered at $c_n$. If

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n - a_n}{c_n} = 0,$$

then 0 is an $\mathcal{I}$-dispersion point of $\bigcup_{n \in \mathbb{N}} [a_n, b_n]$.

**Theorem 2.** Let $B$ be a regular open set. The following statements are equivalent:

(i) 0 is an $\mathcal{I}$-dispersion point of $B$.

(ii) For every increasing sequence $\{t_k\}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{k_i}\}$ such that

$$\limsup_{i \to \infty} t_{k_i} B \cap (-1, 1) \in \mathcal{I}.$$

(iii) For every increasing sequence $\{t_k\}$ of positive numbers diverging to infinity and every nonempty interval $(a, b) \subset (-1, 1)$ there exists a nonempty subinterval $(c, d) \subset (a, b)$ and a subsequence $\{t_{k_i}\}$ such that for every $i \in \mathbb{N}$

$$(c, d) \cap t_{k_i} B = \emptyset.$$

**Proof:** The fact that (i) and (ii) are equivalent is known [3, Theorem 1].

Assume that (ii) is true, but that there exists an interval $(a, b) \subset (-1, 1)$ for which (iii) fails. Then every subinterval $(c, d) \subset (a, b)$ has the property that $\{k : (c, d) \cap t_k B = \emptyset\}$ is finite. From this it is apparent that $\limsup_i t_{k_i} B$ is a dense $\mathbf{G}_\delta$ subset of $(a, b) \subset (-1, 1)$ for every subsequence $\{t_{k_i}\}$ of $\{t_k\}$. This contradicts (1), so (iii) must be true.

Finally, suppose that (iii) is true. Let $d_n$ be a countable dense subset of $(-1, 1)$ and suppose $I_n$ is a sequential representation of the set $\{(d_n, d_m) : n, m \in \mathbb{N}, d_n < d_m\}$. Applying (iii), there must exist an interval $J_1 \subset I_1$ and a subsequence $\{t_{k_1}^{(1)}\}$ of $\{t_k\}$ so that $t_{k_1}^{(1)} B \cap J_1 = \emptyset$ for all $m$. Proceeding inductively, for each $i \in \mathbb{N}$ there must exist an interval $J_{i+1} \subset I_{i+1}$ and a subsequence $t_{k_i}^{(i+1)}$ of $t_{k_i}^{(i)}$ such that $t_{k_i}^{(i+1)} B \cap J_{i+1} = \emptyset$ for each $m$. Since $\{d_n : n \in \mathbb{N}\}$ is dense in $(-1, 1)$ it is clear that $\limsup_i t_{k_i}^{(i)} B \cap (-1, 1) \in \mathcal{I}$, and (ii) follows.

The following theorem is a consequence of [1, Corollary 1].
Theorem 3. If $f: \mathbb{R} \to \mathbb{R}$ is monotone and satisfies the Lipschitz condition

$$0 < \alpha |b - a| < |f(b) - f(a)| < \beta |b - a| < \infty$$

for all distinct $a$ and $b$ in some interval $I$, then $f$ is $I$-density continuous on $I$.

The first order of business is to prove that $A \subset C_{II}$. The following two technical lemmas are needed for the proof.

Lemma 4. Let $f, h: [0, +\infty) \to [0, +\infty)$ be homeomorphisms such that

$$\lim_{x \to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1.$$ 

Then for every $0 < c < c' < d' < d$ there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$,

$$f((\varepsilon c', \varepsilon d')) \subset h((\varepsilon c, \varepsilon d)).$$

Proof: Since $c/c' < 1$ and $d/d' > 1$ we can find $\delta_0 > 0$ such that for every $x \in (0, \delta_0)$

$$\frac{c}{c'} < \frac{h^{-1}(x)}{f^{-1}(x)} < \frac{d}{d'}.$$ \hspace{1cm} (2)

Using the continuity of $f^{-1}$ at 0 we can find $\varepsilon_0 > 0$ such that $f((0, \varepsilon_0 d)) \subset (0, \delta_0)$.

