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On the extremality of regular extensions of contents and measures

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Abstract. Let $\mathcal{A}$ be an algebra and $\mathcal{K}$ a lattice of subsets of a set $X$. We show that every content on $\mathcal{A}$ that can be approximated by $\mathcal{K}$ in the sense of Marczewski has an extremal extension to a $\mathcal{K}$-regular content on the algebra generated by $\mathcal{A}$ and $\mathcal{K}$. Under an additional assumption, we can also prove the existence of extremal regular measure extensions.

Keywords: regular content, lattice, semicompact, sequentially dominated

Classification: 28A12

1. Introduction

If $\mathcal{A}$, $\mathcal{B}$, are algebras of subsets of some set $X$ with $\mathcal{A} \subset \mathcal{B}$, then Plachky [9] has shown by a Krein-Milman argument that every (finite) content on $\mathcal{A}$ has an extremal extension to a content on $\mathcal{B}$. In [2], this result has been generalized in the following way. If $\mathcal{K}$, $\mathcal{L}$ are lattices of subsets of $X$ with $\mathcal{K} \subset \mathcal{L}$, then every $\mathcal{K}$-regular content on $\alpha(\mathcal{K})$, the algebra generated by $\mathcal{K}$, has an extremal extension to an $\mathcal{L}$-regular content on $\alpha(\mathcal{L})$. It is the aim of this note to give the following further generalization. If $\mathcal{A}$ is an algebra and $\mathcal{K}$ a lattice of subsets of $X$, then every content on $\mathcal{A}$ which can be approximated by $\mathcal{K}$ in the sense of Marczewski [7] has an extremal extension to a $\mathcal{K}$-regular content on $\alpha(\mathcal{A} \cup \mathcal{K})$. Under an additional assumption, we can also prove the existence of extremal regular measure extensions. Note that extremal measure extensions are considered always under some additional assumptions ([2]) or for special situations (e.g. if the target $\sigma$-algebra is generated from a given one by adjunction of a family which either consists of pairwise disjoint sets or is well ordered by inclusion [3], [4], [5]), since, in general, extremal measure extensions do not exist (see [9], [11]).

Now we fix the notation. $X$ will always denote an arbitrary set. Let $\mathcal{C}$ be a subset of $\mathcal{P}(X)$, the power set of $X$. We write $\alpha(\mathcal{C})$, $\sigma(\mathcal{C})$ for the algebra, $\sigma$-algebra generated by $\mathcal{C}$, respectively. Furthermore, $\mathcal{C}_\delta$ denotes the family of all countable intersections of sets from $\mathcal{C}$. $\mathcal{C}$ is said to be semicompact if every countable subfamily of $\mathcal{C}$ having the finite intersection property has nonvoid intersection. $\mathcal{C}$ is called a lattice if $\emptyset \in \mathcal{C}$ and $\mathcal{C}$ is closed under finite unions and finite intersections. For a lattice $\mathcal{C}$, we denote by $\mathcal{F}(\mathcal{C}) := \{ F \subset X : F \cap C \in \mathcal{C} \text{ for every } C \in \mathcal{C} \}$ the lattice of so-called “local $\mathcal{C}$-sets”. Obviously, $X \in \mathcal{F}(\mathcal{C})$ and $\mathcal{C} \subset \mathcal{F}(\mathcal{C})$; in addition, we have $\mathcal{C} = \mathcal{F}(\mathcal{C})$ iff $X \in \mathcal{C}$. 
If $D$ is another subset of $\mathcal{P}(X)$, then $\mathcal{C}$ is said to be sequentially dominated by $D$ if whenever $(C_n \in \mathcal{C})_{n \in \mathbb{N}}$ and $C_n \downarrow \emptyset$, there exists a sequence $(D_n \in D)_{n \in \mathbb{N}}$ such that $D_n \downarrow \emptyset$ and $C_n \subset D_n$ for all $n \in \mathbb{N}$. Note that a semicompact family is sequentially dominated by any family $D$ with $X \in D$.

By a content (measure) we always understand a $[0, \infty)$-valued, finitely (countably) additive set function defined on an algebra.

