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Some commutative neutrix convolution products of functions

BRIAN FISHER, ADEM KILIÇMAN

Abstract. The commutative neutrix convolution product of the locally summable functions \( \cos(-\lambda x) \) and \( \cos(\mu x) \) is evaluated. Further similar commutative neutrix convolution products are evaluated and deduced.

Keywords: neutrix, neutrix limit, neutrix convolution product

Classification: 46F10

In the following we let \( \mathcal{D} \) be the space of infinitely differentiable functions with compact support and let \( \mathcal{D}' \) be the space of distributions defined on \( \mathcal{D} \). The convolution product \( f * g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}' \) is then usually defined by the equation

\[
\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle
\]

for arbitrary \( \phi \) in \( \mathcal{D} \), provided \( f \) and \( g \) satisfy either of the conditions

(a) either \( f \) or \( g \) has bounded support,

(b) the supports of \( f \) and \( g \) are bounded on the same side,

see Gel'fand and Shilov [5].

Note that if \( f \) and \( g \) are locally summable functions satisfying either of the above conditions then

\[
(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt = \int_{-\infty}^{\infty} f(x - t)g(t) \, dt.
\]

It follows that if the convolution product \( f * g \) exists by this definition then

\[
(f * g)' = f' * g = f * g'.
\]

This definition of the convolution product is rather restrictive and can only be used for a small class of distributions. In order to extend the convolution product to a larger class of distributions, the commutative neutrix convolution product was introduced in [3] and was extended in [2]. In the following, we give a further
generalization. We first of all let \( \tau \) be a function in \( \mathcal{D} \) satisfying the following conditions:

(i) \( \tau(x) = \tau(-x) \),
(ii) \( 0 \leq \tau(x) \leq 1 \),
(iii) \( \tau(x) = 1 \) for \( |x| \leq \frac{1}{2} \),
(iv) \( \tau(x) = 0 \) for \( |x| \geq 1 \).

The function \( \tau_\nu \) is now defined by

\[
\tau_\nu(x) = \begin{cases} 
1, & |x| \leq \nu, \\
\tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\
\tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu,
\end{cases}
\]

for \( \nu > 0 \).

We now define the extended neutrix convolution product.

**Definition 1.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_\nu(x) = f(x)\tau_\nu(x) \) and \( g_\nu(x) = g(x)\tau_\nu(x) \) for \( \nu > 0 \). Then the neutrix convolution product \( f \ast g \) is defined as the neutrix limit of the sequence \( \{f_\nu \ast g_\nu\} \), provided that the limit \( h \) exists in the sense that

\[
N_{\nu \to \infty} \lim_{\nu \to \infty} \langle f_\nu \ast g_\nu, \phi \rangle = \langle h, \phi \rangle,
\]

for all \( \phi \) in \( \mathcal{D} \), where \( N \) is the neutrix, see van der Corput [1], having domain \( N' \) the positive reals and range \( N'' \) the real numbers, with negligible functions finite linear sums of the functions

\[
\nu^\lambda \ln^{r-1} \nu, \ln^r \nu, \nu^\mu e^{\lambda \nu}, \nu^\mu \cos \lambda \nu, \nu^\mu \sin \lambda \nu \quad (\lambda \neq 0, \ r = 1, 2, \ldots)
\]

and all functions which converge to zero in the usual sense as \( \nu \) tends to infinity.

Note that in this definition the convolution product \( f_\nu \ast g_\nu \) is defined in Gel’fand and Shilov’s sense, the distribution \( f_\nu \) and \( g_\nu \) having bounded support. It is clear that if the neutrix convolution product \( f \ast g \) exists then the neutrix convolution product \( g \ast f \) exists and \( f \ast g = g \ast f \).

