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Differential equations at resonance

DONAL O’REGAN

Abstract. New existence results are presented for the two point singular “resonant” boundary value problem \( \frac{1}{p(t)}(py')' + ry + \lambda_my = f(t, y, py') \) a.e. on \([0, 1]\) with \( y \) satisfying Sturm Liouville or Periodic boundary conditions. Here \( \lambda_m \) is the \((m + 1)\)st eigenvalue of \( \frac{1}{p(t)}[(pu')' + rpu] + \lambda u = 0 \) a.e. on \([0, 1]\) with \( u \) satisfying Sturm Liouville or Periodic boundary data.

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1. Introduction

In this paper we derive some existence results for the second order equation
\[
(1.1) \quad \frac{1}{p(t)}(p(t)y'(t))' + r(t)y(t) + \lambda_m q(t)y(t) = f(t, y(t), p(t)y'(t)) \quad \text{a.e. on } [0, 1]
\]
with \( y \) satisfying either

(i) (Sturm Liouville)

\[
\begin{align*}
-\alpha y(0) + \beta \lim_{t \to 0^+} p(t)y'(t) &= 0, \quad \alpha \geq 0, \beta \geq 0, \alpha^2 + \beta^2 > 0 \\
\alpha y(1) + b \lim_{t \to 1^-} p(t)y'(t) &= 0, \quad a \geq 0, b \geq 0, a^2 + b^2 > 0
\end{align*}
\]

or

(ii) (Periodic)

\[
\begin{align*}
y(0) &= y(1) \\
\lim_{t \to 0^+} p(t)y'(t) &= \lim_{t \to 1^-} p(t)y'(t).
\end{align*}
\]

Remarks. (i) \( \lambda_m \) will be described later.
(ii) The Neumann condition \( \lim_{t \to 0^+} p(t)y'(t) = \lim_{t \to 1^-} p(t)y'(t) = 0 \) is included in (SL) with \( \alpha = a = 0 \).
(iii) If a function \( u \in C[0, 1] \cap C^1(0, 1) \) with \( pu' \in C[0, 1] \) satisfies boundary condition (i) we write \( u \in (SL) \). A similar remark applies for the boundary condition (ii).

Throughout the paper \( p \in C[0, 1] \cap C^1(0, 1) \) together with \( p > 0 \) on \((0, 1)\). Also \( pf : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function. By this we mean:

(i) \( t \to p(t)f(t, y, q) \) is measurable for all \( (y, q) \in \mathbb{R}^2 \);
(ii) \( (y, q) \to p(t)f(t, y, q) \) is continuous for a.e. \( t \in [0, 1] \);
(iii) for any \( r > 0 \) there exists \( h_r \in L^1[0,1] \) such that \( |p(t)f(t,y,q)| \leq h_r(t) \) for a.e. \( t \in [0,1] \) and for all \( |y| \leq r, |q| \leq r \).

For notational purposes let \( w \) be a weight function. By \( L^1_w[0,1] \) we mean the space of functions \( u \) such that \( \int_0^1 w(t)|u(t)| dt < \infty \). \( L^2_w[0,1] \) denotes the space of functions \( u \) such that \( \int_0^1 w(t)|u(t)|^2 dt < \infty \); also for \( u,v \in L^2_w[0,1] \) define \( \langle u,v \rangle = \int_0^1 w(t)u(t)v(t) \, dt \). Let \( AC[0,1] \) be the space of functions which are absolutely continuous on \([0,1]\).

Before we discuss the boundary value problem (1.1) and its appropriate literature we first gather together some facts on second order differential equations ([12], [16]). Consider the linear equation

(1.2) \[
\begin{cases}
\frac{1}{p}(py')' + \tau y = g(t) & \text{a.e. on } [0,1] \\
y \in (\text{SL}) \text{ or (P)}.
\end{cases}
\]

By a solution to (1.2) we mean a function \( y \in C[0,1] \cap C^1(0,1) \) with \( py' \in AC[0,1] \) which satisfies the differential equation in (1.2) a.e. on \([0,1]\) and the stated boundary conditions.

**Theorem 1.1.** Suppose

(1.3) \[ p \in C[0,1] \cap C^1(0,1) \text{ with } p > 0 \text{ on } (0,1) \text{ and } \int_0^1 \frac{ds}{p(s)} < \infty \]

and

(1.4) \[ \tau, g \in L^1_p[0,1] \]

are satisfied. If

(1.5) \[
\begin{cases}
\frac{1}{p}(py')' + \tau y = 0 & \text{a.e. on } [0,1] \\
y \in (\text{SL}) \text{ or (P)}
\end{cases}
\]

has only the trivial solution, then (1.2) has exactly one solution \( y \) given by

\[
y(t) = d_0u_1(t) + d_1u_2(t) + \int_0^t \frac{[u_2(t)u_1(s) - u_1(t)u_2(s)]}{W(s)}g(s) \, ds
\]
where \( u_1 \) is the unique solution to
\[
\begin{cases}
\frac{1}{p}(pu')' + \tau u = 0 \quad \text{a.e. on } [0, 1] \\
u(0) = 1, \lim_{t \to 0^+} p(t)u'(t) = 0
\end{cases}
\]
and \( u_2 \) is the unique solution to
\[
\begin{cases}
\frac{1}{p}(pu')' + \tau u = 0 \quad \text{a.e. on } [0, 1] \\
u(0) = 0, \lim_{t \to 0^+} p(t)u'(t) = 1
\end{cases}
\]
and \( d_0 \) and \( d_1 \) are uniquely determined from the boundary condition; \( W \) of course denotes the Wronskian. In fact
\[
y(t) = \int_0^1 G(t, s)g(s) \, ds
\]
with
\[
G(t, s) = \begin{cases}
\frac{y_1(s)y_2(t)}{W(s)}, & 0 < s \leq t \\
\frac{y_1(t)y_2(s)}{W(s)}, & t \leq s < 1
\end{cases}
\]
where \( y_1 \) and \( y_2 \) are the two "usual" linearly independent solutions i.e. choose \( y_1 \neq 0, y_2 \neq 0 \) so that \( y_1, y_2 \) satisfy \( \frac{1}{p}(py')' + \tau y = 0 \) a.e. on \([0, 1]\) with \( y_1 \) satisfying the first boundary condition and \( y_2 \) satisfying the second boundary condition.

We now state an existence principle ([16]), which was established using fixed point methods, for the second order nonresonant boundary value problem
\[
\begin{cases}
\frac{1}{p}(py')' + \tau y = f(t, y, py') \quad \text{a.e. on } [0, 1] \\
y \in (\text{SL}) \text{ or } (\text{P})
\end{cases}
\] (1.6)

**Theorem 1.2.** Let \( p f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) be an \( L^1 \)-Carathéodory function and assume (1.3) and
\[
\tau \in L^1_{p}[0, 1]
\] (1.7)
hold. In addition suppose (1.5) has only the trivial solution. Now assume there is a constant \( M_0 \), independent of \( \lambda \), with
\[
\|y\|_* = \max\{\sup_{[0, 1]} |y(t)|, \sup_{(0, 1)} |p(t)y'(t)|\} \leq M_0
\]
for any solution \( y \) to
\[
\begin{cases}
\frac{1}{p}(py')' + \tau y = \lambda f(t, y, py') \quad \text{a.e. on } [0, 1] \\
y \in (\text{SL}) \text{ or } (\text{P})
\end{cases}
\]
for each \( \lambda \in (0, 1) \). Then (1.6) has at least one solution \( u \in C[0, 1] \cap C^1(0, 1) \) with \( pu' \in AC[0, 1] \).