Consider a lattice $\mathcal{K} \subset \mathcal{P}(X)$ and a content $\mu$ on the algebra $\mathcal{A} \subset \mathcal{P}(X)$. Under the assumption $\mathcal{K} \subset \mathcal{A}$, $\mu$ is called $\mathcal{K}$-regular if $\mu(A) = \sup \{\mu(K) : K \in \mathcal{K}, K \subset A\}$ for all $A \in \mathcal{A}$. For the following concept going back to Marczewski [7], we will use the terminology of [8]:

$\mathcal{K}$ is said to $\mu$-approximate $\mathcal{A}$ if for every $A \in \mathcal{A}$ and every $\varepsilon > 0$, there exist sets $B \in \mathcal{A}$ and $K \in \mathcal{K}$ such that $B \subset K \subset A$ and $\mu(A - B) < \varepsilon$ hold. Note that in case $\mathcal{K} \subset \mathcal{A}$, $\mathcal{K}$ $\mu$-approximates $\mathcal{A}$ iff $\mu$ is $\mathcal{K}$-regular.

2. The main results

In this section we consider an algebra $\mathcal{A}$ and two lattices $\mathcal{K}, \mathcal{L}$ of subsets of $X$ with $\mathcal{K} \subset \mathcal{L}$ as well as a content $\mu$ on $\mathcal{A}$ such that $\mathcal{K}$ $\mu$-approximates $\mathcal{A}$.

If $\mathcal{B} \supset \mathcal{A}$ is another algebra, then $\text{ba}(\mu, \mathcal{B})$ denotes the family of all contents on $\mathcal{B}$ that extend $\mu$. In addition, we define $\text{ba}(\mu, \mathcal{B}, \mathcal{K}) := \{\nu \in \text{ba}(\mu, \mathcal{B}) : \mathcal{K} \mu$-approximates $\mathcal{B}\}$ and $\text{ca}(\mu, \mathcal{B}, \mathcal{K}) := \{\nu \in \text{ba}(\mu, \mathcal{B}, \mathcal{K}) : \nu$ is a measure$\}$. Note that $\text{ba}(\mu, \mathcal{B})$, $\text{ba}(\mu, \mathcal{B}, \mathcal{K})$ and $\text{ca}(\mu, \mathcal{B}, \mathcal{K})$ are convex sets. If $D$ is any of these sets, then $\text{exD}$ denotes the set of extreme points of $D$.

Lemma 2.1. Let $\mathcal{B} \supset \mathcal{A}$ be another algebra and $\nu \in \text{ba}(\mu, \mathcal{B}, \mathcal{K})$. Then $\nu \in \text{ex ba}(\mu, \mathcal{B}, \mathcal{K})$ iff $\nu \in \text{ex ba}(\mu, \mathcal{B})$.

Proof: Assume $\nu \in \text{ex ba}(\mu, \mathcal{B}, \mathcal{K})$ and let $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ with $\nu_1, \nu_2 \in \text{ba}(\mu, \mathcal{B})$. Since $\frac{1}{2}\nu_i \leq \nu$ and $\nu \in \text{ba}(\mu, \mathcal{B}, \mathcal{K})$ we have $\nu_i \in \text{ba}(\mu, \mathcal{B}, \mathcal{K})$ for $i = 1, 2$. Thus we infer $\nu_1 = \nu_2$ from the extremality of $\nu$. This proves $\nu \in \text{ex ba}(\mu, \mathcal{B})$. The other part of the claim is obvious. \hfill $\square$

Lemma 2.2. If $Q \in \mathcal{F}(\mathcal{K}) - \mathcal{A}$ and $\mathcal{B} := \alpha(\mathcal{A} \cup \{Q\})$ then $\text{ex ba}(\mu, \mathcal{B}, \mathcal{K}) \neq \emptyset$.

Proof: (1) For every $E \in \mathcal{P}(X)$, we define $\mu^*(E) := \inf \{\mu(A) : E \subset A \in \mathcal{A}\}$ and $\mu^*(E) := \sup \{\mu(A) : E \supset A \in \mathcal{A}\}$. It is well known ([6]) that $\mathcal{B} = \{(A_1 \cap Q) \cup (A_2 - Q) : A_1, A_2 \in \mathcal{A}\}$ and $\nu(B) := \mu^*(B \cap Q) + \mu^*(B - Q)$, $B \in \mathcal{B}$, defines an element $\nu$ of $\text{ba}(\mu, \mathcal{B})$.