In the original definition of the neutrix convolution product, the domain of the neutrix \( N \) was the set of positive integers \( N' = \{1, 2, \ldots, n, \ldots\} \) and the negligible functions were finite linear sums of the functions

\[
n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, \ r = 1, 2, \ldots)
\]

and all functions which converge to zero in the usual sense as \( n \) tends to infinity. In [2], the set of negligible functions was extended to include finite linear sums of the functions

\[
n^\lambda e^{\mu n} \quad (\mu > 0).
\]

It is easily seen that any results proved with the original definition hold with the new definition. The following theorem proved in [3] therefore holds.
Theorem 1. Let \( f \) and \( g \) be distributions in \( D' \) satisfying either condition (a) or condition (b) of Gel’fand and Shilov’s definition. Then the neutrix convolution product \( f \ast g \) exists and

\[
f \ast g = f \ast g.
\]

A number of neutrix convolution products have been evaluated. For example, \( x^\lambda \ast x^\mu_+ \) see [3], \( \ln x^- \ast \ln x^+ \) see [6] and \( \ln x^- \ast x^\nu_- \) see [4].

We now define the locally summable functions \( e^\lambda_+ \), \( e^\mu_- \), \( \cos_+ (\lambda x) \), \( \cos_- (\lambda x) \), \( \sin_+ (\lambda x) \) and \( \sin_- (\lambda x) \) by

\[
e^\lambda_+ = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \quad e^\mu_- = \begin{cases} 0, & x > 0, \\ e^{\mu x}, & x < 0, \end{cases}
\]

\[
\cos_+ (\lambda x) = \begin{cases} \cos(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \quad \cos_- (\lambda x) = \begin{cases} 0, & x > 0, \\ \cos(\lambda x), & x < 0, \end{cases}
\]

\[
\sin_+ (\lambda x) = \begin{cases} \sin(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \quad \sin_- (\lambda x) = \begin{cases} 0, & x > 0, \\ \sin(\lambda x), & x < 0. \end{cases}
\]

It follows that

\[
\cos_- (\lambda x) + \cos_+ (\lambda x) = \cos(\lambda x), \quad \sin_- (\lambda x) + \sin_+ (\lambda x) = \sin(\lambda x).
\]

The following theorem was proved in [2]

Theorem 2. The neutrix convolution product \( (x^r e^\lambda_-) \ast (x^s e^\mu_+) \) exists and

\[
(x^r e^\lambda_-) \ast (x^s e^\mu_+) = D^r_\lambda D^s_\mu \frac{e^\mu_+ + e^{-\lambda_-}}{\lambda - \mu},
\]

where \( D_\lambda = \partial / \partial \lambda \) and \( D_\mu = \partial / \partial \mu \), for \( \lambda \neq \mu \) and \( r, s = 0, 1, 2, \ldots \), these neutrix convolution products existing as convolution products if \( \lambda > \mu \) and

\[
(x^r e^\lambda_-) \ast (x^s e^\mu_+) = \omega (r + 1, s + 1) x^{r+s+1} \operatorname{sgn} x \cdot e^{\lambda x},
\]

for all \( \lambda \) and \( r, s = 0, 1, 2, \ldots \), where \( \omega \) denotes the Beta function.

We now prove the following theorem.

Theorem 3. The neutrix convolution products \( \cos_- (\lambda x) \ast \cos_+ (\mu x) \), \( \cos_- (\lambda x) \ast \sin_+ (\mu x) \), \( \sin_- (\lambda x) \ast \cos_+ (\mu x) \) and \( \sin_- (\lambda x) \ast \sin_+ (\mu x) \) exist and

\[
\cos_- (\lambda x) \ast \cos_+ (\mu x) = \frac{\lambda \sin_- (\lambda x) + \mu \sin_+ (\mu x)}{\lambda^2 - \mu^2},
\]

\[
\cos_- (\lambda x) \ast \sin_+ (\mu x) = \frac{-\mu \cos_- (\lambda x) + \mu \cos_+ (\mu x)}{\lambda^2 - \mu^2},
\]

\[
\sin_- (\lambda x) \ast \cos_+ (\mu x) = \frac{-\lambda \cos_- (\lambda x) + \lambda \cos_+ (\mu x)}{\lambda^2 - \mu^2},
\]

\[
\sin_- (\lambda x) \ast \sin_+ (\mu x) = \frac{-\mu \sin_- (\lambda x) + \lambda \sin_+ (\mu x)}{\lambda^2 - \mu^2},
\]
for $\lambda \neq \pm \mu$.