Now let $\varepsilon \in (0, \varepsilon_0)$ and $x \in f((\varepsilon c, \varepsilon d')) \subset f((0, \varepsilon_0 d)) \subset (0, \delta_0)$. So, (2) holds and $f^{-1}(x) \in (\varepsilon c', \varepsilon d')$; i.e.,

$$\varepsilon c' < f^{-1}(x) < \varepsilon d'.$$

Multiplying the above inequality by (2), we obtain

$$\varepsilon c < h^{-1}(x) < \varepsilon d,$$

which implies $x \in h((\varepsilon c, \varepsilon d)).$

Lemma 5. If $f, h: [0, \infty) \to [0, \infty)$ are homeomorphisms satisfying

$$\lim_{x \to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1,$$ \hspace{1cm} (3)

then $h$ is right $I$-density continuous at 0 iff $f$ is right $I$-density continuous at 0.

Proof: Without loss of generality we may assume that both functions are increasing, as the decreasing case is essentially the same.
So assume that $h$ is right $\mathcal{I}$-density continuous at 0. It will be shown that $f$ is right $\mathcal{I}$-density continuous at 0. This will finish the proof, as the converse implication follows by exchanging $f$ with $h$.

Let us choose $B \in \mathcal{B}$, $0 \notin B$, which has 0 as an $\mathcal{I}$-dispersion point. We will use Theorem 2 to show that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$.

First, notice that since $f$ and $h$ are both homeomorphisms, we may assume that $B$ is a regular open set. Choose a divergent increasing sequence of positive real numbers $\{t_k\}_{k \in \mathbb{N}}$ and a nonempty interval $(a, b) \subset (0, 1)$. Since 0 is a right $\mathcal{I}$-dispersion point of $h^{-1}(B)$, there exists a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{t_{k_p}\}_{p \in \mathbb{N}}$ of $\{t_k\}_{k \in \mathbb{N}}$ such that for every $p \in \mathbb{N}$

$$(c, d) \cap t_{k_p} h^{-1}(B) = \emptyset.$$  

But this last condition is equivalent to

$$h \left( \left( \frac{1}{t_{k_p}} c, \frac{1}{t_{k_p}} d \right) \right) \cap B = \emptyset.$$  

Now let $0 < c < c' < d' < d$. Then, by Lemma 4,

$$f \left( \frac{1}{t_{k_p}} c', \frac{1}{t_{k_p}} d' \right) \subset h \left( \frac{1}{t_{k_p}} c, \frac{1}{t_{k_p}} d \right)$$

for almost all $p \in \mathbb{N}$. This implies that for almost all $p \in \mathbb{N}$

$$f \left( \left( \frac{1}{t_{k_p}} c', \frac{1}{t_{k_p}} d' \right) \right) \cap B = \emptyset,$$

or

$$(c', d') \cap t_{k_p} f^{-1}(B) = \emptyset.$$  

This finishes the proof of Lemma 5. \[\square\]

The following theorem, which is interesting in its own right, is also needed in what follows. Its analogue for ordinary density continuity is also known to be true [2].

**Theorem 6.** For any $\alpha \in \mathbb{R}$, the function $f(x) = x^\alpha$ is $\mathcal{I}$-density continuous on its domain.

**Proof:** If $x \neq 0$ and $f(x)$ exists, then it is clear that on a neighborhood of $x$, $f$ satisfies the conditions of Theorem 3, so $f$ is $\mathcal{I}$-density continuous at $x$.

Suppose $x = 0$ and $\alpha > 0$. It suffices to show $f$ is right $\mathcal{I}$-density continuous at 0. Let $B \in \mathcal{B}$ such that 0 is an $\mathcal{I}$-dispersion point of $B$. It must be shown that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$. 


To do this, first note that $f$ is a homeomorphism on $(0,\infty)$, so $f^{-1}(S) \in \mathcal{I}$ whenever $S \in \mathcal{I}$ and there is no generality lost with the assumption that $B$ is a regular open set. Choose any nonempty interval $(a, b) \subset (0, 1)$ and an increasing sequence $\{s_k\}_{k \in \mathbb{N}}$ of positive numbers diverging to infinity. Let $(a', b') = f((a, b))$ and define the increasing sequence

$$t_k = \frac{1}{f(1/s_k)} \to \infty.$$ 

Using Theorem 2, there exists an interval $(c', d') \subset (a', b')$ and a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that

$$(c', d') \cap t_{k_i}B = \emptyset \quad \text{for all} \quad i \in \mathbb{N}.$$ 