(2) To prove $\nu \in \text{ba}(\mu, \mathcal{B}, \mathcal{K})$ let $B \in \mathcal{B}$ and $\varepsilon > 0$ be given. Then $B = (A_1 \cap Q) \cup (A_2 - Q)$ with some $\mathcal{A}$-sets $A_1, A_2$. Since $\mu^*(B - Q) = \mu^*(A_2 - Q) = \sup \{\mu(A) : A \in \mathcal{A}, A \subset A_2 - Q\}$, there is an $\mathcal{A}$-set $C$ satisfying $C \subset A_2 - Q$ and $\mu^*(B - Q) < \mu(C) + \frac{\varepsilon}{2}$. In addition, there exist sets $C_0 \in \mathcal{A}$ and $K_0 \in \mathcal{K}$ such that $C_0 \subset K_0 \subset C$ and $\mu(C) < \mu(C_0) + \frac{\varepsilon}{2}$. This together yields $\mu^*(B - Q) < \mu(C_0) + \frac{\varepsilon}{2}$. Furthermore, one can choose sets $C_1 \in \mathcal{A}$ and $K_1 \in \mathcal{K}$ such that $C_1 \subset K_1 \subset A_1$ and $\mu(A_1 - C_1) < \frac{\varepsilon}{2}$ which implies $\mu^*((A_1 \cap Q) - C_1) \leq \mu(A_1 - C_1) < \frac{\varepsilon}{2}$ and hence $\mu^*(A_1 \cap Q) \leq \mu^*((A_1 \cap Q) - C_1) + \mu^*(A_1 \cap Q \cap C_1) < \mu^*(C_1 \cap Q) + \frac{\varepsilon}{2}$. Now
For an arbitrary $\varepsilon > 0$, choose $A \in A$ such that $Q \subset A$ and $\mu(A) < \mu^*(Q) + \varepsilon$. Then $\nu(A \triangle Q) = \mu(A) - \mu^*(Q) < \varepsilon$. From [9], Theorem 1 and the associated Remark 2, we infer $\nu \in \text{ex ba}(\mu, B, K)$. □

If $B$ is an algebra satisfying $A \cup K \subset B$, then $\text{ba}(\mu, B, K)$ is the family of all $K$-regular contents on $B$ that extend $\mu$. According to [1, Theorem 3.4], $\mu$ can be extended to a $K$-regular content on $\alpha(A \cup F(K))$. The following basic result shows that an extremal extension exists.

**Theorem 2.3.** $\text{ex ba}(\mu, \alpha(A \cup \mathcal{E}), K) \neq \emptyset$ for every sublattice $\mathcal{E}$ of $\mathcal{F}(K)$.

**Proof:** (1) Fix some sublattice $\mathcal{M}, \varrho_i \in I$ in $\Gamma$. Then $\mathcal{M} := \bigcup_{i \in I} \mathcal{M}_i$ is a sublattice of $\mathcal{E}$ and $\varrho \in \text{ex ba}(\mu, \alpha(A \cup \mathcal{M}), K)$. Note that $\{\emptyset\}, \mu \in \Gamma$. We order the elements of $\Gamma$ in the following way: $(\mathcal{M}, \varrho) \leq (\mathcal{M}', \varrho')$ iff $\mathcal{M} \subset \mathcal{M}'$ and $\varrho$ is an extension of $\varrho'$.

(2) Now we show that $\Gamma$ is inductively ordered. Consider a chain $(\mathcal{M}_i, \varrho_i)_{i \in I}$ in $\Gamma$. Then $\mathcal{M} := \bigcup_{i \in I} \mathcal{M}_i$ is a sublattice of $\mathcal{E}$ and $\alpha(A \cup \mathcal{M}) = \bigcup_{i \in I} \alpha(A \cup \mathcal{M}_i)$. For $C \in \alpha(A \cup \mathcal{M})$, define $\varrho(C) := \varrho_i(C)$ provided that $C \in \alpha(A \cup \mathcal{M}_i)$. $\varrho$ is a content on $\alpha(A \cup \mathcal{M})$ that extends every $\varrho_i$. It is easy to see that $\varrho \in \text{ba}(\mu, \alpha(A \cup \mathcal{M}), K)$.

