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Commentationes Mathematicae Universitatis Carolinae, Vol. 36 (1995), No. 4, 805--807

Persistent URL: <http://dml.cz/dmlcz/118807>

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On a condition weaker than insatiability condition

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Abstract. A condition weaker than the insatiability condition is given.

Keywords: economy, attainable state, insatiability condition, satiable consumption

Classification: 90A14, 90D13

An economy ε is defined by: m consumers indexed by $i = 1, 2, \dots, m$; n producers indexed by $j = 1, 2, \dots, n$; for each $i = 1, 2, \dots, m$ a consumption set (X_i, \preceq_i) , where X_i is a nonempty subset of \mathbf{R}^ℓ the production set for the producer j , and a priori vector $w \in \mathbf{R}^\ell$, called the total resources of ε . A state of economy ε is an $(m+n)$ -tuple of \mathbf{R}^ℓ , which can be represented by a point of $\mathbf{R}^{(m+n)\ell}$.

A state $(x, y) = ((x_i), (y_j))$ of ε is called attainable if $\sum_{i=1}^m x_i - \sum_{j=1}^n y_j = w$. The set of all attainable states of an economy ε will be denoted by A . An increasing function $u_i : X_i \rightarrow \mathbf{R}$ is called a utility function (i.e. $x_i, x'_i \in X_i$ with $x_i \preceq_i x'_i \Rightarrow u_i(x_i) \leq u_i(x'_i)$).

In this note we consider the economy $\varepsilon = ((X_i, \preceq_{u_i}), (Y_j), w)$, where X_i, \preceq_{u_i}, Y_j and w are defined as above, i.e. we are assuming that each preference preordering \preceq_i can be represented by a utility function u_i . The utility function u_i is said to satisfy the insatiability condition if u_i has no greatest element with respect to \preceq_{u_i} . The greatest element of \preceq_{u_i} is called a satiation consumption. Finally, a real valued function f defined on a convex set Y is said to be quasiconvex if for each real number t , the set $\{y \in Y : f(y) > t\}$ is either empty or convex.

Any other term or concept which is not defined here can be found in Debreu [1]. In [2] and [3] the author has proved the existence of Pareto optimum of an economy under the following condition (P) instead of insatiability condition:

If $(x, y) = ((x_i), (y_j))$ and $(x', y') = ((x'_i), (y'_j))$ are two attainable states of an economy $\varepsilon = ((X_i, \preceq_{u_i}), (Y_i), w)$ such that $u_i(x_i) \geq u_i(x'_i)$ for all i and $u_i(x_i) > u_i(x'_i)$ for at least one i then there is an attainable state

$(\bar{x}, \bar{y}) = ((\bar{x}_i), (\bar{y}_j))$ of ε such that $u_i(\bar{x}_i) > u_i(x'_i)$ for each $i = 1, 2, \dots, m$.

The object of this note is to prove that under the usual conditions on the economy ε the condition (P) is weaker than the insatiability condition, i.e. the insatiability condition implies the condition (P). Thus the results proved in [2] is more general than the corresponding results of Debreu [1].

We first prove the following lemma.

Lemma 1. *If $(x, y) = ((x_i), (y_j))$ and $(x', y') = ((x'_i), (y'_j))$ are two attainable states of an economy $\varepsilon = ((X_i, \preceq_{u_i}), (Y_j), w)$, where each X_i is connected and no consumption is satiated and each u_i is continuous and if $u_i(x_i) \geq u_i(x'_i)$ for all i and $u_i(x_i) > u_i(x'_i)$ for at least one i , then there is a state $(\bar{x}, \bar{y}) = ((\bar{x}_i, \bar{y}_j))$ such that $u_i(\bar{x}_i) > u_i(x'_i)$ for each $i = 1, 2, \dots, m$.*

PROOF: Let $J \subset \{1, 2, \dots, m\}$ such that $u_i(x_i) > u_i(x'_i)$ for all $i \in J$ and $K \subset \{1, 2, \dots, m\}$ such that $i \notin J$, i.e. $u_i(x) = u_i(x'_i)$ for all $i \in K$. Now we choose a number $\epsilon > 0$ such that $\epsilon < \min\{u_i(x_i) - u_i(x'_i) : i \in J\}$.