Next we gather together some results on the Sturm Liouville eigenvalue problem

\[
\begin{cases}
Lu = \lambda u \text{ a.e. on } [0, 1] \\
u \in (SL) \text{ or } (P)
\end{cases}
\]

where \( Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu] \). Assume (1.3) and

\[ r, q \in L^1_p[0, 1] \text{ with } q > 0 \text{ a.e. on } [0, 1] \]

hold. Let

\[ D(L) = \{ w \in C[0, 1] : w, pw' \in AC[0, 1] \text{ with } w \in (SL) \text{ or } (P) \} \]

Then \( L \) has a countably infinite number ([1], [12], [16]) of real eigenvalues \( \lambda_i \) with corresponding eigenfunctions \( \psi_i \in D(L) \). The eigenfunctions \( \psi_i \) may be chosen so that they form a orthonormal set and we may also arrange the eigenvalues so that

\[ \lambda_0 < \lambda_1 < \lambda_2 < \ldots \]

Remark. The \( \lambda_i \)'s may be estimated numerically ([2]) using SLEIGN.

In addition the set of eigenfunctions \( \psi_i \) form a basis for \( L^2_{pq}[0, 1] \) and if \( h \in L^2_{pq_i}[0, 1] \) then \( h \) has a Fourier series representation and \( h \) satisfies Parseval's equality i.e.

\[
h = \sum_{i=0}^{\infty} \langle h, \psi_i \rangle \psi_i \quad \text{and} \quad \int_0^1 pq|h|^2 \, dt = \sum_{i=0}^{\infty} |\langle h, \psi_i \rangle|^2.
\]

We are concerned with existence results for the nonlinear second order equation

\[
\begin{cases}
\frac{1}{p}(py')' + ry + \lambda_m qy = f(t, y, py') \quad \text{a.e. on } [0, 1] \\
y \in (SL) \text{ or } (P)
\end{cases}
\]

where \( \lambda_m \) is the \((m + 1)^{st}\) eigenvalue of (1.8). In recent years several authors ([4], [7]–[9], [11], [13], [18]–[19]) have examined the boundary value problems

\[
\begin{cases}
y'' + n^2 \pi^2 y = f(t, y) \quad \text{a.e. on } [0, 1] \\
y(0) = y(1) = 0
\end{cases}
\]

and

\[
\begin{cases}
y'' + m^2 \pi^2 y = f(t, y) \quad \text{a.e. on } [0, 1] \\
y(0) = y(1), y'(0) = y'(1)
\end{cases}
\]
where \( n \geq 1, \ m \geq 0 \) are integers. Most of the papers in the literature ([3], [7], [11], [18]–[19]) concentrate on the first eigenvalue \((n = 1 \ or \ m = 0)\). However over the last ten years or so ([6], [10]) the case when \( n > 1 \ or \ m > 0 \) has been discussed. This paper continues this study for the more general problem (1.11); also it provides a new approach to studying the above resonant type problems. We refer the reader to [6]–[9] for many of the motivating ideas in this paper. Finally it is of interest to note that in previous studies ([6], [8], [11]) the nonlinearity \( f \) is required to grow no more than linearly in \( y \) as \(|y| \to \infty\) whereas in this paper solutions will exist provided \( f \) grows fast enough e.g. \( yf(t,y,z) \geq A|y|^\theta+1 \) for some \( A > 0 \) and \( \theta > 0 \).

2. Existence

Existence theory is developed for the second order boundary value problem

\[
\begin{cases}
\frac{1}{p}(py')' + ry + \lambda mqy = f(t,y,py') & \text{a.e. on } [0,1] \\
y \in (\text{SL}) \text{ or } (P)
\end{cases}
\]

where \( \lambda_m \) is the \((m + 1)^{st}\) eigenvalue of

\[
\begin{cases}
Lu = \lambda u & \text{a.e. on } [0,1] \\
u \in (\text{SL}) \text{ or } (P)
\end{cases}
\]

and \( Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu] \).

Two types of existence results are presented, the first examines the problem on the “left” of the eigenvalue whereas the second discusses the problem on the “right” of the eigenvalue.

Existence theory I.

Throughout this subsection let

\[
H_{\alpha_0,\theta}(u_1) = \begin{cases}
|u_1|^\theta+1, & |u_1| \leq 1 \\
|u_1|^\alpha_0+1, & |u_1| > 1.
\end{cases}
\]

**Theorem 2.1.** Let \( pf : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) be an \( L^1\)-Carathéodory function with (1.3) and (1.9) satisfied. Suppose \( f \) has the decomposition \( f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2) \) with \( pg, ph : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \) \( L^1\)-Carathéodory functions and

\[
\phi \in L_p^1[0,1], \phi > 0 \text{ a.e. on } [0,1] \text{ with } u_1g(t,u_1,u_2) \geq A\phi(t)H_{\alpha_0,\theta}(u_1)
\]

for a.e. \( t \in [0,1] \); here \( \alpha_0 \leq \theta \).
there exist \( \phi_i \in L^1_p[0,1], i = 1, 2, 3 \) and constants \( \beta_0 \) and 

\[
(2.4) \left\{ \begin{array}{l}
\text{there exist } \phi_i \in L^1_p[0,1], i = 1, 2, 3 \text{ and constants } \beta_0 \text{ and } \\
\text{and constants } \beta_0 < \alpha_0 \text{ and } 0 \leq \sigma < \frac{\alpha_0}{2} \text{ and } \\
\phi_3 > 0 \text{ a.e. on } [0,1] \text{ or } \phi_3 \equiv 0 \text{ on } [0,1]
\end{array} \right.
\]

\[
(2.5) \left\{ \begin{array}{l}
\text{there exist } \phi_i \in L^1_p[0,1], i = 4, 5 \text{ and a constant } \gamma \leq \alpha_0 \text{ with } \\
|g(t,u_1,u_2)| \leq \phi_4(t) + \phi_5(t)|u_1|^{\gamma} \text{ for a.e. } t \in [0,1]
\end{array} \right.
\]

\[
(2.6) \left\{ \begin{array}{l}
\phi_4^2q^{-1} \in L^1_p[0,1], \phi_5^2q^{-1} \in L^1_p[0,1], \\
(\phi_5^{2(\alpha_0+1)}q^{-\alpha_0+1})^{1-2\gamma} \in L^1_p[0,1], \\
(\phi_2^{2(\alpha_0+1)}q^{-\alpha_0+1})^{1-2\beta_0} \in L^1_p[0,1] \text{ and } \\
(\phi_2^{2(\alpha_0+1)}q^{-\alpha_0+1})^{1-\gamma} \in L^1_p[0,1]
\end{array} \right.
\]

and

\[
(2.7) \left\{ \begin{array}{l}
(\phi_1^{\alpha_0+1}q^{-1})^{\frac{1}{\alpha_0}} \in L^1_p[0,1], \left(\phi_2^{\alpha_0+1}q^{-\alpha_0+1}\right)^{\frac{1}{\alpha_0-\gamma}} \in L^1_p[0,1], \\
(\phi_3^{\alpha_0+1}q^{-1})^{\frac{1}{\alpha_0}} \in L^1_p[0,1], \\
(\phi_5^{\alpha_0+1}q^{-1})^{\frac{1}{\alpha_0+1-\gamma}} \in L^1_p[0,1] \text{ and } \\
(q^{\alpha_0+1}q^{-1})^{\frac{1}{\alpha_0}} \in L^1_p[0,1]
\end{array} \right.
\]

holding. Then (2.1) has at least one solution \( y \in C[0,1] \cap C^1(0,1) \) with \( py' \in AC[0,1] \).

