

Wenge Hao; Salvatore Leonardi; Mark Steinhauer

Examples of discontinuous, divergence-free solutions to elliptic variational problems

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 36 (1995), No. 3, 511--517

Persistent URL: <http://dml.cz/dmlcz/118780>

## Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1995

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## Examples of discontinuous, divergence-free solutions to elliptic variational problems

WENGE HAO, SALVATORE LEONARDI<sup>1</sup>, MARK STEINHAEUER<sup>1</sup>

*Dedicated to Professor Jindřich Nečas on occasion of his sixty-fifth birthday*

*Abstract.* We give an example of a bounded discontinuous divergence-free solution of a linear elliptic system with measurable bounded coefficients in  $\mathbb{R}^3$  and a corresponding example for a **Stokes-like system**.

*Keywords:* divergence-free solutions, elliptic systems, systems of Stokes-type, regularity

*Classification:* 35B65, 35D10, 35J45, 35Q

### 1. Introduction

In this note, we give a simple example of a discontinuous bounded divergence-free weak solution to a three-dimensional linear elliptic system with measurable bounded coefficients of the form

$$(1.1) \quad -D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j) = 0, \quad i = 1, 2, 3$$

(the summation convention has been adopted throughout the paper). We will suppose that

$$(1.2) \quad A := (A_{ij}^{\alpha\beta}) \in L^\infty(B, \mathbb{R}^{81}), \quad B = \{x \in \mathbb{R}^3 : |x| < 1\},$$

$$(1.3) \quad u^j : B \rightarrow \mathbb{R}, \quad j = 1, 2, 3$$

and that there is a constant  $\lambda > 0$  such that

$$(1.4) \quad A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2$$

for every  $\xi \in \mathbb{R}^9$  and almost every  $x \in B$ .

By a (weak) solution to the system (1.1) we understand a function

$$(1.5) \quad u \in W_{\text{loc}}^{1,2}(B, \mathbb{R}^3) = \left\{ u : \int_K |\nabla u|^2 dx < +\infty \quad \forall K \subset\subset B \right\}$$

---

<sup>1</sup> The research of the second and the third author was carried out during their visit to Northern Illinois University at DeKalb. The third author was supported by the Sonderforschungsbereich 256 at the University of Bonn, Germany.

such that (1.1) holds in the sense of distributions.

On account of the famous example of De Giorgi [2],[5] it is well known that solutions of linear elliptic systems with coefficients in  $L^\infty$  are not Hölder-continuous in general.

Generalizing De Giorgi’s example J. Souček in [14] gave a construction of an elliptic system, which has a solution which is discontinuous on a dense countable set. This shows that “partial regularity” (see [5], [12]) does not hold for solutions of linear systems with coefficients, which are only in  $L^\infty$ . Using Souček’s “algorithm”, John, Malý and Stará in [7] constructed a linear elliptic system with  $L^\infty$ -coefficients and its bounded weak solution for any given  $F_\sigma$ -set  $F$  in  $\mathbb{R}^3$ , which is essentially discontinuous on  $F$  and essentially continuous on  $\mathbb{R}^3 \setminus F$ .

Another modification of De Giorgi’s example can be found in Leonardi [10] (see also [13]): this example shows that generally solutions of linear elliptic systems of the type (1.1) have no better integrability properties than the Sobolev-embedding in question tells us (compare also [3] for a nonlinear example). On the other hand the author presents estimates for the gradient of the solution to a system of type (1.1) in Morrey-spaces and weighted Morrey-spaces which are close to the counterexamples and he improves in some sense Koshelev’s condition number result (see [7], [8], [10], [13]).

Further regularity results in Morrey-spaces or  $L^p$ -spaces for solutions of elliptic systems with coefficients in  $L^\infty \cap VMO$  (the space of functions with vanishing mean oscillation) are established in [15]. These results are in some sense “intermediate” between Campanato’s result ([1]) for systems with continuous coefficients and the counterexamples because of the “embedding  $C^0 \hookrightarrow L^\infty \cap VMO$ ” and the fact that the coefficients in De Giorgi’s counterexample have the property that their gradient belongs to the space weak- $L^n$ . As a consequence of Poincaré’s inequality we have that  $W^{1,n}$ -functions have locally vanishing mean oscillation.