**Proof:** We first of all note that since

$$\sin(\lambda x + \mu \nu) = \sin(\lambda x) \cos(\mu \nu) + \cos(\lambda x) \sin(\mu \nu),$$

it follows that

$$N - \lim_{\nu \to \infty} \sin(\lambda x + \mu \nu) = N - \lim_{\nu \to \infty} \nu \sin(\lambda x + \mu \nu) = 0$$

for $\mu \neq 0$. Similarly

$$N - \lim_{\nu \to \infty} \cos(\lambda x + \mu \nu) = N - \lim_{\nu \to \infty} \nu \cos(\lambda x + \mu \nu) = 0$$

for $\mu \neq 0$.

We now put $\[\cos(\lambda t)\]_\nu = \cos(\lambda t) \tau_{\nu}(t)$ and $\[\cos(\mu x)\]_\nu = \cos(\mu x) \tau_{\nu}(x)$. Since $\[\cos(\lambda x)\]_\nu$ and $\[\cos(\mu x)\]_\nu$ are locally summable functions with $\[\cos(\lambda x)\]_\nu$ and $\[\cos(\mu x)\]_\nu$ having compact support, the convolution product $\[\cos(\lambda x)\]_\nu \ast \[\cos(\mu x)\]_\nu$ is defined by equation (1) and so

$$\[\cos(\lambda x)\]_\nu \ast \[\cos(\mu x)\]_\nu = \int_{-\infty}^{\infty} \[\cos(\lambda t)\]_\nu \[\cos(\mu x - t)\]_\nu dt.$$

When $-\nu \leq x \leq 0$,

$$\int_{-\infty}^{\infty} \[\cos(\lambda t)\]_\nu \[\cos(\mu x - t)\]_\nu dt = \int_{-\nu}^{x} \cos(\lambda t) \cos(\mu x - t) dt +$$

$$+ \int_{-\nu}^{0} \cos(\lambda t) \cos(\mu x - t) \tau_{\nu}(t) \tau_{\nu}(x - t) dt$$

$$= \frac{\sin(\lambda x) - \sin(\mu x - (\lambda - \mu)\nu)}{2(\lambda - \mu)} +$$

$$+ \frac{\sin(\lambda x) - \sin(\mu x + (\lambda + \mu)\nu)}{2(\lambda + \mu)} + O(\nu^{-\nu})$$

and it follows that

$$N - \lim_{\nu \to \infty} \int_{-\infty}^{\infty} \[\cos(\lambda t)\]_\nu \[\cos(\mu x - t)\]_\nu dt = \frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2},$$

on using equation (8).

When $\nu \geq x \geq 0$,

$$\int_{-\infty}^{\infty} \[\cos(\lambda t)\]_\nu \[\cos(\mu x - t)\]_\nu dt = \int_{x - \nu}^{0} \cos(\lambda t) \cos(\mu x - t) dt +$$

$$+ \int_{x - \nu}^{x - \nu} \cos(\lambda t) \cos(\mu x - t) \tau_{\nu}(t) \tau_{\nu}(x - t) dt$$

$$= \frac{\sin(\mu x) - \sin(\mu x + (\lambda - \mu)(x - \nu))}{2(\lambda - \mu)} +$$

$$- \frac{\sin(\mu x) - \sin(\mu x - (\lambda + \mu)(x - \nu))}{2(\lambda + \mu)} + O(\nu^{-\nu})$$
and it follows that
\begin{equation}
(12) \quad N - \lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_{\nu} [\cos_+(\mu(x - t))]_{\nu} \, dt = \frac{\mu \sin(\mu x)}{\lambda^2 - \mu^2},
\end{equation}
on using equation (8).

It now follows from equations (10), (11) and (12) that for arbitrary $\phi$ in $D$
\begin{align*}
N - \lim_{\nu \to \infty} \langle [\cos_-(\lambda x)]_{\nu} * [\cos_+(\mu x)]_{\nu}, \phi(x) \rangle &= \frac{\lambda}{\lambda^2 - \mu^2} \langle \sin_-(\lambda x), \phi(x) \rangle + \\
&\quad + \frac{\mu}{\lambda^2 - \mu^2} \langle \sin_+(\mu x), \phi(x) \rangle
\end{align*}
and equation (4) follows.