**Remark.** Typical examples where (2.3) is satisfied are say (i) \( g(t,u_1,u_2) = \frac{m}{n} u_1 \), 

\( m \) odd and \( n \) odd or (ii) \( g(t,u_1,u_2) = \frac{1}{2} u_1 \), \( u_1 \geq 0 \) with \( g(t,u_1,u_2) = -|u_1|^{\frac{1}{2}} \), 

\( u_1 < 0 \).

**Proof:** Consider the family of problems

\[
(2.8)_\lambda \left\{ \begin{array}{l}
\frac{1}{p}(py')' + ry + mu = \lambda[f(t,y,py') + (\mu - \lambda_m)qy] \text{ a.e. on } [0,1] \\
y \in (SL) \text{ or (P)}
\end{array} \right.
\]

where \( 0 < \lambda < 1 \) and \( \lambda_m - 1 < \mu < \lambda_m \); here \( \lambda_1 = -\infty \) (for notational purposes) 

with \( \lambda_i \) as described in (1.10).
Notice $L^2_{pq}[0,1] = \Omega \bigoplus \Omega^\perp$ where $\Omega = \text{span} \{\psi_0, \psi_1, \ldots, \psi_{m-1}\}$; here $\psi_i$ are the eigenfunctions corresponding to the eigenvalues $\lambda_i$ (see Section 1).

Let $y$ be any solution to (2.8)$_\lambda$. Then $y = u + w$ where $u \in \Omega$ and $w \in \Omega^\perp$. Multiply (2.8)$_\lambda$ by $w - u$ and integrate from 0 to 1 to obtain

$$\int_0^1 (w - u)(py')' \, dt + \int_0^1 p[w^2 - u^2] \, dt + \mu \int_0^1 pq[w^2 - u^2] \, dt$$

$$= \lambda \int_0^1 (w - u)pf(t, y, py') \, dt + \lambda (\mu - \lambda_m) \int_0^1 pq[w^2 - u^2] \, dt.$$

Integration by parts yields

$$\int_0^1 (w - u)(py')' \, dt = Q_0 - \int_0^1 p(w')^2 \, dt + \int_0^1 p(u')^2 \, dt$$

where

$$Q_0 = \begin{cases} - \frac{a}{b}[w^2(1) - u^2(1)] - \frac{a}{b}[w^2(0) - u^2(0)] & \text{if } y \in (\text{SL}) \\ 0 & \text{if } y \in (P); \end{cases}$$

here $y(0) = 0$ means $u(0) + w(0) = 0$ and so $u(0) = w(0) = 0$. Thus we have

$$Q_0 + \int_0^1 [-p(w')^2 + prw^2 + \mu pqw^2] \, dt + \int_0^1 [p(u')^2 - pru^2 - \mu pqu^2] \, dt$$

$$= \lambda \int_0^1 (w - u)pf(t, y, py') \, dt + \lambda (\mu - \lambda_m) \int_0^1 pqw^2 \, dt$$

$$- \lambda (\mu - \lambda_m) \int_0^1 pqu^2 \, dt. \tag{2.9}$$

Now since $u \in \Omega$, $w \in \Omega^\perp$ and $y = u + w$ we have

$$u = \sum_{i=0}^{m-1} c_i \psi_i \quad \text{and} \quad w = \sum_{i=m}^{\infty} c_i \psi_i \quad \text{where} \quad c_i = \langle y, \psi_i \rangle;$$

note $u = 0$ if $m = 0$. Then since $(p\psi_i')' + rp\psi_i + \lambda_i pq\psi_i = 0$ we have

$$Q_0 + \int_0^1 [-p(w')^2 + prw^2 + \mu pqw^2] \, dt + \int_0^1 [p(u')^2 - pru^2 - \mu pqu^2] \, dt$$

$$= \sum_{i=m}^{\infty} (\mu - \lambda_i)c_i^2 \int_0^1 pq\psi_i^2 \, dt + \sum_{i=0}^{m-1} (\lambda_i - \mu)c_i^2 \int_0^1 pq\psi_i^2 \, dt$$

$$\leq (\mu - \lambda_m) \int_0^1 pqw^2 \, dt + (\lambda_{m-1} - \mu) \int_0^1 pqu^2 \, dt.$$
Put this into (2.9) to obtain
\[
\lambda \int_0^1 (w - u)pg(t, y, py') \, dt + (1 - \lambda)(\lambda_m - \mu) \int_0^1 pqw^2 \, dt \\
+ (\mu - \lambda_m - 1) \int_0^1 pqu^2 \, dt + \lambda(\lambda_m - \mu) \int_0^1 pqu^2 \, dt \\
\leq -\lambda \int_0^1 (w - u)ph(t, y, py') \, dt.
\]
Consequently
\[
\int_0^1 pyg(t, y, py') \, dt + (\lambda_m - \mu) \int_0^1 pqu^2 \, dt \leq 2 \int_0^1 pqg(t, y, py') \, dt \\
+ \int_0^1 p|y||h(t, y, py')| \, dt + 2 \int_0^1 p|u||h(t, y, py')| \, dt.
\]
Assumption (2.3) yields
\[
\int_0^1 pyg(t, y, py') \, dt \geq A \int_0^1 p\phi H_{\alpha_0, \theta}(y) \, dt \\
= A \int_0^1 p\phi|y|^{\alpha_0+1} \, dt + A \int_{\{t:|y(t)|\leq 1\}} p\phi||y|^{\theta+1} - |y|^{\alpha_0+1} \, dt \\
\geq A \int_0^1 p\phi|y|^{\alpha_0+1} \, dt - A \int_0^1 p\phi \, dt
\]
and put this into (2.10), and use (2.4) and (2.5), to obtain
\[
A \int_0^1 p\phi|y|^{\alpha_0+1} \, dt + (\lambda_m - \mu) \int_0^1 pqu^2 \, dt \leq A \int_0^1 p\phi \, dt + 2 \int_0^1 p\phi_4|u| \, dt \\
+ 2 \int_0^1 p\phi_5|u||y|^{\gamma} \, dt + \int_0^1 p\phi_1|y| \, dt \\
+ \int_0^1 p\phi_2|y|^{\beta_0+1} \, dt + \int_0^1 p\phi_3|y||py'|^{\sigma} \, dt \\
+ 2 \int_0^1 p\phi_1|u| \, dt + 2 \int_0^1 p\phi_2|u||y|^{\beta_0} \, dt \\
+ 2 \int_0^1 p\phi_3|u||py'|^{\sigma} \, dt.
\]
(2.11)
For the remainder of the proof we assume without loss of generality that \(\sigma > 0\) and \(\phi_3 \neq 0\) on \([0, 1]\). Let \(\epsilon > 0\) be given. Hölder’s inequality together with assumption (2.6) immediately yields the following inequalities:
\[
2 \int_0^1 p\phi_4|u| \, dt \leq 2Q_1 \left( \int_0^1 pqu^2 \, dt \right)^{\frac{1}{2}} \leq \epsilon \int_0^1 pqu^2 \, dt + \frac{Q_1}{\epsilon};
\]
\[ 2 \int_0^1 p\phi_1|u| \, dt \leq \epsilon \int_0^1 pqu^2 \, dt + \frac{Q_2}{\epsilon}; \]
\[ 2 \int_0^1 p\phi_3|u||y|^\gamma \, dt \leq 2Q_3 \left( \int_0^1 pqu^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{2\gamma}{\alpha_0+1}}; \]
\[ \leq \epsilon Q_3 \int_0^1 pqu^2 \, dt + \frac{Q_3}{\epsilon} \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{2\gamma}{\alpha_0+1}}; \]
\[ 2 \int_0^1 p\phi_2|u||y|^{\beta_0} \, dt \leq \epsilon Q_4 \int_0^1 pqu^2 \, dt + \frac{Q_4}{\epsilon} \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{2\beta_0}{\alpha_0+1}}; \]
\[ \int_0^1 p\phi_1|y| \, dt \leq Q_5 \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{1}{\alpha_0+1}}; \]
\[ \int_0^1 p\phi_2|y|^{\beta_0+1} \, dt \leq Q_6 \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{\beta_0+1}{\alpha_0+1}}; \]
\[ \int_0^1 p\phi_3|y| |py'|^\sigma \, dt \leq \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{1}{\alpha_0+1}} \times \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'|^\sigma \frac{\alpha_0+1}{\alpha_0} \, dt \right)^{\frac{\alpha_0}{\alpha_0+1}}; \]
\[ 2 \int_0^1 p\phi_3|u||py'|^\sigma \, dt \leq 2Q_7 \left( \int_0^1 pqu^2 \, dt \right)^{\frac{1}{2}} \times \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'|^\sigma \frac{\alpha_0+1}{\alpha_0} \, dt \right)^{\frac{\alpha_0}{\alpha_0+1}} \leq \epsilon Q_7 \int_0^1 pqu^2 \, dt \]
\[ + \frac{Q_7}{\epsilon} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'|^\sigma \frac{\alpha_0+1}{\alpha_0} \, dt \right)^{\frac{2\alpha_0}{\alpha_0+1}}; \]

