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Remarks on bounded sets in \((LF)_{tv}\)-spaces

JERZY KĄKOL*

Abstract. We establish the relationship between regularity of a Hausdorff \((LB)_{tv}\)-space and its properties like \((K)\), \(M.c.c.\), sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff \((LB)_{tv}\)-space to be an \((LS)_{tv}\)-space. A factorization theorem for \((LN)_{tv}\)-spaces with property \((K)\) is also obtained.

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1. Introduction

Let \((E_n)_{n}\) be an increasing sequence of vector subspaces of a vector space \(E\), whose union is \(E\), such that every \(E_n\) is endowed with a vector topology \(\tau_n\) with \(\tau_{n+1}|_{E_n} \leq \tau_n\). The space \((E, \tau)\), where \(\tau\) is the finest vector topology on \(E\) such that \(\tau|_{E_n} \leq \tau_n\), \(n \in \mathbb{N}\), will be called the inductive limit space and \((E_n, \tau_n)_{n}\) its defining sequence. We will say that \((E, \tau)\) is an

\begin{enumerate}
\item \((LM)_{tv}\), if every \((E_n, \tau_n)\) is a metrizable topological vector space (tv); \n\item \((LN)_{tv}\)-space, if every \((E_n, \tau_n)\) is a locally bounded tvs;
\item \((LF)_{tv}\)-space, if every \((E_n, \tau_n)\) is an \(F\)-space, i.e. a metrizable and complete tvs;
\item \((LB)_{tv}\)-space, if every \((E_n, \tau_n)\) is a quasi-Banach space, i.e. a locally bounded and complete tvs.
\end{enumerate}

It is known cf. e.g. [11, Proposition 2.2], that if every \((E_n, \tau_n)\) is a locally convex space (lcs), then \(\tau\) is locally convex. In this case the corresponding inductive limit space will be called respectively \((LM)\), \((LN)\), \((LF)\), \((LB)\). Recall that a topological vector space (tvs) \(E\) is locally bounded if \(E\) has a bounded neighbourhood of zero.

Following Floret [6], [7], and Makarov [18], an inductive limit space \((E, \tau)\) with defining sequence \((E_n, \tau_n)\) will be called

\begin{enumerate}
\item \textit{regular}, if every bounded set in \((E, \tau)\) is contained in some \(E_m\) and is bounded in \((E_m, \tau_m)\);
\item \textit{sequentially retractive}, if every null-sequence in \((E, \tau)\) is contained in some \(E_m\) and is a null-sequence in \((E_m, \tau_m)\).
\end{enumerate}

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Following Grothendieck, a tvs $E$ will be said to satisfy the Mackey convergence condition (M.c.c.) if for every null-sequence $(x_n)_n$ in $E$ there exists a scalar sequence $(t_n)_n$, $t_n \not\to \infty$, with $t_n x_n \to 0$. A regular $(LM)_t$-space is sequentially retractive iff it satisfies M.c.c.

Grothendieck’s factorization theorem [10, p. 16], implies that a Hausdorff $(LF)_t$-space is regular iff it is locally complete. Other criteria for regularity or sequential retractivity of $(LF)_t$, $(LB)_t$-spaces were obtained (among others) by Floret [6], [7], [8], Neus [15], Fernandez [5], Vogt [19]. Recently we have showed [12] (extending Gilsdorf’s result of [9]) that a Hausdorff $(LB)_t$-space $E$ is regular if $E$ has property:

(K) Every null-sequence $(x_n)_n$ in $E$ has a subsequence $(x_{n(k)})_k$ such that the series $\sum_{k=1}^{\infty} x_{n(k)}$ converges in $E$.

Note that there exist a non-sequentially complete (metrizable) tvs with property (K) ([14, Theorem 2]), and a complete tvs without property (K), cf. e.g. [12]. On the other hand every metrizable tvs with property (K) is a Baire tvs [2, 2.2].

Developing the argument used by Gilsdorf in [9] and ideas found in [6], [7], we establish the relationship between regularity of a Hausdorff $(LB)_t$-space and its properties like (K), M.c.c., sequential completeness, local completeness. We give a sufficient and necessary condition for a Hausdorff $(LB)_t$-space to be an $(LS)_t$-space. Moreover a factorization theorem for $(LN)_t$-spaces with property (K) is obtained.