We now prove equation (5). Putting $[\sin_+(\mu x)]_{\nu} = \sin_+(\mu x)\tau_{\nu}(x)$, we have as above
\begin{equation}
(13) \quad [\cos_-(\lambda x)]_{\nu} * [\sin_+(\mu x)]_{\nu} = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_{\nu} [\sin_+(\mu(x - t))]_{\nu} \, dt.
\end{equation}

When $-\nu \leq x \leq 0$,
\begin{align*}
\int_{-\infty}^{x} [\cos_-(\lambda t)]_{\nu} [\sin_+(\mu(x - t))]_{\nu} \, dt &= \int_{-\nu}^{x} \cos(\lambda t) \sin[\mu(x - t)] \, dt + \\
&\quad + \int_{-\nu}^{x-x^\nu} \cos(\lambda t) \sin[\mu(x - t)] \tau_{\nu}(t) \tau_{\nu}(x - t) \, dt \\
&= - \frac{\cos(\lambda x) - \cos[\mu x - (\lambda - \mu)\nu]}{2(\lambda - \mu)} + \\
&\quad + \frac{\cos(\lambda x) - \cos[\mu x + (\lambda + \mu)\nu]}{2(\lambda + \mu)} + O(\nu^{-\nu}),
\end{align*}
and it follows that
\begin{equation}
(14) \quad N - \lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_{\nu} [\sin_+(\mu(x - t))]_{\nu} \, dt = - \frac{\mu \cos(\lambda x)}{\lambda^2 - \mu^2},
\end{equation}
on using equations (9).

When $\nu \geq x \geq 0$,
\begin{align*}
\int_{-\infty}^{x} [\cos_-(\lambda t)]_{\nu} [\sin_+(\mu(x - t))]_{\nu} \, dt &= \int_{x-\nu}^{0} \cos(\lambda t) \sin[\mu(x - t)] \, dt + \\
&\quad + \int_{x-\nu}^{x-x^\nu} \cos(\lambda t) \sin[\mu(x - t)] \tau_{\nu}(t) \tau_{\nu}(x - t) \, dt \\
&= - \frac{\cos(\mu x) - \cos[\mu x + (\lambda - \mu)(x - \nu)]}{2(\lambda - \mu)} + \\
&\quad + \frac{\cos(\mu x) - \cos[\mu x - (\lambda + \mu)(x - \nu)]}{2(\lambda + \mu)} + O(\nu^{-\nu}),
\end{align*}
and it follows that

\[ N - \lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos_{-}(\lambda t)]_{\nu} [\sin_{+}(\mu(x - t))]_{\nu} \, dt = -\frac{\mu \cos(\lambda x)}{\lambda^2 - \mu^2}, \]

on using equations (9).

Equation (5) now follows as above on using equations (13), (14) and (15).

Replacing \( x \) by \(-x\) in equation (5) and interchanging \( \lambda \) and \( \mu \) we get

\[-\cos_{+} (\mu x) \sin_{-} (\lambda x) = -\frac{\lambda \cos_{+} (\mu x) + \lambda \cos_{-} (\lambda x)}{\mu^2 - \lambda^2}.\]

Equation (6) now follows since the convolution is commutative.

We finally prove equation (7). Putting \([\sin_{-} (\lambda x)]_{\nu} = \sin_{-} (\lambda x) \tau_{\nu} (x)\), we have

\[ [\sin_{-} (\lambda x)]_{\nu} * [\sin_{+} (\mu x)]_{\nu} = \int_{-\infty}^{\infty} [\sin_{-} (\lambda t)]_{\nu} [\sin_{+} (\mu(x - t))]_{\nu} \, dt. \]

When \(-\nu \leq x \leq 0\),

\[
\int_{-\infty}^{\infty} [\sin_{-} (\lambda t)]_{\nu} [\sin_{+} (\mu(x - t))]_{\nu} \, dt = \int_{-\nu}^{0} \sin (\lambda t) \sin [\mu(x - t)] \, dt + \\
+ \int_{-\nu}^{-\nu} \sin (\lambda t) \sin [\mu(x - t)] \tau_{\nu} (t) \tau_{\nu} (x - t) \, dt \\
= \frac{\sin (\lambda x) + \sin [\mu x + (\lambda + \mu) \nu]}{2(\lambda + \mu)} + \\
- \frac{\sin (\lambda x) - \sin [\mu x - (\lambda - \mu) \nu]}{2(\lambda - \mu)} + O(\nu^{-\nu})
\]

and it follows that

\[ N - \lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\sin_{-} (\lambda t)]_{\nu} [\sin_{+} (\mu(x - t))]_{\nu} \, dt = -\frac{\mu \sin (\lambda x)}{\lambda^2 - \mu^2}, \]

on using equations (9).