Since for each $i = 1, 2, \dots, m$, X_i is connected and u_i is continuous and no consumption is satiated, it is possible to choose $\bar{x} = (\bar{x}_i)$ such that

$$u_i(\bar{x}_i) = \begin{cases} u_i(x'_i) + \frac{\epsilon}{s} & \text{if } i \in K; \\ u_i(x_i) + \frac{\epsilon}{r} & \text{if } i \in J, \end{cases}$$

where s and r denote the cardinality of K and J respectively.

Now it is clear that $u_i(\bar{x}_i) > u_i(x'_i)$ for each $i = 1, 2, \dots, m$ and also for the sake of interest we note that

$$\begin{aligned} \sum_{i=1}^m \bar{u}_i(\bar{x}_i) &= \sum_{i \in K} u_i(\bar{x}_i) + \sum_{i \in J} u_i(\bar{x}_i) \\ &= \sum_{i \in K} u_i(x'_i) + \epsilon + \sum_{i \in J} u_i(x_i) - \epsilon \\ &= \sum_{i \in K} u_i(x_i) + \sum_{i \in J} u_i(x_i) = \sum_{i=1}^m u_i(x_i). \end{aligned}$$

□

Theorem 1. *Let $\varepsilon = ((X_i, \preceq_{u_i}), (Y_j), w)$ be an economy such that*

- (a) *for each $i = 1, 2, \dots, m$*
 - (i) X_i *is convex;*
 - (ii) u_i *is continuous and quasiconcave;*
 - (iii) u_i *is insatiable;*
- (b) $Y = \sum_{j=1}^n Y_j$ *is convex.*

Then ε satisfies the condition (P).

PROOF: Let $(x, y) = ((x_i), (y_j))$ and $(x', y') = ((x'_i), (y'_j))$ be two attainable states of ε such that $u_i(x_i) \geq u_i(x'_i)$ for all i and $u_i(x_i) > u_i(x'_i)$ for at least one i . For each $i = 1, 2, \dots, m$, let $O_i(x'_i) = \{\bar{x}_i \in X_i : u_i(\bar{x}_i) > u_i(x'_i)\}$. Then for each $i = 1, 2, \dots, m$, $O_i(x'_i)$ is a nonempty open subset of X_i by virtue of the continuity of u_i and the Lemma.

In order to prove the theorem it will suffice to prove that $w \in \sum_{i=1}^m O_i(x'_i) - Y$. We prove it by contradiction.

If possible, let $w \notin \sum_{i=1}^m O_i(x'_i) - Y = Z$. Since by quasi concavity of u_i , $O_i(x'_i)$ is convex and by (b) Y is convex, it follows that Z is convex. Hence by Minkowski's theorem (see Debreu [1, p. 25]) there is a hyperplane H through w bounding Z , i.e. there is $p \in \mathbf{R}^\ell$ such that $p \neq 0$ and $p \cdot a \geq p \cdot w$ for every $a \in Z$ where \cdot is the inner product in \mathbf{R}^ℓ . Now by the continuity of each u_i , it follows that $G = \sum_{i=1}^m C_i(x'_i) - Y$ is contained in $C = \sum_{i=1}^m \overline{O_i(x'_i)} - \sum_{j=1}^n Y_j$ where for each $i = 1, 2, \dots, m$, $C_i(x'_i) = \{\bar{x}_i \in X_i : u_i(\bar{x}_i) \geq u_i(x'_i)\}$. Hence it follows that $\sum_{i=1}^m C_i(x'_i) - Y$ is contained in C and hence in the closed half space above the hyperplane H . Now since $w = x' - y' \in G$, it minimizes $p \cdot a$ on G . Hence x'_i minimizes $p \cdot a$ on $C_1(x'_i)$ for each i and $-y'_j$ minimizes $p \cdot a$ on $-Y_j$ (see e.g. Section 3.4 in [1, p. 45]). Hence by the result stated in [1, p. 93], $((x'_i), (y'_j))$ is an equilibrium with respect to the price p and by (6.3) in Debreu [1, p. 94], $((x'_i), (y'_j))$ is a Pareto optimum which is impossible. Hence $w \in Z$. \square

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(Received June 24, 1994)