To prove $\varrho \in \text{ex ba}(\mu, \alpha(A \cup \mathcal{M}), K)$ consider $\tau_1, \tau_2 \in \text{ba}(\mu, \alpha(A \cup \mathcal{M}), K)$ with $\varrho = \frac{1}{2}(\tau_1 + \tau_2)$. Fix some $i_0 \in I$ and define $\tilde{\tau}_j := \tau_j | \alpha(A \cup \mathcal{M}_{i_0})$ for $j = 1, 2$. Then $\tilde{\tau}_j \in \text{ba}(\mu, \alpha(A \cup \mathcal{M}_{i_0}))$, $j = 1, 2$, and $\varrho_{i_0} = \frac{1}{2}(\tilde{\tau}_1 + \tilde{\tau}_2)$. Since $\varrho_{i_0} \in \text{ex ba}(\mu, \alpha(A \cup \mathcal{M}_{i_0}), K)$, we infer $\tilde{\tau}_1 = \tilde{\tau}_2$ from 2.1.

Now consider an arbitrary $A \in \alpha(A \cup \mathcal{M})$. Then $A \in \alpha(A \cup \mathcal{M}_{i_0})$ for some $i_0 \in I$ and hence $\tau_1(A) = \tilde{\tau}_1(A) = \tilde{\tau}_2(A) = \tau_2(A)$. Thus $\tau_1 = \tau_2$ which proves $\varrho \in \text{ex ba}(\mu, \alpha(A \cup \mathcal{M}), K)$.

Consequently, $(\mathcal{M}_i, \varrho_i) \leq (\mathcal{M}, \varrho) \in \Gamma$ for all $i \in I$. So $\Gamma$ is inductively ordered.

(3) By Zorn’s lemma, there is a maximal element $(\tilde{\mathcal{M}}, \tilde{\varrho})$ in $\Gamma$. We will show $\tilde{\mathcal{M}} = \mathcal{E}$ which implies that $\tilde{\varrho}$ is the desired extremal element of $\text{ba}(\mu, \alpha(A \cup \mathcal{E}), K)$.

Assume that there is a set $Q \in \mathcal{E} - \tilde{\mathcal{M}}$. Denoting by $\tilde{K}$ the lattice generated by $\tilde{\mathcal{M}} \cup \{Q\}$, we have $\alpha(A \cup \tilde{K}) = \alpha(B \cup \{Q\})$ with $B := \alpha(A \cup \tilde{\mathcal{M}})$. It follows $Q \notin B$. By 2.2, there exists an element $\tilde{\mu} \in \text{ex ba}(\tilde{\varrho}, \alpha(A \cup \tilde{K}), K)$.

Next we shall $\tilde{\mu} \in \text{ex ba}(\mu, \alpha(A \cup \tilde{K}), K)$ which implies $(\tilde{K}, \tilde{\mu}) \in \Gamma$. On the other hand, $(\tilde{\mathcal{M}}, \tilde{\varrho}) \leq (\tilde{K}, \tilde{\mu})$ and $\tilde{\mathcal{M}} \neq \tilde{K}$ which, however, is in contrast to the maximality of $(\tilde{\mathcal{M}}, \tilde{\varrho})$.