Suppose that $(c, d) = f^{-1}((c', d'))$. Then a straightforward calculation shows

$$\emptyset = f^{-1}((c', d') \cap t_{k_i}B) = (c, d) \cap f^{-1}\left(\frac{1}{f(1/s_{k_i})}B\right) = (c, d) \cap \left(s_{k_i}^{-\alpha}B\right)^{-1/\alpha} = (c, d) \cap s_{k_i}B^{-1/\alpha} = (c, d) \cap s_{k_i}f^{-1}(B).$$

From Theorem 2, we see that 0 is a right $\mathcal{I}$-dispersion point of $f^{-1}(B)$, and the theorem follows. \hfill \Box

**Theorem 7.** $A \subset \mathcal{C}_{\mathcal{II}}$.

**Proof:** Let $h \in A$. It is enough to prove that $h$ is $\mathcal{I}$-density continuous at 0. We prove that $h$ is right $\mathcal{I}$-density continuous at 0. The left-hand argument is similar.

Let $h(x) = \sum_{n=0}^{\infty} a_n x^n$. We can assume that $a_0 = 0$. Since the $\mathcal{I}$-density topology is closed under homothetic transformations of its open sets, we can also assume that for $i = \min\{n: a_n \neq 0\}$ we have $a_i = 1$. Now let $f(x) = x^i$. Because $h$ is analytic, $h^{-1}$ exists on some right neighborhood of 0. Let us assume that $h^{-1}$ is positive on this neighborhood, the other case being similar. Then

$$1 = \lim_{x \to 0^+} \frac{h(x)}{x^i} = \lim_{x \to 0^+} \frac{h(h^{-1}(x))}{(h^{-1}(x))^i} = \lim_{x \to 0^+} \left(\frac{x^i}{h^{-1}(x)}\right)^i = \left(\lim_{x \to 0^+} \frac{f^{-1}(x)}{h^{-1}(x)}\right)^i.$$ 

Hence,
\[ \lim_{x \to 0^+} \frac{h^{-1}(x)}{f^{-1}(x)} = 1 \]
and, by Lemma 5 and Theorem 6, \( h \) is \( \mathcal{I} \)-density continuous at 0. \( \square \)

After seeing that \( A \subseteq \mathcal{C}_T \), it is natural to ask whether the same can be claimed for \( C^\infty \). This turns out not to be true. The lemma and theorem given below are used to establish this fact.

**Lemma 8.** Let \( f \in C^\infty \) be such that for every \( n \geq 0 \)
\[ f^{(n)}(0) = 0 \quad \text{and} \quad f^{(n)}((0, \varepsilon_n)) \subset (0, \infty), \quad \text{for some } \varepsilon_n > 0. \]

Then
\[ \lim_{x \to 0^+} \frac{f(ax)}{f(x)} = 0, \]
for every \( a \in (0, 1) \).

**Proof:** Let \( a \in (0, 1) \) and \( n \in \mathbb{N} \). Moreover, let us choose \( \varepsilon > 0 \) such that \( 0 < \varepsilon < \varepsilon_k \) for every \( k \leq n + 1 \). In particular, \( f^{(n)} \) is increasing on \((0, \varepsilon)\), and so
\[ \left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < 1 \quad \text{for every } \xi \in (0, \varepsilon). \]

Now let \( x \in (0, \varepsilon) \) and let \( g(x) = f(ax) \). Using Cauchy’s Theorem \( n \)-times we can find \( \xi \in (0, x) \) such that
\[ \left| \frac{f(ax)}{f(x)} \right| = \left| \frac{g(x)}{f(x)} \right| = \left| \frac{g^{(n)}(\xi)}{f^{(n)}(\xi)} \right| = |a^n| \left| \frac{f^{(n)}(a\xi)}{f^{(n)}(\xi)} \right| < a^n. \]

Thus,
\[ \lim_{x \to 0^+} \frac{f(ax)}{f(x)} = 0. \]

**Theorem 9.** Let \( f \in C^\infty \) be such that for every \( n \geq 0 \)
\[ f^{(n)}(0) = 0 \quad \text{and} \quad f^{(n)}((0, \varepsilon_n)) \subset (0, \infty) \quad \text{for some } \varepsilon_n > 0. \]

Then \( f \) is not \( \mathcal{I} \)-density continuous.