for some constants \( Q_1, \ldots, Q_7 \). Put these into (2.11) to obtain
\[ A \int_0^1 p\phi|y|^{\alpha_0+1} \, dt + (\lambda_m - \mu - 2\epsilon - \epsilon Q_3 - \epsilon Q_4 - \epsilon Q_7) \int_0^1 pqu^2 \, dt \]
\[ \leq Q_8 + \frac{Q_3}{\epsilon} \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{2\gamma}{\alpha_0+1}} + \frac{Q_4}{\epsilon} \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{2\beta_0}{\alpha_0+1}} \]
\[ + Q_5 \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{1}{\alpha_0+1}} + Q_6 \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{\beta_0+1}{\alpha_0+1}} \]
\[ + \left( \int_0^1 p\phi|y|^{\alpha_0+1} \, dt \right)^{\frac{1}{\alpha_0+1}} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'|^\sigma \frac{\alpha_0+1}{\alpha_0} \, dt \right)^{\frac{\alpha_0}{\alpha_0+1}} \]
for some constant $Q_8$. We may choose $\epsilon$ so that $\lambda_m - \mu - 2\epsilon - \epsilon Q_3 - \epsilon Q_4 - \epsilon Q_7 > 0$ and we have

$$A \int_0^1 p \phi |y|^{\alpha_0+1} dt \leq Q_8 + \frac{Q_3}{\epsilon} \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{2\gamma}{\alpha_0+1}}$$

$$+ \frac{Q_4}{\epsilon} \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{2\beta_0}{\alpha_0+1}} + Q_5 \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{1}{\alpha_0+1}}$$

$$+ Q_6 \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{1}{\alpha_0+1}} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \sigma \left( \alpha_0+1 \right) \alpha_0 dt \right)^{\frac{\alpha_0}{\alpha_0+1}}$$

$$+ \frac{Q_7}{\epsilon} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \sigma \left( \alpha_0+1 \right) \alpha_0 dt \right)^{\frac{2\alpha_0}{\alpha_0+1}}.$$

We now consider two cases $\int_0^1 p \phi |y|^{\alpha_0+1} dt > 1$ and $\int_0^1 p \phi |y|^{\alpha_0+1} dt \leq 1$ separately.

**Case (i).** $\int_0^1 p \phi |y|^{\alpha_0+1} dt > 1$.

Divide (2.12) by $\left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{1}{\alpha_0+1}}$ and use $\int_0^1 p \phi |y|^{\alpha_0+1} dt > 1$ to obtain

$$A \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{\alpha_0}{\alpha_0+1}} \leq Q_8 + \frac{Q_3}{\epsilon} \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{2\gamma-1}{\alpha_0+1}}$$

$$+ \frac{Q_4}{\epsilon} \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{2\beta_0-1}{\alpha_0+1}}$$

$$+ Q_5 + Q_6 \left( \int_0^1 p \phi |y|^{\alpha_0+1} dt \right)^{\frac{\beta_0}{\alpha_0+1}}$$

$$+ \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \sigma \left( \alpha_0+1 \right) \alpha_0 dt \right)^{\frac{\alpha_0}{\alpha_0+1}}$$

$$+ \frac{Q_7}{\epsilon} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \sigma \left( \alpha_0+1 \right) \alpha_0 dt \right)^{\frac{2\alpha_0}{\alpha_0+1}}.$$

Now since $\max\{2\gamma - 1, 2\beta_0 - 1, \beta_0\} < \alpha_0$ there exist constants $Q_9, Q_{10}$ and $Q_{11}$
with
\[
\left( \int_0^1 p|y|^{\alpha_0+1} dt \right)^{\alpha_0 \over \alpha_0+1} \leq Q_9 + Q_{10} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right)^{1 \over \alpha_0} |y'|^{\sigma(\alpha_0+1) \over \alpha_0} dt \right)^{\alpha_0 \over \alpha_0+1} + Q_{11} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right)^{1 \over \alpha_0} |y'|^{\sigma(\alpha_0+1) \over \alpha_0} dt \right)^{2\alpha_0 \over \alpha_0+1}.
\]

Using the inequality \((a + b)^c \leq 2^c(a^c + b^c)\) for \(a \geq 0, b \geq 0, c \geq 0\) we see that there exist constants \(Q_{12}\) and \(Q_{13}\) with
\[
(2.13) \quad \int_0^1 p|y|^{\alpha_0+1} dt \leq Q_{12} + Q_{13} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right)^{1 \over \alpha_0} |y'|^{\sigma(\alpha_0+1) \over \alpha_0} dt \right)^2.
\]

Case (ii). \(\int_0^1 p|y|^{\alpha_0+1} dt \leq 1\).

In this case (2.13) is clearly true with \(Q_{12} = 1\).

Consequently in all cases (2.13) is true. Returning to (2.8) we have
\[
y(t) = \lambda \int_0^1 G(t,s)[f(s,y(s),p(s)y'(s)) + (\mu - \lambda m)q(s)y(s)] ds
\]
and
\[
(2.15) \quad p(t)y'(t) = \lambda \int_0^1 p(t)G(t,s)[f(s,y(s),p(s)y'(s)) + (\mu - \lambda m)q(s)y(s)] ds
\]
where \(G(t,s)\) is the Green’s function associated with \(1 \over p(pv')' + rv + \mu qv = 0\) a.e. on \([0,1]\), \(v \in (SL)\) or \((P)\).