When \( \nu \geq x \geq 0 \),

\[
\int_{-\infty}^{\infty} [\sin_{-} (\lambda t)]_{\nu} [\sin_{+} (\mu(x - t))]_{\nu} \, dt = \int_{x - \nu}^{0} \sin (\lambda t) \sin [\mu(x - t)] \, dt + \\
+ \int_{x - \nu}^{x - \nu} \sin (\lambda t) \sin [\mu(x - t)] \tau_{\nu} (t) \tau_{\nu} (x - t) \, dt \\
= -\frac{\sin (\mu x) - \sin [\mu x - (\lambda + \mu) (x - \nu)]}{2(\lambda + \mu)} + \\
- \frac{\sin (\mu x) - \sin [\mu x + (\lambda - \mu) (x - \nu)]}{2(\lambda - \mu)} + O(\nu^{-\nu}),
\]
and it follows that

\[
N-\lim_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_{\nu} [\sin_+((x-t))]_{\nu} dt = -\frac{\lambda \sin(\mu x)}{\lambda^2 - \mu^2}
\]

on using equations (9).

Equation (7) now follows as above on using equations (16), (17) and (18).

**Corollary.** The neutrix convolution products 
\([1 - H(x)] \textcircled{âˆš} \cos_+(\mu x), \cos_-(\lambda x) \textcircled{âˆš} H(x), [1 - H(x)] \textcircled{âˆš} \sin_+(\mu x) \text{ and } \sin_-(\lambda x) \textcircled{âˆš} H(x)\) exist and

\[
\begin{align*}
(19) \quad [1 - H(x)] \textcircled{âˆš} \cos_+(\mu x) &= -\mu^{-1} \sin_+(\mu x), \\
(20) \quad \cos_-(\lambda x) \textcircled{âˆš} H(x) &= \lambda^{-1} \sin_-(\lambda x), \\
(21) \quad [1 - H(x)] \textcircled{âˆš} \sin_+(\mu x) &= \mu^{-1}[1 - H(x) + \cos_+(\mu x)], \\
(22) \quad \sin_-(\lambda x) \textcircled{âˆš} H(x) &= -\lambda^{-1}[H(x) + \cos_-(\lambda x)],
\end{align*}
\]

for \(\lambda, \mu \neq 0\), where \(H\) denotes Heaviside's function.

**Proof:** Equations (19) and (20) follow from equations (4) and (5) respectively on putting \(\lambda = 0\) and equations (20) and (21) follow from equations (4) and (6) respectively on putting \(\mu = 0\). \(\square\)

Further results can be easily deduced. For example, it is easily proved that

\[\cos_+(\lambda x) \ast \cos_+(\mu x) = \frac{\lambda \sin_+(\lambda x) - \mu \sin_+(\mu x)}{\lambda^2 - \mu^2},\]

for \(\lambda \neq \pm \mu\), and it follows that

\[\cos(\lambda x) \textcircled{âˆš} \cos_+(\mu x) = \cos_-(\lambda x) \textcircled{âˆš} \cos_+(\mu x) + \cos_+(\lambda x) \ast \cos_+(\mu x) = \frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}.
\]