In this situation a regularity result of De Giorgi-Moser-Nash-type for solutions of linear elliptic systems with  $L^\infty$  — coefficients can be expected only if one finds new additional structure conditions for such systems and/or their solutions. For example one can ask whether weak solutions to our elliptic system which satisfy in addition an “incompressibility condition”

$$\operatorname{div} u = D_i u^i = 0$$

are regular, or, more general, whether all solutions of linear elliptic systems with  $L^\infty$ -coefficients are regular, if the mean flux of the given boundary function through the boundary of the given domain is zero (see [13,14]). Our example shows that the answer to these two questions must be negative.

Another still open question (at least to the authors’ knowledge) is, whether minima of “isotropic” quadratic functionals of the type

$$J(u) := \frac{1}{2} \int_B W(Du) dx = \frac{1}{2} \int_B A_{ij}^{\alpha\beta}(x) D_\beta u^j D_\alpha u^i dx, \quad u \in W^{1,2}(B, \mathbb{R}^3)$$

are regular (see again [13]).

**Acknowledgements:** The authors would like to thank Professor J. Nečas for bringing this problem to their attention and for stimulating discussions and inspiration in connection with this work.

## 2. The examples

For our examples we use Souček’s algorithm to construct for a given  $W^{1,2}$ -function an elliptic system to which it is the solution. To construct a divergence-free  $W^{1,2}$ -function in the unit ball  $B$  we call to mind that such a function must be a curl because of Poincaré’s Lemma for star-convex open sets (see for example [11]).

Our divergence-free function in  $B$  is

$$\begin{aligned}
 (2.1) \quad u(x) &:= \operatorname{curl} \begin{pmatrix} |x| \\ |x| \\ |x| \end{pmatrix} = \frac{1}{|x|} \begin{pmatrix} x_2 & - & x_3 \\ x_3 & - & x_1 \\ x_1 & - & x_2 \end{pmatrix} \\
 &= \frac{1}{|x|} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =: \frac{1}{|x|} T x,
 \end{aligned}$$

which is an element of  $L^\infty \cap W^{1,2}(B, \mathbb{R}^3)$ , but is not continuous at  $x = 0$  (in fact we have  $Du \in L^p(B, \mathbb{R}^{3 \times 3})$  for every  $p < 3$ ). Notice that the mean flux of  $u$  through  $\partial B$  is zero.

Defining

$$(2.2) \quad b_\alpha^i(x) := \frac{F}{|x|} \left\{ t_\alpha^i + \frac{x_\alpha t_\alpha^i x_l}{|x|^2} \right\}$$

for some positive constant  $F$  and the matrix  $T = (t_\alpha^i)$  from (2.1), one verifies

$$(2.3) \quad b_\alpha^i \in L^p(B) \quad \forall p < 3,$$

and

$$(2.4) \quad D_\alpha b_\alpha^i = 0 \quad \text{for } x \neq 0.$$

Setting

$$\begin{aligned}
 (2.5) \quad d_\alpha^i &:= b_\alpha^i - D_\alpha u^i \\
 &= \frac{1}{|x|} \left\{ (F - 1)t_\alpha^i + (F + 1) \frac{x_\alpha t_\alpha^i x_l}{|x|^2} \right\},
 \end{aligned}$$

elementary calculations show that for  $F > 1$

$$(2.6) \quad d_\alpha^i D_\alpha u^i \geq \frac{1}{|x|^2} (2F - 2) > 0,$$

$$(2.7) \quad \frac{b_\alpha^i d_\alpha^i}{d_\alpha^i D_\alpha u^i} \leq \frac{18F^2 - 2F}{2F - 2} =: M < +\infty.$$

