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A note on existence and uniqueness of solutions of neutral functional-differential equations with state-dependent delays

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Abstract. Existence and uniqueness theorem for state-dependent delay-differential equations of neutral type is given. This theorem generalizes previous results by Grimm and the author.

Keywords: functional-differential equation, existence and uniqueness of solutions

Consider the scalar initial-value problem for state-dependent delay-differential equations of neutral type

\[ \begin{align*}
    y'(t) &= f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))), \\
    y(t) &= g(t),
\end{align*} \]

\( t \in [\gamma, \alpha], \quad \gamma \leq a < b, \) where \( \gamma \leq \alpha(t, y) \leq t, \ \gamma \leq \beta(t, y) \leq t, \) and \( g \) is a given initial function. We assume the following:

(i) \( g \) and \( g' \) are Lipschitz-continuous with constants \( L_g \) and \( L_{g'} \) respectively;

(ii) \( f(a, g(a), g(\alpha(a, g(a))), g'(\beta(a, g(a)))) = g'(a), \) \( g'(a) \) denotes the left hand side derivative.

Moreover, suppose that in their respective domains \( f, \alpha \) and \( \beta \) satisfy the following conditions with nonnegative Lipschitz constants:

(iii) \( |f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)| \leq L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|, \) \( L_2 < 1; \)

(iv) \( |\alpha(t_1, y_1) - \alpha(t_2, y_2)| \leq A_1|t_1 - t_2| + A_2|y_1 - y_2|; \)

(v) \( |\beta(t_1, y_1) - \beta(t_2, y_2)| \leq B_1|t_1 - t_2| + B_2|y_1 - y_2|. \)

The problem (1) with \( \gamma = a \) was studied by Grimm [1]. He proved an existence result for (1) assuming that \( f \) is bounded by some constant \( M, \) \( L_2 < 1, \) and \( B_1 + B_2M \leq 1. \) He also proved a uniqueness result when \( \beta \) is independent of \( y. \) In the recent paper [2] the author relaxed this very restrictive assumption at the expense of the additional condition \( L_2(1 + B_1 + B_2G) < 1, \) where \( G \) is some constant depending on \( f \) and \( g. \) This condition means that the dependence of \( f \) on the last argument is not too strong. It is the purpose of this note to improve
further the results given in [1] and [2]. We prove the existence and uniqueness theorem for (1) (with \( \beta \) depending both on \( t \) and \( y \)), where the inequality \( L_2(1 + B_1 + B_2G) < 1 \) is replaced by the weaker conditions \( L_2 < 1 \) and \( B_1 + B_2G \leq 1 \).

For any continuous functions \( y \) and \( z \) on \([\gamma, b] \), put

\[
F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t)))),
\]

and define

\[
M := \sup\{|F(t, 0, 0)| : t \in [a, b]\}; \quad C_1 := (g'_{[\gamma, a]} + M)/(1 - L_2);
\]

\[
C_2 := 2L_1/(1 - L_2); \quad Y := (g_{[\gamma, a]} + C_1/C_2) \exp((b - a)C_2);
\]

\[
Z := \max\{C_1 + C_2Y, (M + 2L_1Y)/(1 - L_2)\}; \quad G := \max\{Lg, Z\};
\]

\[
D := \max\{Lg', L_1(1 + G(1 + A_1 + A_2G))/(1 - L_2(B_1 + B_2G))\}.
\]

Here \( x_{[c, d]} := \sup\{|x(t)| : t \in [c, d]\} \) for any function \( x \). We have the following:

**Theorem.** Assume that (i)--(v) hold, \( L_2 < 1 \), and \( B_1 + B_2G \leq 1 \). Then (1) has a solution \( y \) whose derivative is Lipschitz-continuous. Moreover, this solution is unique in the space of continuously differentiable functions on \([\gamma, a] \).