Replacing \(x\) by \(-x\) in this equation we get

\[\cos(\lambda x) \textcircled{âˆš} \cos_-(\mu x) = -\frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}
\]

and so

\[\cos(\lambda x) \textcircled{âˆš} \cos(\mu x) = \cos(\lambda x) \textcircled{âˆš} \cos_-(\mu x) + \cos(\lambda x) \textcircled{âˆš} \cos_+(\mu x) = 0.
\]
Theorem 4. The neutrix convolution products $\cos-(\lambda x) \boxtimes \cos+(\lambda x)$, $\sin-(\lambda x) \boxtimes \cos+(\lambda x)$ and $\sin-(\lambda x) \boxtimes \sin+(\lambda x)$ exist and

\begin{align}
\cos-(\lambda x) \boxtimes \cos+(\lambda x) &= \frac{2\lambda x[\cos-(\lambda x) - \cos+(\lambda x)] + \sin-(\lambda x) - \sin+(\lambda x)}{4\lambda}, \\
\cos-(\lambda x) \boxtimes \sin+(\lambda x) &= \frac{2\lambda x[\sin-(\lambda x) - \sin+(\lambda x)] + \cos(\lambda x)}{4\lambda}, \\
\sin-(\lambda x) \boxtimes \cos+(\lambda x) &= -\frac{2\lambda x[\sin+(\lambda x) - \sin-(\lambda x)] + \cos(\lambda x)}{4\lambda}, \\
\sin-(\lambda x) \boxtimes \sin+(\lambda x) &= \frac{2\lambda x[\cos+(\lambda x) - \cos-(\lambda x)] + \sin-(\lambda x) - \sin+(\lambda x)}{4\lambda},
\end{align}

for $\lambda \neq 0$.

Proof: We have

\begin{equation}
[\cos-(\lambda x)]_{\nu} \ast [\cos+(\lambda x)]_{\nu} = \int_{-\infty}^{\infty} [\cos-(\lambda t)]_{\nu}[\cos+(\lambda(x - t))]_{\nu} dt.
\end{equation}

When $-\nu \leq x \leq 0$,

\begin{align}
\int_{-\infty}^{\infty} [\cos-(\lambda t)]_{\nu}[\cos+(\lambda(x - t))]_{\nu} dt &= \int_{-\nu}^{x} \cos(\lambda t) \cos[\lambda(x - t)] dt + \int_{-\nu}^{-\nu} \cos(\lambda t) \cos[\lambda(x - t)] \tau_{\nu}(t) \tau_{\nu}(x - t) dt \\
&= \frac{(x + \nu) \cos(\lambda x)}{2} + \frac{\sin(\lambda x) + \sin(\lambda x + 2\lambda \nu)}{4\lambda} + O(\nu^{-\nu})
\end{align}

and it follows that

\begin{equation}
\lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos-(\lambda t)]_{\nu}[\cos+(\lambda(x - t))]_{\nu} dt = \frac{2\lambda x \cos(\lambda x) + \sin(\lambda x)}{4\lambda},
\end{equation}

on using equation (8).

When $\nu \geq x \geq 0$,

\begin{align}
\int_{-\infty}^{\infty} [\cos-(\lambda t)]_{\nu}[\cos+(\lambda(x - t))]_{\nu} dt &= \int_{x-\nu}^{0} \cos(\lambda t) \cos[\lambda(x - t)] dt + \int_{x-\nu}^{x-\nu} \cos(\lambda t) \cos[\lambda(x - t)] \tau_{\nu}(t) \tau_{\nu}(x - t) dt \\
&= \frac{(x - \nu) \cos(\lambda x)}{2} - \frac{\sin(\lambda x) + \sin(\lambda x - 2\lambda \nu)}{4\lambda} + O(\nu^{-\nu})
\end{align}
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and it follows that

\[
\lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos(\lambda t)]_{\nu} [\cos(\lambda(x-t))]_{\nu} \, dt = -\frac{2\lambda x \cos(\lambda x) + \sin(\lambda x)}{4\lambda},
\]

on using equation (8).

Equation (23) now follows as above on using equations (27), (28) and (29).

We now prove equation (24). We have as above

\[
\lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos(\lambda x)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt.
\]

When \(-\nu \leq x \leq 0,

\[
\int_{-\infty}^{\infty} [\cos(\lambda t)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt = \int_{-\nu}^{x} [\cos(\lambda t)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt + \int_{-\nu}^{-\nu} [\cos(\lambda t)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt + \int_{x}^{\infty} [\cos(\lambda t)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt
\]

and it follows that

\[
\lim_{\nu \to \infty} \int_{-\infty}^{\infty} [\cos(\lambda x)]_{\nu} [\sin(\lambda(x-t))]_{\nu} \, dt = \frac{2\lambda x \sin(\lambda x) + \cos(\lambda x)}{4\lambda},
\]

on using equations (9).

