Zbigniew S. Kowalski
Minimal generators for aperiodic endomorphisms

Commentationes Mathematicae Universitatis Carolinae, Vol. 36 (1995), No. 4, 721--725

Persistent URL: http://dml.cz/dmlcz/118799

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1995

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
Minimal generators for aperiodic endomorphisms

Zbigniew S. Kowalski

Abstract. Every aperiodic endomorphism \( f \) of a nonatomic Lebesgue space which possesses a finite 1-sided generator has a 1-sided generator \( \beta \) such that \( k_f \leq \text{card } \beta \leq k_f + 1 \). This is the best estimate for the minimal cardinality of a 1-sided generator. The above result is the generalization of the analogous one for ergodic case.

Keywords: aperiodic endomorphism, 1-sided generator
Classification: 28D05

0. Introduction

Let \( f \) be an aperiodic endomorphism of a nonatomic Lebesgue space \((X, \mathcal{B}, \mu)\). Let \( f^{-1}\varepsilon \) denote the partition \( \{f^{-1}(x) : x \in X\} \) and let \( \{m_{f^{-1}(x)}\}_{x \in X} \) be the canonical system of measures. Denote by \( h(f) \) the entropy of \( f \). If \( H(\varepsilon | f^{-1}\varepsilon) = h(f) < \infty \), then the canonical measures are purely atomic. In this case we define a number \( k_f \) in the following way:

\[
k_f = \min \{ k : \text{card } \{ y : y \in f^{-1}(x) \text{ and } m_{f^{-1}(x)}(y) > 0 \} \leq k \text{ a.e.} \}.
\]

The aim of this paper is to prove Theorem 1 [2] without assumptions of ergodicity of \( f \).

Theorem 1. An aperiodic endomorphism \( f \) has a finite 1-sided generator iff \( H(\varepsilon | f^{-1}\varepsilon) = h(f) < \infty \text{ and } k_f < \infty \). Moreover, if \( f \) admits a finite 1-sided generator, then there exists a 1-sided generator \( \beta \) such that \( k_f \leq \text{card } \beta \leq k_f + 1 \).

We prove out theorem by using the ergodic decomposition of \( \mu \) and some ideas of [1] and [2].

1. Preparation for the proof

We say that a set \( B \in \mathcal{B} \) is invariant if \( \mu(f^{-1}B \triangle B) = 0 \). Let \( \mathcal{A} \) be the \( \sigma \)-field of invariant sets and let \( \alpha \) be the measurable partition of \( X \) determined by \( \mathcal{A} \). Let \( \{\mu_A\}_{A \in \alpha} \) be the canonical system of measures for \( \alpha \). The family of dynamical systems \( (A, \mathcal{B}_A, \mu_A, f \mid A)_{A \in \alpha} \) is called the ergodic decomposition of \((X, \mu, f)\). For the next considerations we need the following lemma.
Lemma 1. Let $\gamma$ and $\beta$ be measurable partitions such that $\gamma < \beta$. Moreover, let $\{\mu_G\}_{G \in \gamma}$, $\{\mu_B\}_{B \in \beta}$ be the canonical system of measures for $\gamma$, $\beta$ respectively. If $\{\tilde{\mu}_B\}_{B \in \beta \cap G}$ denotes the canonical system of measures with respect to $\mu_G$ then $\tilde{\mu}_B = \mu_B$ a.e. for a.e. $G$.

Proof: Let $G$ and $B$ be the $\sigma$-fields for $\gamma$ and $\beta$ respectively. When we ignore a set of atoms of $\mu$ measure zero then

$$\int f \, d\mu_G = E(f \mid G) \mid G \quad \text{and} \quad \int f \, d\mu_B = E(f \mid B) \mid B,$$

for every $f \in L^1(\mu)$. We have also

$$E(E(f \mid B) \mid G) = E(f \mid G),$$

by $G \subset B$. Therefore

$$\int f \, d\mu_G = E(f \mid G) \mid G = E(E(f \mid B) \mid G) \mid G = \int E(f \mid B) \, d\mu_G =$$

$$= \int E(f \mid B) \mid B \, d\mu_G(B) = \int \int f \, d\mu_B \, d\mu_G(B).$$

The above implies that $\{\mu_B\}_{B \in \beta \cap G}$ is the canonical system of measures with respect to $\mu_G$ and therefore $\tilde{\mu}_B = \mu_B$ a.e. $\square$

Let $\beta$ denote the finite partition of $X$ and $h(\beta, f)$ the entropy of $f$ with respect to $\beta$. Besides, let us denote by $h_A(\beta, f)$ the entropy of $f \mid A$ with respect to $\beta \mid A$.