Notice ([16], [17]) that \(\sup_{t \in [0,1]} |p(t)G(t,s)| \leq Q_{14}p(s)\) for some constant \(Q_{14}\). Now (2.15) together with (2.4) and (2.5) imply for \(t \in (0,1)\) that
\[
|p(t)y'(t)| \leq Q_{15} \int_0^1 p\phi_1 ds + Q_{15} \int_0^1 p\phi_2 |y|^{\beta_0} ds + Q_{15} \int_0^1 p\phi_3 |y'|^{\sigma} ds + Q_{15} \int_0^1 p\phi_4 ds + Q_{15} \int_0^1 p\phi_5 |y|^{\gamma} ds + Q_{16} \int_0^1 pq |y| ds
\]
for some constants \(Q_{15}\) and \(Q_{16}\). Hölder’s inequality together with (2.6) implies
\[
|p(t)y'(t)| \leq Q_{17} + Q_{18} \left( \int_0^1 p\phi|y|^{\alpha_0+1} dt \right)^{\beta_0 \over \alpha_0+1} + Q_{19} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right)^{1 \over \alpha_0} |y'|^{\sigma(\alpha_0+1) \over \alpha_0} dt \right)^{\alpha_0 \over \alpha_0+1} + Q_{20} \left( \int_0^1 p\phi|y|^{\alpha_0+1} dt \right)^{\gamma \over \alpha_0+1} + Q_{21} \left( \int_0^1 p\phi|y|^{\alpha_0+1} dt \right)^{1 \over \alpha_0+1}.
\]
for some constants $Q_{17}, \ldots, Q_{21}$. Thus for $t \in (0, 1)$ we have

$$
|p(t)y'(t)| \frac{\sigma(\alpha_0+1)}{\alpha_0} \leq Q_{22} + Q_{23} \left( \int_0^1 p\phi |y|^{\alpha_0+1} \right)^{\frac{\sigma_0}{\alpha_0}} \\
\quad + Q_{24} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \right)^{\sigma} \\
\quad + Q_{25} \left( \int_0^1 p\phi |y|^{\alpha_0+1} \right)^{\frac{\sigma_0}{\alpha_0}} \\
\quad + Q_{26} \left( \int_0^1 p\phi |y|^{\alpha_0+1} \right)^{\frac{\sigma_0}{\alpha_0}}
$$

(2.16)

for some constants $Q_{22}, \ldots, Q_{26}$. This together with (2.13) implies

$$
\int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \\
\leq Q_{27} + Q_{28} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \right)^{\frac{2\sigma_0}{\alpha_0}} \\
\quad + Q_{29} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \right)^{\sigma} \\
\quad + Q_{30} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \right)^{\frac{2\sigma_0}{\alpha_0}} \\
\quad + Q_{31} \left( \int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \right)^{\frac{2\sigma_0}{\alpha_0}}
$$

for some constants $Q_{27}, \ldots, Q_{31}$. Finally since $\max \left\{ \frac{2\sigma_0}{\alpha_0}, \sigma, \frac{2\sigma_0}{\alpha_0}, \frac{2\sigma}{\alpha_0} \right\} < 1$ there exists a constant $Q_{32}$ with

$$
\int_0^1 p \left( \phi_3^{\alpha_0+1} \phi^{-1} \right) \frac{1}{\alpha_0} |py'| \frac{\sigma(\alpha_0+1)}{\alpha_0} \ dt \leq Q_{32}
$$

(2.17)

and this together with (2.13) implies that there exists a constant $Q_{33}$ with

$$
\int_0^1 p\phi |y|^{\alpha_0+1} \ dt \leq Q_{33}.
$$

(2.18)

Putting these inequalities into (2.16) establishes the existence of a constant $Q_{34}$ with

$$
\sup_{t \in (0, 1)} |p(t)y'(t)| \leq Q_{34}.
$$

(2.19)
Now (2.14) together ([16], [17]) with sup_{t \in [0,1]} |G(t, s)| \leq Q_{35}p(s), for some constant Q_{35}, and Hölder’s inequality implies for t \in [0,1] that

\begin{align*}
|y(t)| \leq Q_{36} + Q_{37} \left( \int_0^1 p(t)s^{\alpha_0+1} \sigma(t) \right)^{\beta_0 / \alpha_0+1} \\
+ Q_{38} \left( \int_0^1 p(t)s^{\alpha_0+1} \sigma(t) \right)^{\gamma_0 / \alpha_0+1} \\
+ Q_{39} \left( \int_0^1 p(t)s^{\alpha_0+1} \sigma(t) \right)^{\delta_0 / \alpha_0+1} \\
+ Q_{40} \left( \int_0^1 p(t)s^{\alpha_0+1} \sigma(t) \right)^{\epsilon_0 / \alpha_0+1}
\end{align*}

for some constants Q_{36}, \ldots, Q_{40}. This together with (2.17) and (2.18) implies that there is a constant Q_{41} with

\begin{equation}
\sup_{t \in [0,1]} |y(t)| \leq Q_{41}.
\end{equation}

Now (2.19), (2.20) together with Theorem 1.2 establish the result. \hfill \square

**Example.** Theorem 2.1 (here $H_{\alpha_0,\theta}(u) = H_{\alpha_0,\theta}(u)$) immediately guarantees that

\begin{equation}
\begin{cases}
y'' + n^2 \pi^2 y = y^{1/3} + [y']^{1/7} + 1 \quad \text{a.e. on } [0,1] \\
y(0) = y(1) = 0, \quad n \in \{1, 2, \ldots\}
\end{cases}
\end{equation}

has a solution.

One can improve considerably the above theorem if $m = 0$ (at the first eigen-value). In particular the condition $0 < \alpha_0 < 1$ is replaced by $\alpha_0 > 0$ in this case; also condition (2.5) can be improved and the condition $\sigma < \alpha_0^2$ can be relaxed.

We present two existence results.

Consider

\begin{equation}
\begin{cases}
\frac{1}{p}(py')' + ry + \lambda_0 qy = f(t, y, py') \quad \text{a.e. on } [0,1] \\
y \in (\text{SL}) \text{ or } (P)
\end{cases}
\end{equation}

where $\lambda_0$ is the first eigenvalue of (2.2).

**Theorem 2.2.** Let $pf : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be an $L^1$-Carathéodory function with (1.3) and (1.9) satisfied. Suppose $f$ has the decomposition $f(t, y, py') = g(t, u_1, u_2) + h(t, u_1, u_2)$ with $pg, ph : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ $L^1$-Carathéodory functions and

\begin{equation}
\begin{cases}
\text{there exist constants } A > 0, \alpha_0 > 0 \text{ and a function } \phi \in L^1_p[0,1], \\
\phi > 0 \text{ a.e. on } [0,1] \text{ with } u_1 g(t, u_1, u_2) \geq A \phi(t) H_{\alpha_0,\theta}(u_1) \\
\text{for a.e. } t \in [0,1]; \text{ here } \alpha_0 \leq \theta
\end{cases}
\end{equation}
there exist \( \phi_i \in L^1_p[0,1] \), \( i = 1, 2, 3 \) and constants \( \beta_0 \) and\n
\[
\begin{align*}
(2.23) \\
\text{\begin{cases} \\
age t \in [0,1]; \text{ here } \beta_0 < \alpha_0 \text{ and } \phi_3 > 0 \\
\text{a.e. on } [0,1] \text{ or } \phi_3 \equiv 0 \text{ on } [0,1] \\
\end{cases}}
\end{align*}
\]

Hölder’s inequality implies

\[
(2.28)
\]

