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A note on Banach spaces with ℓ^1 -saturated duals

DENNY H. LEUNG

Abstract. It is shown that there exists a Banach space with an unconditional basis which is not c_0 -saturated, but whose dual is ℓ^1 -saturated.

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Let E and F be Banach spaces. We say that E is F -saturated if every infinite dimensional closed subspace of E contains an isomorphic copy of F . In [2], it is shown that there exists a c_0 -saturated Banach space with an unconditional basis whose dual contains an isomorphic copy of ℓ^2 . In this note, we give an example where the dual situation occurs. It is shown that there is a Banach space with an unconditional basis which contains an isomorphic copy of ℓ^2 , and whose dual is ℓ^1 -saturated.

We follow standard Banach space terminology as used in [3]. Our example is a certain subspace of the weak L^2 space $L^{2,\infty}[0, \infty)$. Recall that this is the space of all measurable functions f on $[0, \infty)$ such that

$$(1) \quad \|f\| = \sup_{c>0} c(\lambda\{|f| > c\})^{1/2} < \infty,$$

where λ is the Lebesgue measure on $[0, \infty)$. Although equation (1) only defines a quasi-norm on $L^{2,\infty}[0, \infty)$, it is well known that it is equivalent to a norm on $L^{2,\infty}[0, \infty)$, and that $L^{2,\infty}[0, \infty)$ is norm complete. The reader may consult [4] for further information concerning the family of Lorentz spaces, of which $L^{2,\infty}[0, \infty)$ is a member. Finally, for a measurable function f , we let f^* be the decreasing rearrangement of $|f|$, as defined in §2a of [4].

Proposition 1. *For each $n \in \mathbb{N}$, define*

$$f_n(t) = \begin{cases} \min(2^{n/2}, (t - n + 1)^{-1/2}) & \text{if } n - 1 < t \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $m \in \mathbb{N}$, and any sequence of scalars (a_n) ,

$$\left(\frac{1}{2} \sum_{n=1}^m a_n^2\right)^{1/2} \leq \left\| \sum_{n=1}^m a_n f_n \right\| \leq \left(\sum_{n=1}^m a_n^2\right)^{1/2},$$

where $\|\cdot\|$ refers to the quasi-norm defined by (1).

PROOF: Given a scalar sequence $(a_n)_{n=1}^m$, let $a = \max\{|a_n| : 1 \leq n \leq m\}$, and suppose this maximum is attained at n_0 . Let N be the set of all natural numbers $\leq m$ such that $a_n > 2^{-n/2}a$. If $0 < c < a$, and $n \in N$, then $c < 2^{n/2}a_n$. Hence $\lambda\{|a_n f_n| > c\} = \min(a_n^2/c^2, 1)$. Therefore,

$$\begin{aligned} \left\| \sum_{n=1}^m a_n f_n \right\| &\geq \sup_{0 < c < a} c \left(\lambda \left\{ \sum_{n \in N} |a_n f_n| > c \right\} \right)^{1/2} \\ &= \sup_{0 < c < a} \left(\sum_{n \in N} \min(a_n^2, c^2) \right)^{1/2} \\ &= \left(\sum_{n \in N} a_n^2 \right)^{1/2}. \end{aligned}$$

If $1 \leq n \leq m$, and $n \notin N$, then $a_n \leq 2^{-n/2}a$. Thus

$$\sum_{\substack{1 \leq n \leq m \\ n \notin N}} a_n^2 \leq a^2 \sum_{n=1}^m 2^{-n} < a^2.$$

Hence

$$\begin{aligned} \left\| \sum_{n=1}^m a_n f_n \right\| &\geq \left(\sum_{n=1}^m a_n^2 - \sum_{\substack{1 \leq n \leq m \\ n \notin N}} a_n^2 \right)^{1/2} \\ &\geq \left(\sum_{n=1}^m a_n^2 - a^2 \right)^{1/2} \\ &= \left(\sum_{\substack{1 \leq n \leq m \\ n \neq n_0}} a_n^2 \right)^{1/2}. \end{aligned}$$

Clearly, $\|\sum_{n=1}^m a_n f_n\| \geq \|a_{n_0} f_{n_0}\| = |a_{n_0}|$ as well. This proves the first half of the inequality. Observe that for any $c > 0$,

$$\lambda \left\{ \left| \sum_{n=1}^m a_n f_n \right| > c \right\} = \sum_{n=1}^m \lambda \left\{ |f_n| > \frac{c}{|a_n|} \right\} \leq \frac{1}{c^2} \sum_{n=1}^m a_n^2.$$

The second half of the inequality follows. □

For $n \in \mathbb{N}$, and $1 \leq j \leq 2^n$, let $g_{n,j}$ be the characteristic function of the interval $[n-1 + (j-1)2^{-n}, n-1 + j2^{-n})$. Let E be the closed linear span of the sequence $(g_{n,j})_{j=1}^{2^n}_{n=1}^\infty$. Clearly $(g_{n,j})_{j=1}^{2^n}_{n=1}^\infty$ is an unconditional basis of E .

Proposition 2. *The space E contains an isomorphic copy of ℓ^2 .*

PROOF: Recall the sequence (f_n) defined in Proposition 1. For each n , let $h_n = \sum_{j=1}^{2^n} \sqrt{2^n/j} g_{n,j}$. Then $0 \leq h_n \leq f_n \leq \sqrt{2} h_n$ for all n . It follows from Proposition 1 that the subspace $[\{h_n\}]$ of E is isomorphic to ℓ^2 . \square

It remains to show that E' is ℓ^1 -saturated. Let F be the closed subspace of $L^{2,\infty}[0, \infty)$ generated by $L^1 \cap L^\infty$. It is well known that F' is canonically isomorphic to $L^{2,1}[0, \infty)$, where the latter is the space of all measurable functions f on $[0, \infty)$ such that

$$\|f\|_{2,1} = \int_0^\infty \frac{f^*(t)}{\sqrt{t}} dt < \infty.$$

Let Σ be the σ -algebra generated by the sets $[n-1 + (j-1)2^{-n}, n-1 + j2^{-n})$, $1 \leq j \leq 2^n$, $n \in \mathbb{N}$. Then clearly E is the subspace of F consisting of all Σ -measurable functions. It follows easily that E' can be identified canonically with the subspace of $L^{2,1}[0, \infty)$ consisting of all Σ -measurable functions. Now if G is a subspace of E' , then it contains a basic sequence equivalent to a normalized disjointly supported sequence (u_n) in $L^{2,1}[0, \infty)$. By [1, Corollary 2.4], $\{\{u_n\}\}$, and hence G , contains a copy of ℓ^1 .

We end this note with the following problem.

Problem. Suppose E is a Banach space (with or without an unconditional basis) such that E' has the Schur property. Is E necessarily c_0 -saturated?

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