Paolo Cubiotti

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# A counter-example to some recent existence results on implicit variational inequalities

Paolo Cubiotti

*Abstract.* In this note we prove that some recent results on an implicit variational inequality problem for multivalued mappings, which seem to extend and improve some well-known and celebrated results, are not correct.

*Keywords:* quasi-variational inequalities, lower semicontinuity, partition of unity, minimax

*Classification:* 49J40

## 1. Introduction

Very recently, in [1], J. Fu introduced the following implicit variational inequality problem for multivalued mappings; given two topological vector spaces $X$ and $Y$, two nonempty subsets $C$ and $D$ of $X$ and $Y$, respectively, two multivalued mappings $E : C \to 2^C$ and $F : C \to 2^D$, two real functions $f : C \times C \times D \to \mathbf{R}$ and $g : C \times C \to \mathbf{R}$, with $f(x, x, y) \geq 0$ for any $x \in C$ and any $y \in F(x)$, find $(v, u) \in C \times D$ such that

$$(1)\quad v \in E(v),\ u \in F(v) \ \text{ and } \ g(v, v) \leq f(v, w, u) + g(v, w) \ \text{ for all } \ w \in E(v).$$

Such problem extends an implicit variational problem studied by Mosco [2].

In [1], Fu stated the following assertion which he employed to obtain existence results for problem (1) and for some special cases as quasi-variational inequalities (for the basic definitions, we refer to [1]).

**Assertion A** (Theorem 1 of [1]). *Let $X$, $Y$ be Hausdorff locally convex topological vector spaces, $C$ be a nonempty compact convex set of $X$ and $D$ be a nonempty closed convex set of $Y$. Let $E : C \to 2^C$ be upper hemicontinuous with nonempty closed convex values and $F : C \to 2^D$ be a mapping with nonempty values. Suppose that $f : C \times C \times D \to \mathbf{R}$ satisfies the following conditions:*

(i) *for each $x \in C$ and $y \in F(x)$, $f(x, x, y) \geq 0$;*
(ii) *for any fixed $x \in C$ and $y \in D$, the function $f(x, u, y)$ of $u$ is convex;*
(iii) *for any fixed $u \in C$, the function $\sup_{y \in F(x)} f(x, u, y)$ of $x$ is upper semicontinuous.*

*Then there exists* $x^* \in C$ *such that*

$$x^* \in E(x^*) \ \text{ and } \sup_{y \in F(x^*)} f(x^*, u, y) \geq 0 \ \text{ for all } \ u \in E(x^*).$$

The aim of this note is to point out that Assertion A, in general, is false, together with several of its consequences obtained in [1]. We shall do this by means of a simple counter-example. We shall also illustrate in detail the gap in the original proof of Assertion A.

## 2. The counter-example

The following example shows that Assertion A, in general, is false.

**Example 2.1.** Let $X = Y = D = \mathbf{R}$, $C = [0, 1]$, $F(x) \equiv \{1\}$,

$$E(x) = \begin{cases} \left[\frac{3}{4}, 1\right] & \text{if } x \in \left[0, \frac{1}{2}\right[ \\ \left[0, 1\right] & \text{if } x = \frac{1}{2} \\ \left[0, \frac{1}{4}\right] & \text{if } x \in \left]\frac{1}{2}, 1\right], \end{cases}$$

$f(x, u, y) = y(u - x)$. It is immediate to realize that all the assumptions of Assertion A are satisfied. In particular, we note that the graph of $E$ is closed, hence, since $C$ is compact, $E$ is upper semicontinuous. Therefore, by Lemma 1 of [3], $E$ is upper hemicontinuous. We note that the only fixed point of $E$ is $x^* = \frac{1}{2}$. However, we have $\sup_{y \in F(x^*)} f(x^*, u, y) = u - x^* < 0$ for all $u \in [0, \frac{1}{2}[ \subseteq E(x^*)$. Thus, Assertion A fails.