Therefore we can apply Lemma 1 of Souček’s paper [14] to conclude that our function  $u$  is the solution of the elliptic system

$$(2.8) \quad \int_B A_{ij}^{\alpha\beta}(x) D_\beta u^j D_\alpha \varphi^i dx = 0 \quad \forall \varphi \in C_0^\infty(B, \mathbb{R}^3),$$

where the coefficients are defined by

$$(2.9) \quad A_{ij}^{\alpha\beta}(x) := \delta_{ij} \delta^{\alpha\beta} + \frac{d_\alpha^i d_\beta^j}{D_\rho u^s d_\rho^s}, \quad \alpha, \beta, i, j = 1, 2, 3$$

( $\delta_{ij}, \delta^{\alpha,\beta}, \delta_\alpha^i$  are always Kronecker symbols).

We remark that one can construct corresponding functions  $u$  in higher dimensions using the calculus of differential forms (replacing the special cross product of  $\mathbb{R}^3$ ).

Furthermore a slight modification of Souček’s above quoted Lemma 1 enables us to construct for given functions  $(u, p)$  an elliptic system of Stokes-type

$$\begin{aligned} -D_\alpha (A_{ij}^{\alpha\beta} D_\beta u^j) + D_i p &= 0, \\ \operatorname{div} u &= 0, \end{aligned}$$

such that  $(u, p)$  is the solution of this system (for further details on these systems see [4], [6], [9]).

**Lemma.** *Let  $b = (b_\alpha^i), \alpha, i = 1, \dots, n$ , be a matrix of  $L^2$ -functions such that*

$$(2.10) \quad \int_B b_\alpha^i D_\alpha \varphi^i dx = 0 \quad \forall \varphi \in C_0^\infty(B, \mathbb{R}^n).$$

*Further assume that  $p \in L^2 \cap W^{1,1}(B)$  and define the matrix  $c$  as*

$$(2.11) \quad c_\alpha^i := b_\alpha^i + p \delta_\alpha^i,$$

*and the matrix  $d$  as*

$$(2.12) \quad d_\alpha^i := c_\alpha^i - D_\alpha u^i, \quad u \in W^{1,2}(B, \mathbb{R}^n).$$

Moreover we suppose, that

$$(2.13) \quad d_\alpha^i D_\alpha u^i > 0$$

and that there is a positive constant  $M$ , such that

$$(2.14) \quad \frac{c_\alpha^i d_\alpha^i}{D_\alpha u^i d_\alpha^i} \leq M .$$

If  $u$  is divergence-free, it is a solution to the elliptic system of Stokes-type:

$$(2.15) \quad \int_B A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \varphi^i \, dx = 0 \quad \forall \varphi \in C_0^\infty(B, \mathbb{R}^n), \quad \operatorname{div} \varphi = 0, \\ \operatorname{div} u = 0,$$

where the coefficients are given once more by (2.9).

PROOF: This is done by easy calculations. The ellipticity corresponds to condition (2.13) and the boundedness to (2.14).  $\square$

Using this lemma we get an example of a singular solution to a system of Stokes-type in  $\mathbb{R}^3$ . We take  $u$  as in (2.1) and define  $p$  by

$$(2.17) \quad p(x) := \frac{1}{|x|} \in L^r \cap W^{1,s}(B) \quad \forall r < 3, s < \frac{3}{2} .$$

Defining  $b$  as before, i.e.

$$(2.18) \quad b_\alpha^i := \frac{1}{|x|} \left\{ Ft_\alpha^i + F \frac{x_\alpha t_l^i x_l}{|x|^2} \right\}$$

we get

$$(2.19) \quad c_\alpha^i := \frac{1}{|x|} \left\{ Ft_\alpha^i + F \frac{x_\alpha t_l^i x_l}{|x|^2} + \delta_\alpha^i \right\}$$

and

$$(2.20) \quad d_\alpha^i := \frac{1}{|x|} \left\{ (F - 1)t_\alpha^i + (F + 1) \frac{x_\alpha t_l^i x_l}{|x|^2} + \delta_\alpha^i \right\} .$$