It is obvious that $\tilde{\mu} \in \text{ba}(\mu, \alpha(A \cup \tilde{K}), K)$. To prove the extremality of $\tilde{\mu}$, let $\tilde{\mu} = \frac{1}{2}(\mu_1 + \mu_2)$ with $\mu_1, \mu_2 \in \text{ba}(\mu, \alpha(A \cup \tilde{K}), K)$ and define $\tilde{\mu}_i := \mu_i | \alpha(A \cup \tilde{\mathcal{M}})$, $i = 1, 2$. For $B \in B$, $\tilde{\varrho}(B) = \tilde{\mu}(B) = \frac{1}{2}(\tilde{\mu}_1(B) + \tilde{\mu}_2(B))$, i.e. $\tilde{\varrho} = \frac{1}{2}(\tilde{\mu}_1 + \tilde{\mu}_2)$.
Since $\tilde{\theta} \in \text{ex ba}(\mu, \alpha(A \cup \tilde{M}))$ by 2.1, we infer $\tilde{\mu}_1 = \tilde{\mu}_2 = \tilde{\theta}$. Consequently, $\mu_1, \mu_2 \in \text{ba}(\tilde{\theta}, \alpha(A \cup \tilde{K}))$. As $\tilde{\mu} \in \text{ex ba}(\tilde{\theta}, \alpha(A \cup \tilde{K}))$ by 2.1, we obtain $\mu_1 = \mu_2$ proving $\tilde{\mu} \in \text{ex ba}(\mu, \alpha(A \cup \tilde{K}), \mathcal{K})$.

**Corollary 2.4.** $\text{ex ba}(\mu, \alpha(A \cup \mathcal{E}), \mathcal{L}) \neq \emptyset$ for every sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{L})$.

**Proof:** Since $\mathcal{K} \subset \mathcal{L}$ and $\mathcal{K}$ $\mu$-approximates $A$, so does $\mathcal{L}$. Thus our claim follows from 2.3 (with $\mathcal{L}$ instead of $\mathcal{K}$).

In case $A = \alpha(\mathcal{K})$, the assumption that $\mathcal{K}$ $\mu$-approximates $A$ is equivalent to $\mathcal{K}$-regularity of $\mu$. Thus we obtain from 2.4

**Corollary 2.5** ([2, Theorem 2.3]). Every $\mathcal{K}$-regular content on $\alpha(\mathcal{K})$ admits an extremal extension to an $\mathcal{L}$-regular content on $\alpha(\mathcal{L})$.

Our next result is concerned with the existence of extremal measure extensions.

**Theorem 2.6.** If $\mu$ is a measure and $\mathcal{K}$ is sequentially dominated by $A$, then $\text{ex ca}(\mu, \sigma(A \cup \mathcal{E}), \mathcal{K}_\delta) \neq \emptyset$ for every sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{K}_\delta)$.

**Proof:** Fix some sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{K}_\delta)$ and define $\mathcal{B} := \alpha(A \cup \mathcal{E})$. By 2.4, there exists an element $\varrho \in \text{ex ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$. To show the countable additivity of $\varrho$, consider a sequence $(B_n)$ of sets from $\mathcal{B}$ with $B_n \downarrow \emptyset$. For any $\varepsilon > 0$ and $n \in N$, choose $C_n \in \mathcal{B}$ and $K_n \in \mathcal{K}_\delta$ such that $C_n \subset K_n \subset B_n$ and $\varrho(B_n - C_n) < \varepsilon \cdot 2^{-n}$. Then $D_n := \bigcap_{i=1}^n C_i \subset \bigcap_{i=1}^n K_i \subset B_n$ and $\varrho(B_n - D_n) \leq \varrho(\bigcup_{i=1}^n (B_i - C_i)) \leq \sum_{i=1}^n \varrho(B_i - C_i) < \varepsilon$ for $n \in N$. Furthermore, $K'_n := \bigcap_{i=1}^n K_i \in \mathcal{K}_\delta$ and $K'_n \downarrow \emptyset$. Since also $\mathcal{K}_\delta$ is sequentially dominated by $A$, there is a sequence $(A_n)$ of $A$-sets satisfying $A_n \downarrow \emptyset$ and $K'_n \subset A_n$ for $n \in N$. This implies $\varrho(B_n) \leq \varrho((B_n - D_n) \cup A_n) \leq \varrho(B_n - D_n) + \varrho(A_n) < \varepsilon + \mu(A_n) < 2\varepsilon$ for all sufficiently large $n$. Therefore $\varrho$ is a measure.