Proof: \text{Let } \\AC

\[
(2.24)
\]

there exist \( \phi_i \in L^1_p[0,1] \), \( i = 4, 5, 6 \) and constants \( \gamma \leq \alpha_0, \tau > \sigma \)

with \( g(t,u_1,u_2) \leq \phi_4(t) + \phi_5(t)|u_1|^\gamma + \phi_6(t)|u_2|^\tau \)

for a.e. \( t \in [0,1] \);

here \( \phi_6 > 0 \) a.e. on \([0,1]\) or \( \phi_6 \equiv 0 \) on \([0,1]\)

\[
(2.25)
\]

\[
\sigma < \min\{1, \frac{\alpha_0}{\gamma}, \alpha_0\} \text{ and } \tau < 1
\]

\[
(2.26)
\]

\[
\begin{align*}
&\phi_1^{\alpha_0+1} - \phi^{-1} \in L^1_p[0,1], \quad \left(\phi_2^{\alpha_0+1} - \phi^{-(\beta_0+1)}\right)^\frac{1}{\alpha_0-\beta_0} \in L^1_p[0,1], \\
&\phi_5^{\alpha_0+1} - \phi^{-\gamma} \in L^1_p[0,1] \text{ and } (q^{\alpha_0+1} - \phi^{-1})^\frac{1}{\alpha_0} \in L^1_p[0,1]
\end{align*}
\]

and

\[
(2.27)
\]

\[
\begin{align*}
&\phi_6^\kappa \in L^1_p[0,1] \text{ and } \left(\phi_3^\kappa - \phi^{-1}\right)^\frac{\kappa}{\alpha_0+1} (\phi_6)^{-\kappa} \in L^1_p[0,1]
\end{align*}
\]

\text{with } \kappa = \max\{\alpha_0+1, 2\}, \left(\phi_3^\kappa - \phi^{-1}\right)^\frac{1}{\alpha_0+1} \in L^1_p[0,1]. \text{ Also need}

\text{if } \phi_6 > 0 \text{ a.e. on } [0,1]

holding. Then (2.21) has at least one solution \( y \in C[0,1] \cap C^1(0,1) \) with \( py' \in AC[0,1] \).}

\text{PROOF: Let } y \text{ be a solution of (2.8)}_\lambda \text{ with } m = 0. \text{ Following the ideas of Theorem 2.1 with } u = 0 \text{ and } y = w \text{ we obtain the analogue of (2.11), namely}

\[
A \int_0^1 p\phi|y|^\alpha dt \leq A \int_0^1 p\phi dt + \int_0^1 p\phi_1|y| dt + \int_0^1 p\phi_2|y|^\beta dt
\]

\[
(2.28)
\]

\[
+ \int_0^1 p\phi_3|y||py'|^\sigma dt.
\]

Hölder’s inequality implies

\[
A \int_0^1 p\phi|y|^\alpha dt \leq N_0 + N_1 \left( \int_0^1 p\phi|y|^\alpha dt \right)^\frac{1}{\alpha_0+1}
\]

\[
(2.29)
\]

\[
+ N_2 \left( \int_0^1 p\phi|y|^\alpha dt \right)^\frac{\beta_0+1}{\alpha_0+1} + \int_0^1 p\phi_3|y||py'|^\sigma dt.
\]
for some constants $N_0$, $N_1$ and $N_2$. Let $\kappa = \max\{2, \frac{\alpha_0 + 1}{\alpha_0}\}$. Hölder’s inequality together with assumption (2.27) implies

$$
\int_0^1 p\phi_3 |y||py'|^\sigma \, dt
\leq \left( \int_0^1 p|y|^\alpha \, dt \right)^{\frac{1}{\alpha + 1}} \left( \int_0^1 p \left( \phi_3 \phi - \frac{1}{\alpha + 1} \right)^\kappa |py'|^{\kappa \sigma} \, dt \right)^{\frac{1}{\kappa}}
$$

if $\kappa = \frac{\alpha_0 + 1}{\alpha_0}$ whereas

$$
\int_0^1 p\phi_3 |y||py'|^\sigma \, dt \leq \left( \int_0^1 p|y|^\alpha \, dt \right)^{\frac{1}{\alpha + 1}} \times \left( \int_0^1 p \left( \phi_3 \phi - \frac{1}{\alpha + 1} \right)^\kappa |py'|^{\kappa \sigma} \, dt \right)^{\frac{1}{\kappa}} \left( \int_0^1 p(t) \, dt \right)^{\frac{\alpha_0 - 1}{2(\alpha_0 + 1)}}
$$

if $\kappa = 2$. Put this into (2.29) and essentially the same reasoning as in Theorem 2.1 establishes the existence of constants $N_3$ and $N_4$ with

$$
\int_0^1 p|y|^{\alpha + 1} \, dt \leq N_3 + N_4 \left( \int_0^1 p \left( \phi_3 \phi - \frac{1}{\alpha + 1} \right)^\kappa |py'|^{\kappa \sigma} \, dt \right)^{\frac{\alpha_0 + 1}{\alpha_0 \kappa}}.
$$

Also (2.15) implies (as in Theorem 2.1) for $t \in (0, 1)$ that

$$
|p(t)y'(t)| \leq N_5 + N_6 \left( \int_0^1 p|y|^{\alpha + 1} \, dt \right)^{\frac{1}{\alpha + 1}} + N_7 \int_0^1 p\phi_3 |py'|^\sigma \, dt
$$

$$
+ N_8 \left( \int_0^1 p|y|^{\alpha + 1} \, dt \right)^{\frac{1}{\alpha + 1}} + N_9 \int_0^1 p\phi_6 |py'|^{\tau} \, dt
$$

$$
+ N_{10} \left( \int_0^1 p|y|^{\alpha + 1} \, dt \right)^{\frac{1}{\alpha + 1}}
$$

for some constants $N_5, \ldots, N_{10}$. Again with $\kappa = \max\{2, \frac{\alpha_0 + 1}{\alpha_0}\}$ we have

$$
\int_0^1 p\phi_3 |py'|^\sigma \, dt \leq \left( \int_0^1 p \left( \phi_3 \phi - \frac{1}{\alpha + 1} \right)^\kappa |py'|^{\kappa \sigma} \, dt \right)^{\frac{1}{\kappa}} \left( \int_0^1 p\phi \, dt \right)^{\frac{1}{\alpha + 1}}
$$

if $\kappa = \frac{\alpha_0 + 1}{\alpha_0}$

$$
\int_0^1 p\phi_3 |py'|^\sigma \, dt \leq \left( \int_0^1 p \left( \phi_3 \phi - \frac{1}{\alpha + 1} \right)^\kappa |py'|^{\kappa \sigma} \, dt \right)^{\frac{1}{\kappa}} \left( \int_0^1 p\phi^{\alpha_0 + 1} \, dt \right)^{\frac{1}{2}}
$$

if $\kappa = 2$

$$
\int_0^1 p\phi_6 |py'|^{\tau} \, dt \leq \left( \int_0^1 p\phi_6^{\kappa}|py'|^{\tau \kappa} \, dt \right)^{\frac{1}{\kappa}} \left( \int_0^1 p(t) \, dt \right)^{1 - \frac{1}{\kappa}}.
$$
There are two cases to consider, namely $\phi_6 > 0$ a.e. on $[0, 1]$ or $\phi_6 \equiv 0$ on $[0, 1]$.

Case (i). $\phi_6 > 0$ a.e. on $[0, 1]$.