Theorem 2 [1]. $h(\beta, f) = \int h_A(\beta, f) \, d\mu_\alpha(A)$,

$h(f) = \int h_A(f) \, d\mu_\alpha(A)$.

Let $J_f(x)$ denote the Jacobian for $f$, i.e.

$$J_f(x) = (m_f^{-1}(f(x))(x))^{-1} \quad (\text{see [5]}).$$

Then

$$H(\varepsilon \mid f^{-1}\varepsilon) = \int \log J_f \, d\mu = \int \int_A \log J_f \, d\mu_A \, d\mu_\alpha$$

$$= \quad \text{(by Lemma 1)} \quad = \int H_A(\varepsilon \mid f^{-1}\varepsilon) \, d\mu.$$

If $h(f) < \infty$ then due to Theorem 2 we get the following equivalence

(1) $H(\varepsilon \mid f^{-1}\varepsilon) = h(f)$ iff $H_A(\varepsilon \mid f^{-1}\varepsilon) = h_A(f)$ a.e.
2. Proof of Theorem 1

We are in the position to prove the part of Theorem 1 (the necessity). Namely, if \( f \) possesses a 1-sided finite generator \( \beta \) then \( h(f) < \infty \) and \( \beta | A \) is the 1-sided generator for a.e. \( A \in \alpha \). Therefore \( H_A(\varepsilon \mid f^{-1}\varepsilon) = h_A(f) < \infty \) for a.e. \( A \) ([3, p. 97]) and by (1) \( H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty \). Since \( \beta \) is the 1-sided generator, \( f | B \) is 1–1 for every \( B \in \beta \). Therefore \( k_f \leq \text{card} \ \beta \).

In order to prove the sufficiency part of Theorem 1 we show that the conditions \( H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty \) and \( k_f < \infty \) imply that \( f \) possesses a 1-sided generator \( \beta \) such that \( k_f \leq \text{card} \ \beta \leq k_f + 1 \). Let \( \beta = \{B_1, \ldots, B_{k_f}\} \) be a partition such that \( \beta \cup f^{-1}\varepsilon = \varepsilon \). We obtain the partition as above by using the following construction of Rohlin [4]: \( B_1 \cap f^{-1}(x) \) consists of an atom of the greatest \( m_{f^{-1}} \) measure, next \( B_2 \cap f^{-1}(x) \) consists of an atom of the greatest measure in \( f^{-1}(x) - B_1 \), etc.

If \( \beta \cup f^{-1}\varepsilon = \varepsilon \) then \( \beta | A \) has the same property for a.e. \( A \). By Lemma 1 [2] we have also

\[
(2) \quad h_A(\beta, f) = h_A(f) \text{ a.e.}
\]

Due to (1) and by our assumptions

\[
h_A(\beta, f) = h_A(f) = H_A(\varepsilon \mid f^{-1}\varepsilon) \text{ a.e.}
\]

Let \( \overline{f} \) denote the natural extension of \( f \) to an automorphism. The transformation \( \overline{f} \) is an aperiodic automorphism of the measurable space \((\overline{X}, \overline{B}, \overline{\mu})\), where \( \overline{B} \) is an exhaustive \( \sigma \)-algebra of \( \overline{B} \). The ergodic decomposition \( \{\mu_A\}_{A \in \alpha} \) lifts to the ergodic decomposition of \( \overline{f} \). We will denote it by \( \{\overline{\mu}_x\}_{x \in \overline{X}} \). Here \( \overline{\mu}_x = \overline{\mu}_A \) for \( x \in \overline{A} \). To obtain the sufficiency part of Theorem 1 we need (as in [2]) the following lemma.

**Lemma 2.** Let \( \beta = \{B_1, \ldots, B_{k_f}\} \) be a partition such that \( h(\beta, \overline{f}) = h(\overline{f}) \) and \( \beta \subseteq B \). Then there exists a partition \( \{A_1, A_2\} \) of \( B_1 \) such that \( \{A_1, A_2\} \subseteq B \) and \( \gamma = \{A_1, A_2, B_2, \ldots, B_{k_f}\} \) is a generator for \( \overline{f} \).