Using these definitions and observing  $t_\alpha^i \delta_\alpha^i = 0$  and  $x_i t_l^i x_l = 0$  we calculate

$$(2.21) \quad d_\alpha^i D_\alpha u^i = \frac{1}{|x|^2} \left\{ 6(F - 1) - (F - 1) \frac{(t_l^i x_l)^2}{|x|^2} \right\} \\ \geq \frac{1}{|x|^2} \{2F - 2\} > 0 ,$$

provided  $F > 1$ , and

$$(2.22) \quad \begin{aligned} b_\alpha^i d_\alpha^i &= \frac{1}{|x|^2} \left\{ 6F(F-1) + [F(F-1) + 2F(F+1)] \frac{(t_t^i x_l)^2}{|x|^2} + 3 \right\} \\ &\leq \frac{1}{|x|^2} \{18F^2 - 2F + 3\}. \end{aligned}$$

So we finally arrive at

$$(2.23) \quad \frac{b_\alpha^i d_\alpha^i}{d_\alpha^i D_\alpha u^i} \leq \frac{18F^2 - 2F + 3}{2F - 2} =: M < +\infty$$

and in view of the variant of Souček's lemma we have proved that

$$(2.24) \quad (u, p) = \frac{1}{|x|} (Tx, 1)$$

is a singular solution to a system of Stokes-type, where  $A_{ij}^{\alpha\beta}$  are defined as in (2.9).

#### REFERENCES

- [1] Campanato S., *Sistemi ellittici in forma divergenza. Regolarità all'interno*, Quaderno Sc. Norm. Sup. Pisa, Pisa, 1980.
- [2] De Giorgi E., *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. U.M.I. **4** (1968), 135–137.
- [3] Frehse J., *Una generalizzazione di un controesempio di De Giorgi nella teoria delle equazioni ellittiche*, Boll. U.M.I. **6** (1970), 998–1002.
- [4] Galdi G.P., *An introduction to the mathematical theory of the Navier-Stokes equations, Volume I: Linearized steady problems*, Springer tracts in natural philosophy vol. 38, Springer-Verlag, New York Berlin Heidelberg et alii, 1994.
- [5] Giaquinta M., *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies, vol. 105, Princeton University Press, Princeton, 1983.
- [6] Giaquinta M., Modica G., *Non linear systems of the type of the Navier-Stokes system*, J. Reine Angewandte Math. **330** (1982), 173–214.
- [7] John O., Malý J., Stará J., *Nowhere continuous solutions to elliptic systems*, Comment. Math. Univ. Carolinae **30** (1989), 33–43.
- [8] Koshelev A.I., Chelkak S.I., *Regularity of solutions of quasilinear elliptic systems*, Teubner, Leipzig, 1985.
- [9] Ladyzhenskaya O. A., *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1969.
- [10] Leonardi S., *On constants of some regularity theorems. De Giorgi counterexamples*, Preprint N.I.U (1993) [Thesis (1995-96)], to appear.
- [11] Munkres J.R., *Analysis on manifolds*, Addison-Wesley, Redwood City, 1991.
- [12] Nečas J., *Introduction to the theory of nonlinear elliptic equations*, Teubner, Leipzig, 1983, or J. Wiley, Chichester, 1986.
- [13] ———, *Notes for a talk given at the Equadiff 8 in Bratislava 1993*, (personal communication).

- [14] Souček J., *Singular solutions to linear elliptic systems*, Comment. Math. Univ. Carolinae **25** (1984), 273–281.
- [15] Steinhauer M., *Funktionenräume vom Campanato-Typ und Regularitätseigenschaften elliptischer Systeme zweiter Ordnung*, Diplomarbeit (1993), Univ. Bonn.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB,  
ILLINOIS 60115, USA

*E-mail:* hao@math.niu.edu

DIPARTIMENTO DI MATEMATICA, CITTA' UNIVERSITARIA, VIALE A. DORIA 6, 95125 CATANIA,  
ITALY

*E-mail:* leonardi@dipmat.unict.it

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN, BERINGSTRASSE 4-6,  
D-53115 BONN GERMANY

*E-mail:* mark@fraise.iam.uni-bonn.de

(Received March 27, 1995)