We shall need the following factorization theorem, see [1, (11), pp. 57–58], and its proof.

(0) Let $F$ be a Baire tvs and $E$ a Hausdorff $(LF)_t$-space with defining sequence $(E_n)_n$ of $F$-spaces. If $T : F \to E$ is a continuous linear map, there exists $p \in \mathbb{N}$ such that $T(F) \subset E_p$ and $T : F \to E_p$ is continuous.

By $Bd(\tau)$ we shall denote the set of all $\tau$-bounded subsets of a tvs $(E, \tau)$; $\mathcal{F}(\tau)$ will denote the filter of all $\tau$-neighbourhoods of zero. A sequence $(V_n)_n$ of balanced and absorbing subsets of $E$ will be called a string if $V_{n+1} + V_{n+1} \subset V_n$, $n \in \mathbb{N}$; $(V_n)_n$ is topological if $V_n \in \mathcal{F}(\tau)$ for all $n \in \mathbb{N}$. A subset $A$ of $E$ will be said pseudo-convex if there exists a scalar $t > 0$ such that $A + A \subset tA$.

A tvs $E$ will be called locally complete if for every balanced pseudo-convex bounded and closed set $B$ in $E$ the linear span $[B]$ endowed with the locally bounded topology generated by $B$ is complete. It is easy to see that for lcs this definition is equivalent to the Grothendieck’s one of local completeness (cf. [3, p. 152]). $E$ will be called locally Baire if every bounded subset of $E$ is contained in a bounded set $B$ as above such that $[B]$ is a Baire tvs, cf. e.g. [9]. Every locally bounded non-complete tvs which is Baire is locally Baire but not locally complete. Every locally complete tvs with a fundamental family of pseudo-convex bounded sets is locally Baire.

All tvs given in this paper are assumed to be Hausdorff.
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2. Results

We start with the following proposition; its proof is due to S. Dierolf [4].

**Proposition 2.1.** Every sequentially retractive inductive limit space is regular.

**Proof:** Let \((E_n, \tau_n)\) be a defining sequence of a tvs \(E\) under which \(E\) is sequentially retractive. Let \(B\) be a bounded subset of \(E\); we may assume that \(B \subset E_1\).

Assume that \(B\) is not bounded in \((E_n, \tau_n), n \in \mathbb{N}\). Then for every \(n \in \mathbb{N}\) there exists \(U_n \in \mathcal{F}(\tau_n)\) such that \(B\) is not absorbed by \(U_n\). Thus for every \(m \geq n\) there exists \(b_{n,m} \in B\) with \(m^{-1}b_{n,m} \notin U_n\). Consider the following sequence

\[
b_{11}, 2^{-1}b_{12}, 2^{-1}b_{22}, 3^{-1}b_{13}, 3^{-1}b_{23}, 3^{-1}b_{33}, \ldots .
\]

This sequence converges to zero in \(E\), hence it converges to zero in some \(E_n\); consequently it is residually contained in \(U_n\), a contradiction. \(\square\)

**Lemma 2.2.** Let \((E, \tau)\) be the inductive limit space of the sequence \((E_n, \tau_n)\) of tvs such that

\[
(*) \quad Bd(\tau_n) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.
\]

If \((E, \tau)\) has property (K), then there exists for \((E, \tau)\) a defining sequence \((G_n, \gamma_n)\) of locally bounded Baire tvs under which \((E, \tau)\) is regular. Moreover, if every \((E_n, \tau_n)\) is locally convex, then the same is true (with \((G_n, \gamma_n)\) normed and Baire) when \((*)\) is replaced by

\[
(**) \quad Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.
\]

**Proof:** Let \((S_n)_n\) be a sequence of balanced subsets of \(E\) such that \(S_n + S_n \subset S_{n+1}\) and \(S_n \in Bd(\tau_n) \cap \mathcal{F}(\tau_n), n \in \mathbb{N}\). Set \(A_n := S_n^\tau, G_n := \text{lin} A_n, n \in \mathbb{N}\). Let \(K_n^j := (\alpha_n)^{-j}A_n, j \in \mathbb{N}\), where \(\alpha_n\) are chosen such that \(S_n + S_n \subset \alpha_n S_n\), \(\alpha_n > 1, n \in \mathbb{N}\). Clearly \((K_n^j)_j\) forms a basis of neighbourhoods of zero for a locally bounded vector topology \(\gamma_n\) on \(G_n\) such that \(\tau|G_n \leq \gamma_n\). Fix \(n \in \mathbb{N}\). In order to prove that \((G_n, \gamma_n)\) is Baire, it is enough to show that \((G_n, \gamma_n)\) has property (K), cf. Introduction.