Denote by $\tilde{\varrho}$ the unique measure extension of $\varrho$ to $\sigma(\mathcal{B}) = \sigma(A \cup \mathcal{E})$. Then $\tilde{\varrho} \in \text{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ by [8, (2.10)]. To prove $\tilde{\varrho} \in \text{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ consider $\tilde{\varrho}_1, \tilde{\varrho}_2 \in \text{ca}(\mu, \sigma(\mathcal{B}), \mathcal{K}_\delta)$ with $\tilde{\varrho} = \frac{1}{2} (\tilde{\varrho}_1 + \tilde{\varrho}_2)$. Let $\varrho_i := \tilde{\varrho}_i \mid \mathcal{B}$ for $i = 1, 2$. Then $\varrho = \frac{1}{2} (\varrho_1 + \varrho_2)$. As $\mathcal{K}_\delta$ $\varrho$-approximates $\mathcal{B}$ and $\frac{1}{2} \varrho_i \leq \varrho$, $\mathcal{K}_\delta$ also $\varrho_i$-approximates $\mathcal{B}$ which implies $\varrho_i \in \text{ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$ for $i = 1, 2$. Since $\varrho \in \text{ex ba}(\mu, \mathcal{B}, \mathcal{K}_\delta)$, we conclude $\varrho_1 = \varrho_2$ and hence $\tilde{\varrho}_1 = \tilde{\varrho}_2$.

**Corollary 2.7.** If $\mathcal{K}$ is semicompact, then $\text{ex ca}(\mu, \sigma(A \cup \mathcal{E}), \mathcal{K}_\delta) \neq \emptyset$ for every sublattice $\mathcal{E}$ of $\mathcal{F}(\mathcal{K}_\delta)$.

**Proof:** The semicompactness of $\mathcal{K}$ implies that both $\mu$ is a measure and $\mathcal{K}$ is sequentially dominated by $A$. Thus the assertion follows from 2.6.

Under the additional assumption $\mathcal{K} \subset A$, the previous results can be strengthened in the following way, thus obtaining an “extremal version” of the extension theorem 3.6 of [1].

**Theorem 2.8.** Assume $\mathcal{K} \subset A$.

(a) Then $\text{ex ba}(\mu, \mathcal{B}, \mathcal{L}) \neq \emptyset$ for every algebra $\mathcal{B}$ satisfying $A \cup \mathcal{L} \subset \mathcal{B} \subset \alpha(A \cup \mathcal{F}(\mathcal{K}) \cup \mathcal{F}(\mathcal{L}))$. 

(b) If, in addition, $\mu$ is a measure and $\mathcal{L}$ is sequentially dominated by $\sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta))$, then $\text{exca}(\mu, \mathcal{B}, \mathcal{L}_\delta) \neq \emptyset$ for every $\sigma$-algebra $\mathcal{B}$ satisfying $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$.

**Proof:** We only prove (b), since the (simpler) proof of (a) can be performed in the same way.

(1) We first consider the special case $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$. Define $\mathcal{C} = \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta))$, and let $\nu$ be the $\mathcal{K}_\delta$-regular measure on $\mathcal{C}$ extending $\mu$ that has been constructed in the proof of [1, 3.6(b)]. Since $\mathcal{L}$ is sequentially dominated by $\mathcal{C}$, so is $\mathcal{L}_\delta$. In addition, $\mathcal{K}_\delta \subset \mathcal{L}_\delta$ and $\mathcal{B} = \sigma(\mathcal{C} \cup \mathcal{F}(\mathcal{L}_\delta))$. Thus, by 2.6, there exists an element $\tau \in \text{exca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$. Clearly $\tau \in \text{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$. To prove $\tau \in \text{exca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ consider $\tau_1, \tau_2 \in \text{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ with $\tau = \frac{1}{2}(\tau_1 + \tau_2)$. Then

\[ \nu(C) \leq \tau_i(C) \quad \text{for} \quad C \in \mathcal{C} \quad \text{and} \quad i = 1, 2. \]

Assume that (2.1) fails to be true. Then $\nu(C) > \tau_i(C)$ for some $C \in \mathcal{C}$ and some $i \in \{1, 2\}$. Thus we can find a $\mathcal{K}_\delta$-set $\overline{K}$ satisfying $\overline{K} \subset C$ and $\nu(\overline{K}) > \tau_i(C)$. Choosing a sequence $(K_n)$ in $\mathcal{K}$ such that $K_n \uparrow K$, we obtain the contradiction $\inf_n \mu(K_n) = \inf_n \nu(K_n) = \nu(\overline{K}) > \tau_i(C) \geq \nu(K) = \inf_n \nu_i(K_n) = \inf_n \mu(K_n)$. Thus (2.1) holds true.