Putting the above into (2.31) and using (2.30) leads to

$$
\int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \leq N_{11} + N_{12} \left( \int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt \right)^{\frac{\beta_0 \tau}{\alpha_0}}
$$

$$
+ N_{13} \left( \int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt \right)^\tau
$$

$$
+ N_{14} \left( \int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt \right)^{\frac{\sigma \tau}{\alpha_0}}
$$

$$
+ N_{15} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^\tau
$$

$$
+ N_{16} \left( \int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt \right)^{\frac{\tau - \sigma}{\alpha_0}}
$$

(2.32)

for some constants $N_{11}, \ldots, N_{16}$. Also Hölder’s inequality implies

$$
\int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt
$$

$$
\leq \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^\sigma \left( \int_0^1 p \left( \phi_3^\phi - \frac{1}{\alpha_0 + 1} \right)^\kappa |p'y'|^{\sigma \kappa} \, dt \right)^{\frac{\sigma \tau}{\alpha_0}}
$$

and putting this into (2.32) yields

$$
\int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \leq N_{17} + N_{18} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^{\frac{\beta_0 \sigma}{\alpha_0}}
$$

$$
+ N_{19} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^\sigma + N_{20} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^{\frac{\sigma \tau}{\alpha_0}}
$$

$$
+ N_{21} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^\tau + N_{22} \left( \int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \right)^{\frac{\sigma}{\alpha_0}}
$$

for some constants $N_{17}, \ldots, N_{22}$. Now since $\max \{ \sigma \beta_0 \alpha_0, \sigma, \sigma \gamma \alpha_0, \tau, \sigma / \alpha_0 \} < 1$ then there exists a constant $N_{23}$ with

$$
\int_0^1 p\phi_6^\kappa |p'y'|^{\tau \kappa} \, dt \leq N_{23}
$$

and essentially the same reasoning as in Theorem 2.1 establishes the result.
Case (ii). \( \phi_6 \equiv 0 \) on \([0, 1]\).

We may assume without loss of generality that \( \sigma > 0 \) and \( \phi_3 > 0 \) a.e. on \([0, 1]\); otherwise the result is easy. Then (2.31) for \( t \in (0, 1) \) becomes

\[
|p(t)y'(t)| \leq N_{23} + N_{24} \left( \int_0^1 p\phi|y|^{\alpha_0+1} |y'| dt \right)^{\frac{1}{\alpha_0+1}}
\]

\[
+ N_{25} \left( \int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \right)^{\frac{1}{\kappa}}
\]

\[
+ N_{26} \left( \int_0^1 p\phi|y|^{\alpha_0+1} dt \right)^{\frac{\gamma}{\alpha_0+1}} + N_{27} \left( \int_0^1 p\phi|y|^{\alpha_0+1} dt \right)^{\frac{1}{\alpha_0+1}}
\]

for some constants \( N_{23}, \ldots, N_{27} \). This together with (2.30) leads to

\[
\int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt
\]

\[
\leq N_{28} + N_{29} \left( \int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \right)^{\frac{\beta_0 \sigma}{\alpha_0}}
\]

\[
+ N_{30} \left( \int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \right)^{\sigma}
\]

\[
+ N_{31} \left( \int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \right)^{\frac{\gamma}{\alpha_0}}
\]

\[
+ N_{32} \left( \int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \right)^{\frac{\sigma}{\alpha_0}}
\]

for some constants \( N_{28}, \ldots, N_{32} \). Thus there exists a constant \( N_{33} \) with

\[
\int_0^1 p \left( \frac{\phi_3 \phi - 1}{\alpha_0+1} \right)^{\kappa} |y'|^{\sigma \kappa} dt \leq N_{33}
\]

and the result follows as in Theorem 2.1. \( \square \)

The next theorem establishes the existence of a nonnegative solution to

\[
\begin{align*}
\frac{1}{p}(py')' + \lambda_0 qy &= \psi(t)f(t, y, py'), \quad 0 < t < 1 \\
y &\in (SL) \text{ or } (P)
\end{align*}
\]

where \( \lambda_0 \) is the first eigenvalue of (2.2) with \( r \equiv 0 \) and \( q, \psi \) satisfies

\[
q, \psi \in L_p^1[0, 1] \text{ with } q, \psi > 0 \text{ on } (0, 1).
\]

Let

\[
H^*_{\alpha_0, \theta}(u_1) = \begin{cases} 
  u_1^{\theta+1}, & 0 \leq u_1 \leq 1 \\
  u_1^{\alpha_0+1}, & 1 < u_1 < \infty.
\end{cases}
\]
Theorem 2.3. Let \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) be continuous with (1.3), (2.34) and

\[
(2.35) \quad f(t, 0, 0) \leq 0
\]

holding. Suppose \( \psi f \) has the decomposition \( \psi(t)f(t, u_1, u_2) = g(t, u_1, u_2) + h(t, u_1, u_2) \) with \( pg, ph : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) \( L^1 \)-Carathéodory functions and

\[
(2.36) \quad \begin{cases}
\text{there exist constants } A > 0, \alpha_0 > 0 \text{ and a function } \phi \in L^1_p[0, 1], \\
\phi > 0 \text{ on } (0, 1) \text{ with } u_1 g(t, u_1, u_2) \geq A \phi(t) H_{\alpha_0, \theta}^*(u_1) \\
\text{for } t \in (0, 1), u_1 \geq 0 \text{ and } u_2 \in \mathbb{R}; \text{ here } \alpha_0 \leq \theta
\end{cases}
\]

\[
(2.37) \quad \begin{cases}
\text{there exist } \phi_i \in L^1_p[0, 1], i = 1, 2, 3 \text{ and constants } \beta_0 \text{ and } \\
\text{and } \sigma \text{ with } |h(t, u_1, u_2)| \leq \phi_1(t) + \phi_2(t) u_1^{\beta_0} + \phi_3(t) u_2^\sigma \text{ for } \\
t \in (0, 1), u_1 \geq 0 \text{ and } u_2 \in \mathbb{R}; \text{ here } \beta_0 < \alpha_0 \\
\text{and } \phi_3 > 0 \text{ on } (0, 1) \text{ or } \phi_3 \equiv 0
\end{cases}
\]

and

\[
(2.38) \quad \begin{cases}
\text{there exist } \phi_i \in L^1_p[0, 1], i = 4, 5, 6 \text{ and constants } \gamma \leq \alpha_0, \tau > \sigma \\
\text{with } |g(t, u_1, u_2)| \leq \phi_4(t) + \phi_5(t) u_1^\gamma + \phi_6(t) u_2^\tau \text{ for } t \in (0, 1), u_1 \geq 0 \\
\text{and } u_2 \in \mathbb{R} \text{ and } \phi_6 > 0 \text{ on } (0, 1) \text{ or } \phi_6 \equiv 0
\end{cases}
\]

hold. Finally suppose (2.25) and (2.26) are satisfied. Then (2.33) has at least one nonnegative solution \( y \in C[0, 1] \cap C^1(0, 1) \) with \( py' \in AC[0, 1] \).

**Proof:** Consider the family of problems

\[
(2.39)_\lambda \quad \begin{cases}
\frac{1}{p}(py')' + \mu qy = \lambda f^*(t, y, py'), 0 < t < 1 \\
y \in (SL) \text{ or } (P)
\end{cases}
\]

where \( 0 < \lambda < 1 \) and

\[
\mu = \begin{cases}
0 & \text{if } y \in (SL) \text{ and } \alpha^2 + a^2 > 0 \\
-1 & \text{if } y \in (P) \text{ or } y \in (SL) \text{ with } \alpha = a = 0.
\end{cases}
\]

Also

\[
f^*(t, u_1, u_2) = \begin{cases}
\psi(t)f(t, u_1, u_2) + (\mu - \lambda_0)qu_1, \ u_1 \geq 0 \\
\psi(t)f(t, 0, u_2) + (\mu + 1)qu_1, \ u_1 < 0.
\end{cases}
\]

Notice \( pf^* : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) is an \( L^1 \)-Carathéodory function.