**Proof:** For \( h \in J := \{h \mid h = (b - a)/n, n \geq n_0\} \), where \( n_0 \) is a positive integer, put \( t_i = a + ih, i = 0, 1, \ldots, n, \) and as in [2] define the modified Euler sequences \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) by

\[
yh(t_i + rh) = yh(t_i) + rhzh(t_i),
\]

\[
z_h(t_i + rh) = (1 - r)zh(t_i) + rzh(t_{i+1}),
\]

\[
z_h(t_{i+1}) = F(t_{i+1}, y_h, z_h),
\]

\( i = 0, 1, \ldots, n - 1, r \in (0, 1], \) where \( y_h(t) = g(t) \) and \( z_h(t) = g'(t) \) for \( t \in [\gamma, a] \). Note that (2) is, in general, implicit in \( z_h \), but in view of \( L_2 < 1 \) it has a unique solution \( (y_h, z_h) \) for any \( h \in J \). We will first show that \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) are relatively compact in the space \( C[\gamma, b] \) of continuous functions on \([\gamma, b] \). Proceeding as in [2] it follows that \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) are uniformly bounded by \( Y \) and \( Z \), respectively, and that \( \{y_h\}_{h \in J} \) are uniformly Lipschitz-continuous with the constant \( G \). The proof that \( \{z_h\}_{h \in J} \) are also uniformly Lipschitz-continuous is more delicate than in [2]. The proof is by induction. Assume that

\[
|z_h(t_1) - z_h(t_2)| \leq D|t_1 - t_2|, \quad t_1, t_2 \in [\gamma, t_i],
\]

and we will show that this inequality is also true for \( t_1, t_2 \in [\gamma, t_{i+1}] \) (obviously (3) holds for \( t_1, t_2 \in [\gamma, t_0] \)). Define on \([\gamma, t_{i+1}] \) the iterations \( z^\nu_h(t) = z_h(t) \) for \( t \in [\gamma, t_i], \nu = 0, 1, \ldots, \), and

\[
z^0_h(t_i + rh) = z_h(t_i),
\]

\[
z_{h}^{[\nu + 1]}(t_{i+1}) = F(t_{i+1}, y_h, z^\nu_h),
\]

\[
z_{h}^{[\nu + 1]}(t_i + rh) = (1 - r)z_h(t_i) + rz_{h}^{[\nu+1]}(t_{i+1}),
\]
It follows by the induction with respect to $\nu$ that 
\[ \{z_h^{[\nu]}\}_{\nu=0}^\infty \]
are uniformly bounded by $Z$ and uniformly Lipschitz-continuous on $[\gamma, t_{i+1}]$ with the same constant $D$. Indeed, this is true for $\nu = 0$ and, assuming that it is true for $\nu$, routine manipulations yield
\[
|z_h^{[\nu+1]}(t_{i+1})| \leq M + 2L_1Y + L_2Z \leq Z,
\]
and
\[
|z_h^{[\nu+1]}(t_{i+1}) - z_h^{[\nu+1]}(t_i)| 
\leq L_1(1 + G(1 + A_1 + A_2G))h + L_2D(B_1 + B_2G)h \leq Dh.
\]
The last inequality follows from the definition of $D$. In view of the Ascoli-Arzela theorem the sequence \( \{z_h^{[\nu]}\}_{\nu=0}^\infty \) is relatively compact in $C[\gamma, t_{i+1}]$ and since the solution \((y_h, z_h)\) of (2) is unique, we have $z_h^{[\nu]} - z_h^{[\nu]}|_{[\gamma, t_{i+1}]} \to 0$ as $\nu \to \infty$. Therefore, \( \{z_h\}_{h \in J} \) are uniformly Lipschitz-continuous on $[\gamma, t_{i+1}]$ with the same constant $D$. By induction with respect to $i$, this is also true on $[\gamma, b]$. Consequently, \( \{y_h\}_{h \in J} \) and \( \{z_h\}_{h \in J} \) are relatively compact in $C[\gamma, b]$ and from this point the proof is exactly the same as the proof of Theorem 2 in [2]. We prove the existence by showing that there is a subsequence of \( \{y_h\}_{h \in J} \) convergent to the solution $y$ of (1) and we prove the uniqueness by contradiction.

\[ \square \]

References


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