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## On $\mathcal{L}_{loc}^{2,n}$ -regularity for the gradient of a weak solution to nonlinear elliptic systems

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*Abstract.* Interior  $\mathcal{L}_{loc}^{2,n}$ -regularity for the gradient of a weak solution to nonlinear second order elliptic systems is investigated.

*Keywords:* nonlinear elliptic system, regularity, Campanato-Morrey spaces

*Classification:* 35J60

### 1. Introduction

In this paper we consider the problem of the regularity of the first derivatives of weak solutions to a nonlinear elliptic system

$$(1) \quad -D_\alpha (A_i^\alpha (Du)) = 0, \quad (i = 1, \dots, N)$$

in a bounded open set  $\Omega \subset R^n$ . Throughout the whole text we use the summation convention over repeated indexes.

If  $n \geq 3$ , it is known that  $Du$  may not be continuous. Examples are provided by nonregular solutions of elliptic systems presented by Nečas in [8]. Campanato in [2] proved that  $Du \in \mathcal{L}_{loc}^{2,\lambda}(\Omega, R^N)$  with  $\lambda(n) < n$ , and  $u \in C_{loc}^{0,\alpha}(\Omega, R^N)$  for some  $\alpha < 1$  if  $n = 3, 4$ . In this paper we give sufficient condition on  $\mathcal{L}_{loc}^{2,n}$ -regularity for the gradient of a weak solution to (1). Recall that if  $Du \in \mathcal{L}_{loc}^{2,n}$ , then  $u$  is locally Zygmund continuous.

### 2. Preliminaries

Let  $\Omega$  be a bounded open set in  $R^n$  with points  $x = (x_1, \dots, x_n)$ ,  $n \geq 3$ . The notation  $\Omega_0 \Subset \Omega$  means that the closure of  $\Omega_0$  is contained in  $\Omega$ , i.e.  $\overline{\Omega_0} \subset \Omega$ . For the sake of simplicity we denote by  $|\cdot|$  and  $(\cdot, \cdot)$  the norm and scalar product in  $R^n$ ,  $R^N$  and  $R^{nN}$ . If  $x \in R^n$  and  $r$  is a positive real number, we set  $B(x, r) = \{y \in R^n : |y - x| < r\}$ , i.e. the open ball in  $R^n$ ,  $\Omega(x, r) = B(x, r) \cap \Omega$ . By  $\mu(\Omega(x, r))$  we denote the  $n$ -dimensional Lebesgue measure of  $\Omega(x, r)$ . A bounded domain  $\Omega \subset R^n$  is said to be of type  $\mathcal{A}$  if there exists a constant  $\mathcal{A} > 0$  such that for every  $x \in \overline{\Omega}$  and all  $0 < r < \text{diam } \Omega$  it holds  $\mu(\Omega(x, r)) \geq \mathcal{A}r^n$ .

Let  $u: \Omega \rightarrow R^N$ ,  $N \geq 1$ ,  $u(x) = (u^1(x), \dots, u^N(x))$  be a vector-valued function and  $Du = (D_1u, \dots, D_nu)$ ,  $D_\alpha = \partial/\partial x_\alpha$ .

By  $u_{x,r} = \mu^{-1}(\Omega(x,r)) \int_{\Omega(x,r)} u(y) dy = \int_{\Omega(x,r)} u(y) dy$  we denote mean value of  $u$  over the set  $\Omega(x,r)$  provided that  $u \in L^1(\Omega, R^N)$ . Besides usually used spaces as  $C_0^\infty(\Omega, R^N)$ , the Hölder spaces  $C^{0,\alpha}(\overline{\Omega}, R^N)$  and the Sobolev spaces  $H^{k,p}(\Omega, R^N)$ ,  $H_{loc}^{k,p}(\Omega, R^N)$ ,  $H_0^{k,p}(\Omega, R^N)$  (see e.g. [1], [6], [7] for definitions and basic properties) we use the following Campanato and Morrey spaces.

**Definition 1** (Campanato and Morrey spaces). Let  $\lambda \in [0, n]$ ,  $q \in [1, \infty)$ . The Morrey space  $L^{q,\lambda}(\Omega, R^N)$  is the subspace of such functions  $u \in L^q(\Omega, R^N)$  for which  $\|u\|_{L^{q,\lambda}(\Omega, R^N)}^q = \sup\{r^{-\lambda} \int_{\Omega(x,r)} |u(y)|^q dy : r > 0, x \in \Omega\}$  is finite.