Let \((x_p)_p\) be a null-sequence in \((G_n, \gamma_n)\). We may assume that \(x_j \in K_1^n, j \in \mathbb{N}\). There exists a subsequence \((x_{p(k)})_k\) such that \(\sum_{k=1}^{\infty} x_{p(k)}\) converges in \(\tau\). Since \(y_m := \sum_{k=1}^{m} x_{p(k)}, m \in \mathbb{N}\), is \(\gamma_n\)-Cauchy, \(y_m \in K_1^n + K_1^n \subset A_n, m \in \mathbb{N}\), and \((y_m)_m\) converges in \(\tau|G_n\), the series \(\sum_{k=1}^{\infty} x_{p(k)}\) converges in \((G_n, \gamma_n)\). Consequently, \((G_n, \gamma_n)\) is Baire, by [2, 2.2]. Let \((E, \gamma)\) be the inductive limit space of the sequence \((G_n, \gamma_n)\). Then \(\tau \leq \gamma\). Let \(U \in \mathcal{F}(\gamma)\) and \((U_n)_n\), be a \(\gamma\)-topological
string with $U_1 + U_1 \subset U$. For every $m \in \mathbb{N}$ there exists $j_m \in \mathbb{N}$ such that $U_m \cap G_m \supset K^m_{j_m}$. Hence

$$U \supset U_1 + U_1 \supset \bigcup_{m=1}^{\infty} (K^1_{j_1} + K^2_{j_2} + \cdots + K^m_{j_m}) \supset \bigcup_{m=1}^{\infty} ((\alpha_1)^{-j_1} S_1 + \cdots + (\alpha_m)^{-j_m} S_m).$$

The last set belongs to $\mathcal{F}(\tau)$. [11, Proposition 2.2]; hence $\tau = \gamma$. To see that $(E, \tau)$ is regular with respect to the sequence $(G_n, \gamma_n)_n$, it is enough to show that $(A_n)_n$ is a fundamental sequence of $\tau$-bounded sets; By [1, 16 (6)], the sequence $(A_n)_n$ is a fundamental sequence of bounded sets for the strongest vector topology $\vartheta$ on $E$ which agrees with $\tau$ on every $A_n$. On the other hand $\tau = \vartheta$, cf. [13, proof of Theorem 2].

If every $(E_n, \tau_n)$ is locally convex and $(\ast \ast)$ is satisfied, we choose absolutely convex sets $S_n \in Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n)$ such that $S_n + S_n \subset S_{n+1}$, $n \in \mathbb{N}$. Set $K^n_j := (2)^{-j} A_n$, $n, j \in \mathbb{N}$. To complete the proof of this case we proceed as above.

Note that condition $(\ast \ast)$ is satisfied when every $(E_n, \tau_n)$ is normed or when the inclusion map of $(E_n, \tau_n)$ into $(E_{n+1}, \tau_{n+1})$ is compact (or precompact), $n \in \mathbb{N}$.

**Corollary 2.3.** Let $E$ be an $(LN)_{tv}$-space with property (K) and $F$ an $(LF)_{tv}$-space with defining sequence $(F_n)_n$ of $F$-spaces. If $T : E \to F$ is a linear map with closed graph, then:

1. $T$ is continuous.
2. For every bounded sets $B$ in $E$ there exists $m \in \mathbb{N}$ such that $T(B) \subset F_m$ and $T(B)$ is bounded in $F_m$.

**Proof:** Combining our Lemma 2.2 with the closed graph theorem [1, (11), p. 57], one obtains the continuity of $T$. Now (2) follows from Lemma 2.2 and (0). □

For locally convex spaces we have even the following

**Corollary 2.4.** Let $(E, \tau)$ be a lcs with property (K). Assume that at least one of the following conditions is satisfied.

(a) $(E, \tau)$ is bornological.

(b) $(E, \tau)$ is the inductive limit space of the sequence $(E_n, \tau_n)_n$ of lcs such that

$$Bd(\tau_{n+1}) \cap \mathcal{F}(\tau_n) \neq \emptyset, \quad n \in \mathbb{N}.$$  

If $F, T$ are defined as in Corollary 2.3, the conclusion of Corollary 2.3 is also true.