Replacing \(x\) by \(-x\) in equation (24) we get

\[
-\cos_+(\lambda x) \mathbb{N} \sin_-(\lambda x) = \frac{2\lambda x [\sin_+(\lambda x) - \sin_-(\lambda x)] + \cos(\lambda x)}{4\lambda}
\]
and equation (25) follows since the convolution is commutative.

We finally prove equation (26). We have

\[
[\sin^{-}(\lambda x)]_{\nu} \ast [\sin^{+}(\lambda x)]_{\nu} = \int_{-\infty}^{\infty} [\sin^{-}(\lambda t)]_{\nu} [\sin^{+}(\lambda (x-t))]_{\nu} dt.
\]

When \(-\nu \leq x \leq 0,\)

\[
\int_{-\infty}^{\infty} [\sin^{-}(\lambda t)]_{\nu} [\sin^{+}(\lambda (x-t))]_{\nu} dt = \int_{-\nu}^{x} \sin(\lambda t) \sin(\lambda (x-t)) dt + \int_{-\nu}^{-\nu} \sin(\lambda t) \sin[\lambda (x-t)] \tau_{\nu}(t) \tau_{\nu}(x-t) dt = \frac{\sin(\lambda x) + \sin(\lambda x + 2\nu x)}{4\lambda} - \frac{(x-\nu) \cos(\lambda x)}{2} + O(\nu^{-\nu})
\]

and it follows that

\[
N_{\nu \to \infty} \int_{-\infty}^{\infty} [\sin^{-}(\lambda t)]_{\nu} [\sin^{+}(\lambda (x-t))]_{\nu} dt = \frac{\sin(\lambda x) - 2\lambda x \cos(\lambda x)}{4\lambda},
\]

on using equation (8).

When \(\nu \geq x \geq 0,\)

\[
\int_{-\infty}^{\infty} [\sin^{-}(\lambda t)]_{\nu} [\sin^{+}(\lambda (x-t))]_{\nu} dt = \int_{x-\nu}^{x} \sin(\lambda t) \sin[\lambda (x-t)] dt + \int_{x-\nu}^{-\nu} \sin(\lambda t) \sin[\lambda (x-t)] \tau_{\nu}(t) dt = \frac{(x-\nu) \cos(\lambda x)}{2} - \frac{\sin(\lambda x) + \sin(\lambda x - 2\nu \lambda)}{4\lambda} + O(\nu^{-\nu})
\]

and it follows that

\[
N_{\nu \to \infty} \int_{-\infty}^{\infty} [\sin^{-}(\lambda t)]_{\nu} \sin^{+}[\lambda (x-t)] dt = \frac{2\lambda x \cos(\lambda x) - \sin(\lambda x)}{4\lambda},
\]

on using equation (8).

Equation (26) now follows as above on using equations (33), (34) and (35).

Further results can again be easily deduced. For example, since,

\[
\cos_{+}(\lambda x) \ast \cos_{+}(\lambda x) = \frac{\sin_{+}(\lambda x) + \lambda x \cos_{+}(\lambda x)}{2\lambda},
\]

for \(\lambda \neq 0,\) it follows as above that

\[
\cos(\lambda x) \ast \cos_{+}(\lambda x) = \cos_{-}(\lambda x) \ast \cos_{+}(\lambda x) + \cos_{+}(\lambda x) \ast \cos_{+}(\lambda x) = \frac{\sin(\lambda x) + 2\lambda x \cos_{-}(\lambda x)}{4\lambda},
\]

\[
\cos(\lambda x) \ast \cos_{-}(\lambda x) = -\frac{\sin(\lambda x) + 2\lambda x \cos_{+}(\lambda x)}{4\lambda},
\]

\[
\cos(\lambda x) \ast \cos(\lambda x) = \frac{1}{2} x \cos(\lambda x),
\]

for \(\lambda \neq 0.\)
Some commutative neutrix convolution products of functions

References


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