Since also $\tau_i(X) = \mu(X) = \nu(X)$ for $i = 1, 2$, we infer from (2.1) $\tau_1 | C = \tau_2$. Thus $\tau_1, \tau_2 \in \text{ca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$ which together with $\tau \in \text{exca}(\nu, \mathcal{B}, \mathcal{L}_\delta)$ implies $\tau_1 = \tau_2$. So $\tau \in \text{exca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$.

(2) Now we consider an arbitrary $\sigma$-algebra $\mathcal{B}$ satisfying $\mathcal{A} \cup \mathcal{L} \subset \mathcal{B} \subset \mathcal{E}$ where $\mathcal{E} := \sigma(\mathcal{A} \cup \mathcal{F}(\mathcal{K}_\delta) \cup \mathcal{F}(\mathcal{L}_\delta))$. By the special case (1), there exists an element $\varrho \in \text{exca}(\mu, \mathcal{E}, \mathcal{L}_\delta)$. Then $\nu := \varrho | \mathcal{B} \in \text{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$. To prove $\nu \in \text{exca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ consider $\nu_1, \nu_2 \in \text{ca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$ with $\nu = \frac{1}{2}(\nu_1 + \nu_2)$. For every $E \in \mathcal{E}$, $\varrho(E) = \sup\{\varrho(L) : L \in \mathcal{L}_\delta, L \subset E\} = \nu(L) : L \in \mathcal{L}_\delta, L \subset E\} = \frac{1}{2}(\sup\{\nu_1(L) : L \in \mathcal{L}_\delta, L \subset E\} \leq \nu_1(E) + \nu_2(E))$ where $\nu_i$ denotes an arbitrary content on $\mathcal{E}$ that extends $\nu_i$, $i = 1, 2$. It follows $\varrho \leq \frac{1}{2}(\tilde{\nu}_1 + \tilde{\nu}_2)$ as well as $\frac{1}{2}(\tilde{\nu}_1(X) + \tilde{\nu}_2(X)) = \frac{1}{2}(\nu_1(X) + \nu_2(X)) = \nu(X) = \varrho(X)$ which implies

\[ \varrho = \frac{1}{2}(\tilde{\nu}_1 + \tilde{\nu}_2). \]

From (2.2) we infer both the countable additivity and the $\mathcal{L}_\delta$-regularity of $\tilde{\nu}_i$, $i = 1, 2$. Therefore $\varrho \in \text{exca}(\mu, \mathcal{E}, \mathcal{L}_\delta)$ and (2.2) imply $\tilde{\nu}_1 = \tilde{\nu}_2$ and hence $\nu_1 = \nu_2$. So $\nu \in \text{exca}(\mu, \mathcal{B}, \mathcal{L}_\delta)$.

An immediate consequence of 2.8 (b) is [2, Theorem 2.4], various applications of which are gathered in Section 3 of [2].

The assumptions of 2.8 (b) are, in particular, satisfied if the lattice $\mathcal{L}$ is semicompact. Thus we obtain
Corollary 2.9. If $L$ is semicompact and $K \subset A$ holds, then $\text{exca}(\mu, B, L_\delta) \neq \emptyset$ for every $\sigma$-algebra $B$ satisfying $A \cup L \subset B \subset \sigma(A \cup F(K_\delta) \cup F(L_\delta))$.

The following result is an application of 2.9.

Corollary 2.10. Let $C, D$ be lattices of subsets of $X$ such that $C \subset D \subset F(C_\delta)$. If $C$ is semicompact and $A \subset \sigma(D)$, then every $C \cap A$-regular content on $A$ admits an extremal extension to a $C_\delta$-regular measure on $\sigma(D)$.

**Proof:** The claim follows with $K = C \cap A$ and $L = C$ from 2.9.

The assumptions of 2.10 are, in particular, satisfied if $C, D$ are the lattices of compact, respectively closed, subsets of a Hausdorff topological space. Thus one obtains from 2.10 an “extremal version” of Henry’s extension theorem (cf. [10, Theorem 16, p. 51]).

**References**


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