**Proof:** Let us begin with presentation of the general idea of the proof. Let \( H_0 = B_1 \). We take certain Rohlin tower in \( H_0 \) with basis from \( B \). The tower is given by induced transformation \( \overline{f}_{H_0} \). We suitably label the part of levels of the tower by elements of the set \( \{0, 1\} \). The union of remaining levels will be denoted by \( H_1 \). Next, we repeat the same reasoning with \( H_1 \) and etc. Here we care for the measure of \( H_i \) to tend to zero as \( i \) tends to infinity. For coding we use ergodic theorem, Shannon-McMillan-Breiman theorem and the equality \( h_x(\beta, \overline{f}) = h(\overline{f}) \) a.e. Consequently the set \( A_i \) is the union of levels with label \( i \) for \( i = 0, 1 \). This construction is modification of the proof of Theorem 30.1 [1] and in consequence of the proof of Theorem 28.1 [1]. The detailed presentation of the construction needs the reproduction of the proofs of these theorems. Therefore we enclose below only necessary modifications of the proofs of Theorems 28.1 and 30.1 [1].
At first we observe by (2) that $h_x(f) \leq \log kf$ a.e. Now, we specify our modifications.

(i) For a sequence $(\gamma_i)$ of partitions we assume additionally that $\beta < \gamma_1$ and $\gamma_i \subseteq B$ for $i = 1, 2, \ldots$.

(ii) $Z_i = \emptyset$ for $i = 1, 2, \ldots$.

(iii) We start with $G_0 = \{B_2 \cup \cdots \cup B_{k_f}\}$, $H_0 = B_1$.

(iv) Let $M = \{0, 1\}^Z$ and $K = \{1\}^Z$. There exists a subshift $M$ of a finite type (see Lemma 26.17 [1]) such that $h(M) \geq \frac{1}{2}h(M) = \frac{1}{2}\log 2$ and $M \cap K = \emptyset$. Therefore there exists $L$ such that $a = [1 \ldots 1] \notin M$. As the blocks $U^1_p, U^2_p$ (see [1, p. 283]) we take block $[0]$. We start with $f^{-1}H_0$-Rohlin set $F_1 \subset H_0$ such that $F_1 \in B$ and $f^{-1}(F_1) = \text{const.}$ a.e. We use the same coding method as in the proof of Theorem 28.1 [1] with respect to $M, M$ and $a, U^i_p, i = 1, 2$, as above. By the definition of the first step we have $\gamma \cap G_i > \beta$ for $i = 1, 2, \ldots$ and hence $h_x(\gamma \cap G_i, f) = h_x(f)$ a.e. Therefore the condition (a) ([1, p. 311]) is always satisfied.

(v) In the step (i), for $i \geq 2$, we take $f^{-1}H_i$-Rohlin set $F_i \subset H_i$ such that $F_i \in B$. For coding we use $\overline{f^{-1}H_i}$ instead of $F_i$.

For the proof it suffices to apply the reasoning from the proof of Theorem 30.1 with the above modifications (i)-(v). Consequently we get the generator $\gamma = \{A_0, A_1, B_2, \ldots, B_{k_f}\}$ for $f$. It remains only to show that $\gamma \subseteq B$. Assume that $\gamma \cap G_{i-1} \subseteq B$ for some $i \geq 1$. Then $H_{i-1} \in B$, $\bigvee_{i=0}^{q_i-1} f^{-i} \gamma_i \subseteq B$, $\bigvee_{i=0}^{q_i-1} f^{-i} (\gamma \cap G_{i-1}) \subseteq B$. We code $F_i = \overline{f^{-n_i}H_i} F_i$ by adjoining to every $\overline{f^{-n_i}H_i} \gamma' \cap F_i \subset \overline{f^{-n_i}H_i} \gamma' \cap F_i$ a different $M$-block of length $n_i - k_i - c$ for $\gamma' \in S'_i \subseteq B$, $A'' \in S''_i \subseteq B$. Therefore $\gamma \cap G_i \subseteq B$. It follows that $\gamma \subseteq B$.  

By Lemma 2 we conclude (as in [2]) that $\gamma$ such that $\gamma \vee f^{-1} \epsilon = \epsilon$ is a 1-sided generator for $f_A$ a.e. and consequently is a 1-sided generator for $f$.

**References**


Institute of Mathematics, Technical University of Wroclaw, 50-370 Wroclaw, Poland

E-mail: kowalski@im.pwr.wroc.pl