**Proof:** We start with a proof that \( f \) is not right \( \mathcal{I} \)-density continuous at 0. Let \( D_n = \{ \frac{i}{2^n} : i = 1, 2, \ldots, 2^n \} \) for \( n \in \mathbb{N} \). First notice that if a sequence \( \{n_k\}_{k \in \mathbb{N}} \) is such that
\[ n_{k+1} > 2^k n_k \quad \text{for every } k \in \mathbb{N}, \]

\[ (4) \]
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then

$$\min \frac{1}{n_k} D_k = \frac{1}{n_k} \frac{1}{2^k} > \frac{1}{n_{k+1}} = \max \frac{1}{n_{k+1}} D_{k+1}.$$ 

This means that if $\{s_i\}_{i>1}$ is a decreasing ordering of $D = \bigcup_{k \in \mathbb{N}} \frac{1}{n_k} D_k$, then

$$\frac{1}{n_k} D_k = \{ s_i : 2^k \leq i < 2^{k+1} \}.$$ 

We also define a sequence $\{n_k\}_{k \in \mathbb{N}}$ by induction on $k$ such that it will satisfy condition (4) and for every $k > 0$

$$(5) \quad \frac{f(s_i)}{f(s_{i-1})} \leq \frac{1}{k} \quad \text{for} \quad 2^k \leq i < 2^{k+1}.$$ 

Put $n_1 = 1$ and assume that $n_{k-1}$ has already been chosen for some $k > 1$. Choose $n_k > 2^{k-1} n_{k-1}$ such that

$$\frac{f(2^{k-1} x)}{f(x)} < \frac{1}{k}, \quad \text{for all} \quad x \in (0, \frac{1}{n_k}).$$ 

Such a choice is possible by Lemma 8. Then, the above condition obviously implies condition (5) for $2^k < i < 2^{k+1}$. Increasing $n_k$, if necessary, we can also obtain condition (5) for $i = 2^k$. This finishes the construction of $D$.

Now let $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint intervals such that every interval $(a_n, b_n)$ is centered at $c_n = f(s_n)$ and that

$$\lim_{n \to \infty} \frac{b_n - a_n}{c_n} = 0.$$ 

By (5),

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 0$$

so, by Lemma 1, 0 is an $\mathcal{I}$-dispersion point of the interval set

$$E = \bigcup_{n \in \mathbb{N}} (a_n, b_n).$$ 

On the other hand, we notice that for every subsequence $\{n_{k_i}\}_{i \in \mathbb{N}}$ of $\{n_k\}_{k \in \mathbb{N}}$, the set

$$\bigcup_{i \in \mathbb{N}} n_{k_i} f^{-1}(E) \supset \bigcup_{i \in \mathbb{N}} D_{k_i}$$

is dense and open in $[0, 1]$. So, 0 is not a right $\mathcal{I}$-dispersion point of $f^{-1}(E)$ and $f$ is not $\mathcal{I}$-density continuous at 0. □
Example 10. There exists a convex $C^\infty$ function that is not $I$-density continuous.

**Proof:** Define $g: (-\infty, 0.5) \to \mathbb{R}$ by

$$g(x) = \begin{cases} e^{-x^2} & x \in (0, 1/2) \\ 0 & x \in (-\infty, 0] \end{cases}$$

Examining the second derivative of $g$ it is easy to see that $g$ is convex on $(-\infty, 1/2)$. It is well-known that $f \in C^\infty$ and that $f^{(n)}(0) = 0$ for all $n$. Repeated differentiation of $f$ makes it apparent that for each $n$ there is an $\varepsilon_n > 0$ such that $f^{(n)}(x) > 0$ whenever $0 < x < \varepsilon_n$. Now an application of Theorem 9 finishes the argument. \hfill \Box

It is also not difficult to see that the function described in Theorem 9 does not preserve $I$-density points. In particular, the function $g$ from Example 10 does not preserve $I$-density points.

**References**


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(Received July 8, 1991, revised April 15, 1993)