Let \( y \) be a solution to \((2.39)_\lambda\) for some \( 0 < \lambda < 1 \). We **claim** that \( y \geq 0 \) on \([0, 1]\). If not then \( y \) would have a negative absolute minimum somewhere on \([0, 1]\),
say at $t_0$. If $t_0 \in (0, 1)$ then $y'(t_0) = 0$ and this together with the differential equation and (2.35) yields

$$y''(t_0) = \frac{1}{p(t_0)} (p(t_0)y'(t_0))' = \lambda (\psi(t_0)f(t_0, 0, 0) + q(t_0)y(t_0)) + (\lambda - 1)\mu q(t_0)y(t_0) < 0,$$

a contradiction. Next suppose the negative absolute minimum were to occur at $t_0 = 0$. Consider first the Sturm Liouville boundary condition. Of course we need only consider $\beta \neq 0$. If $\alpha \neq 0$ as well then

$$y(0) \lim_{t \to 0^+} p(t)y'(t) = \frac{\alpha}{\beta} y^2(0) > 0,$$

which implies $y^2(t)$ is an increasing function near 0, a contradiction. So it remains to consider the case $\alpha = 0$ and $\beta \neq 0$. The boundary condition is $\lim_{t \to 0^+} p(t)y'(t) = 0$. Now $f(0, 0, 0) \leq 0$ and this together with the differential equation and (2.34) implies there exists $\delta > 0$ with $(p(t)y'(t))' < 0$ for $t \in (0, \delta)$. Thus the boundary condition implies $p(t)y'(t) < 0$ for $t \in (0, \delta)$, a contradiction. Consequently $t_0 \neq 0$. A similar argument shows $t_0 \neq 1$. Thus our claim is established for Sturm Liouville boundary data.

Consider now Periodic boundary data. If the absolute minimum of $y$ occurs at $t_0 = 0$ then, since $y(0) = y(1)$, it must also occur at 1. Thus $\lim_{t \to 0^+} p(t)y'(t) \geq 0$ and $\lim_{t \to 1^-} p(t)y'(t) \leq 0$. Consequently

$$\lim_{t \to 0^+} p(t)y'(t) = \lim_{t \to 1^-} p(t)y'(t) = 0$$

because of the second boundary condition. As above there exists $\delta > 0$ with $(p(t)y'(t))' < 0$ for $t \in (0, \delta)$ and so $p(t)y'(t) < 0$ for $t \in (0, \delta)$, a contradiction.

Thus $y \geq 0$ on $[0, 1]$ for any solution $y$ to (2.39). Consequently $y$ satisfies

$$\frac{1}{p}(py')' + \mu qy = \lambda (\psi(t)f(t, y, py') + (\mu - \lambda_0)qy), \ 0 < t < 1.$$

Essentially the same reasoning as in Theorem 2.2 (in this case we look at $\int_0^1 p\phi y^{\alpha_0+1} \, dt$) guarantees the existence of a solution $y$ to (2.39)$_1$. Of course $y$ is automatically a solution of (2.33) since $y \geq 0$ on $[0, 1]$. $\square$

Existence theory II.

In this subsection we examine the resonant problem (2.1) on the “right” of the eigenvalue.

**Theorem 2.4.** Let $p f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be an $L^1$-Carathéodory function with (1.3) and (1.9) holding. Suppose $f$ has the decomposition $f(t, u_1, u_2) =$
Consider the family of problems

\[(2.40) \begin{cases} \text{there exist constants } A > 0, 0 < \alpha_0 < 1 \text{ and a function} \\ \phi \in L^1_{\mu}[0,1], \phi > 0 \text{ a.e. on } [0,1] \text{ with} \\ u_1 g(t, u_1, u_2) \leq -A\phi(t)H_{\alpha_0, \theta}(u_1) \text{ for a.e. } t \in [0,1]; \text{ here } \alpha_0 \leq \theta \end{cases} \]

holds. In addition assume (2.4), (2.5), (2.6) and (2.7) are satisfied. Then (2.1) has at least one solution \( y \in C[0,1] \cap C^1(0,1) \) with \( py' \in AC[0,1] \).

**Proof:** Consider the family of problems

\[(2.41) \begin{cases} \frac{1}{p}(py')' + ry + \mu qy = \lambda[f(t, y, py') + (\mu - \lambda_m)qy] \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (P) \end{cases} \]

where \( 0 < \lambda < 1 \) and \( \lambda_m < \mu < \lambda_{m+1} \).

Notice \( L^2_{pq}[0,1] = \Gamma \oplus \Gamma^\perp \) where \( \Gamma = \text{span} \{ \psi_0, \psi_1, \ldots, \psi_m \} \). Multiply (2.41) by \( w - u \) and integrate from 0 to 1 to obtain as in Theorem 2.1 (\( Q_0 \) is as in Theorem 2.1)

\[
Q_0 + \int_0^1 \left[ -p(w')^2 + prw^2 + \mu pqw^2 \right] dt + \int_0^1 \left[ p(u')^2 - pru^2 - \mu pu^2 \right] dt \\
= \lambda \int_0^1 (w - u)pf(t, y, py') dt + \lambda(\mu - \lambda_m) \int_0^1 pqw^2 dt \\
- \lambda(\mu - \lambda_m) \int_0^1 pqu^2 dt.
\]

Now since \( u \in \Gamma, w \in \Gamma^\perp \) and \( y = u + w \) we have

\[ u = \sum_{i=0}^{m} c_i \psi_i \text{ and } w = \sum_{i=m+1}^{\infty} c_i \psi_i \text{ where } c_i = \langle y, \psi_i \rangle. \]

Also as before

\[
Q_0 + \int_0^1 \left[ -p(w')^2 + prw^2 + \mu pqw^2 \right] dt + \int_0^1 \left[ p(u')^2 - pru^2 - \mu pu^2 \right] dt \\
\leq (\mu - \lambda_{m+1}) \int_0^1 pqw^2 dt + (\lambda_m - \mu) \int_0^1 pqu^2 dt
\]

so putting this into (2.42) yields

\[
\lambda \int_0^1 (w - u)pg(t, y, py') dt + (1 - \lambda)(\mu - \lambda_m) \int_0^1 pqw^2 dt \\
+ (\lambda_{m+1} - \mu) \int_0^1 pqw^2 dt + \lambda(\mu - \lambda_m) \int_0^1 pqw^2 dt \\
\leq -\lambda \int_0^1 (w - u)ph(t, y, py') dt.
\]
Now \( w - u = -y + 2w \) and \(-yg(t, y, py') \geq A \phi(t) H_{\alpha_0, \theta}(y) \) for a.e. \( t \in [0, 1] \) so with the above we have

\[
A \int_0^1 p \phi H_{\alpha_0, \theta}(y) \, dt + (\mu - \lambda_m) \int_0^1 pqw^2 \, dt \leq -2 \int_0^1 pwg(t, y, py') \, dt \\
+ \int_0^1 p |y| |h(t, y, py')| \, dt \\
+ 2 \int_0^1 p |w| |h(t, y, py')| \, dt.
\]

Essentially the same reasoning as in Theorem 2.1 (the only difference is that we use \( \int_0^1 pqw^2 \, dt \) in place of \( \int_0^1 pqu^2 \, dt \)) establishes the result. \( \square \)

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