**Remarks.** (i) Example 2.1 also shows that Theorems 2, 3 and 9 of [1] are false.

(ii) We note that Theorem 2 of [1], if correct, would imply, in particular (taking into account Theorem 1.4.16 of [4]), that the celebrated Chan and Pang's existence theorem for generalized quasi-variational inequalities (see Theorem A in [5]) would be true without assuming the lower semicontinuity of the multifunction $E : C \to 2^C$. Example 2.1 shows that such improvement of Chan and Pang's result is not possible.

We note that the original proof of Assertion A (proof of Theorem 1 in [1]) is arranged as follows.

*First step.* By assuming that the conclusion is false, it is shown that there exists a finite set $\{p_1, \ldots, p_n\} \subseteq X^*$ (where $X^*$ denotes the topological dual space of $X$) such that

$$C = V_0 \cup \bigcup_{i=1}^{n} V(p_i),$$

where we put

$$V(p_i) = \Big\{x \in C : \mathrm{Re}\langle p_i, x\rangle - \sup_{z\in E(x)} \mathrm{Re}\langle p_i, z\rangle > 0\Big\},$$

and

$$V_0 = \bigcup_{u\in A} \Big\{x \in C : \sup_{y\in F(x)} f(x,u,y) < 0\Big\},$$

with

$$A = \Big\{u \in C : \exists\, x \in C \ \text{ such that } \ u \in E(x) \ \text{ and } \ \sup_{y\in F(x)} f(x,u,y) < 0\Big\},$$

and $\langle\cdot,\cdot\rangle$ is the usual pairing between $X^*$ and $X$.

*Second step.* The author considers a continuous partition of unity $\{\beta_0, \beta_1, \dots, \beta_n\}$ subordinated to the open covering $\{V_0, V(p_1), \dots, V(p_n)\}$ of $C$.

*Third step.* The author defines a function $\varphi : C \times C \to \mathbf{R}$ by setting

$$\varphi(x,u) = -\beta_0(x) \sup_{y\in F(x)} f(x,u,y) + \sum_{i=1}^{n} \beta_i(x)\mathrm{Re}\langle p_i, x-u\rangle$$

and observes that by the Ky Fan minimax principle (Theorem A of [3]) there exists $\hat{x} \in C$ such that

$$\varphi(\hat{x},u) \le 0 \ \text{ for all } \ u \in C. \tag{2}$$

*Fourth step.* The author claims that inequality (2) is a contradiction. In particular, he claims that if $\beta_0(\hat{x}) > 0$, then there exists $u_0 \in A \subseteq C$ such that $\varphi(\hat{x}, u_0) > 0$.

But the gap is exactly here, since the inequality (2) does not imply, in general, any contradiction. To see this, take $X, Y, D, C, F, f$, and $E$, as in Example 2.1. The reader can easily check that in this case we have $A = [0, \frac{1}{2}[$, $V_0 = ]0,1]$. Since $V(-1) = [0, \frac{1}{2}[$, we can take $n = 1$, $p_1 = -1$,

$$\beta_0(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ 1 & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$\beta_1(x) = \begin{cases} 1-2x & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ 0 & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

According to Ky Fan minimax principle, there exists $\hat{x} \in [0,1]$ such that

$$\varphi(\hat{x},u) = (\beta_0(\hat{x}) - \beta_1(\hat{x}))(\hat{x} - u) \le 0 \ \text{ for all } \ u \in [0,1].$$

In fact, one can take $\hat{x} = \frac{1}{4}$ since $\beta_0(\frac{1}{4}) = \beta_1(\frac{1}{4}) = \frac{1}{2}$, hence $\varphi(\frac{1}{4}, u) = 0$ for all $u \in [0,1]$. Thus, the contradiction claimed in the final part of the original proof of Assertion A does not hold.

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Department of Mathematics, University of Messina, 98166 Sant' Agata – Messina, Italy

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