**Proof:** (a): Since $(E, \tau)$ is bornological with property (K), it is the inductive limit space of normed Baire spaces $[B]$, where $B$ run over the family of absolutely convex bounded and closed subsets of $E$. We complete the proof as in Corollary 2.3.
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(b): See the proof of Corollary 2.2. □

The following extends Theorem 5.5 of [7].

**Theorem 2.5.** Let $E$ be an $(LB)_{tv}$-space and $(E_n, \tau_n)_n$ its defining sequence consisting of quasi-Banach spaces. Consider the following conditions:

(a) $E$ is sequentially retractive;
(b) $E$ is sequentially complete;
(c) $E$ is locally complete;
(d) $E$ is locally Baire;
(e) $E$ is regular;
(f) $E$ has property (K).

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\iff$ (d) $\Rightarrow$ (e) $\iff$ (f). If $E$ satisfies M.c.c., then all the conditions are equivalent.

**Proof:** (a) $\Rightarrow$ (b): Follows from Corollary 5.3 of [7] (which also holds for $(LF)_{tv}$-spaces). (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious. (d) $\Rightarrow$ (c): This follows from the following:

If $B$ is a balanced pseudo-convex bounded and closed subset of $E$ such that $\overline{B}$ is Baire, then $[B]$ is continuously included in some $(E_m, \tau_m)$ (by using (0)). Since $B$ is closed in $E$, it follows that $[B]$ is complete. (d) $\Rightarrow$ (e): Follows by using (0). (f) $\Rightarrow$ (e): Corollary 2.3. If $E$ satisfies M.c.c., then (e) $\Rightarrow$ (f) $\Rightarrow$ (a) hold. □

An $(LB)_{tv}$-space $(E, \tau)$ with defining sequence $(E_n, \tau_n)_n$ of quasi-Banach spaces will be called an $(LS)_{tv}$-space if for every $n \in \mathbb{N}$ there exists $m > n$ such that the inclusion $(E_n, \tau_n) \to (E_m, \tau_m)$ is compact. By [17], an $(LS)_{tv}$-space is a regular $B$-complete (hence complete) space; hence such a space is Montel (= barrelled, see [1] for definition) for which every bounded closed set is compact and sequentially retractive.

The following extends Proposition 8.5.36 of [3].

**Proposition 2.6.** Let $(E, \tau)$ be a tvs with an increasing sequence $(S_n)_n$ of balanced pseudo-convex bounded sets covering $E$. Then the following assertions are equivalent:

(i) $(E, \tau)$ is an $(LS)_{tv}$-space,
(ii) $(E, \tau)$ is Montel and satisfies M.c.c.

**Proof:** We have only to show (ii) $\Rightarrow$ (i). Since $(E, \tau)$ is barrelled, then $(A_n)_n$, where $A_n := \overline{S_n}$, $n \in \mathbb{N}$, is a fundamental sequence of $\tau$-bounded sets [1, 16 (6), (7)]. Since (by assumption) every $A_n$ is $\tau$-compact [1, 18 (8) and 18 (3)] apply to show that $(E, \tau)$ is a $B$-complete bornological $DF$-space. Let $(E, \vartheta)$ be the inductive limit space of quasi-Banach spaces $[A_n]$, $n \in \mathbb{N}$. Then $\tau \leq \vartheta$. Since $Bd(\tau) = Bd(\vartheta)$ and $(E, \tau)$ is bornological, then $\tau = \vartheta$, [1, 11 (3)]. For every $n \in \mathbb{N}$ there exists $m > n$ such that $A_n$ is compact in $[A_m]$. In fact, since every $A_n$ is $\tau$-compact, then (by [16]) every $A_n$ is metrizable in $\tau$. The assumption of Grothendieck’s lemma (cf. [7, p. 86]) are satisfied for $\mathcal{F} := \mathcal{F}(\gamma_k) \mid A_n, k > n, \mathcal{F} := \mathcal{F}(\tau)[A_n$, where $\gamma_k$ is the original topology of $[A_k]$. By Grothendieck’s
lemma [7, p. 86], there exists \( k > n \) such that \( F_k \) is weaker than \( F \); this applies to complete the proof.

\[ \square \]

References


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