Let  $\lambda \in [0, n + q]$ ,  $q \in [1, \infty)$ . The Campanato spaces  $\mathcal{L}^{q,\lambda}(\Omega, R^N)$  and  $\mathcal{L}_1^{q,\lambda}(\Omega, R^N)$  are subspaces of such functions  $u \in L^q(\Omega, R^N)$  for which  $[u]_{\mathcal{L}^{q,\lambda}(\Omega, R^N)}^q = \sup\{r^{-\lambda} \int_{\Omega(x,r)} |u(y) - u_{x,r}|^q dy : r > 0, x \in \Omega\}$  is finite and  $[u]_{\mathcal{L}_1^{q,\lambda}(\Omega, R^N)}^q = \sup\{\inf\{r^{-\lambda} \int_{\Omega(x,r)} |u(y) - P(y)|^q dy : P \in \mathcal{P}_1\} : r > 0, x \in \Omega\}$  is finite. Here  $\mathcal{P}_1$  is the set of all polynomials in  $n$  variables and of degree  $\leq 1$ . Let us denote  $\|u\|_{L^{q,\lambda}}$ ,  $\|u\|_{\mathcal{L}^{q,\lambda}} = \|u\|_{L^q} + [u]_{\mathcal{L}^{q,\lambda}}$  and  $\|u\|_{\mathcal{L}_1^{q,\lambda}} = \|u\|_{L^q(\Omega, R^N)} + [u]_{\mathcal{L}_1^{q,\lambda}}$ .

*Remark 1.* It is worth to recall a trivial however important property saying that  $\int_{\Omega} |u - u_{\Omega}|^2 dx = \min\{\int_{\Omega} |u - c|^2 dx : c \in R^N\}$  for every  $u \in L^2(\Omega, R^N)$ .

**Definition 2.** The Zygmund class  $\Lambda^1(\overline{\Omega}, R^N)$  is the subspace of such functions  $u \in C^0(\overline{\Omega}, R^N)$  for which  $[u]_{\Lambda^1(\overline{\Omega}, R^N)} = \sup\{|u(x) + u(y) - 2u((x+y)/2)| : |x - y| : x, y, (x+y)/2 \in \overline{\Omega}\}$  is finite.

For more details see [1], [4], [6], [7]. In particular, we will use the following result.

**Proposition 1.** *Let  $\Omega$  be of type  $\mathcal{A}$  and  $1 \leq q < \infty$ . Then it holds*

- (a)  $L^{q,\lambda}(\Omega, R^N)$ ,  $\mathcal{L}^{q,\lambda}(\Omega, R^N)$  and  $\mathcal{L}_1^{q,\lambda}(\Omega, R^N)$  equipped with norms  $\|u\|_{L^{q,\lambda}}$ ,  $\|u\|_{\mathcal{L}^{q,\lambda}}$  and  $\|u\|_{\mathcal{L}_1^{q,\lambda}}$  are Banach spaces.
- (b)  $\mathcal{L}^{q,\lambda}(\Omega, R^N)$  is isomorphic to the  $C^{0,(\lambda-n)/q}(\overline{\Omega}, R^N)$  if  $n < \lambda \leq n + q$ ,
- (c)  $L^{q,n}(\Omega, R^N)$  is isomorphic to the  $L^\infty(\Omega, R^N) \subsetneq \mathcal{L}^{q,n}(\Omega, R^N)$ ,
- (d)  $\mathcal{L}_1^{2,n+2}(\Omega, R^N)$  is isomorphic to the  $\Lambda^1(\overline{\Omega}, R^N)$ ,
- (e)  $C^{0,1}(\overline{\Omega}, R^N) \subsetneq \Lambda^1(\overline{\Omega}, R^N) \subsetneq \bigcap_{0 < \alpha < 1} C^{0,\alpha}(\overline{\Omega}, R^N)$ .

Further, we suppose

- (i) there is an  $M > 0$  such that for every  $p \in R^{nN}$

$$(2) \quad |A_i^\alpha(p)| \leq M(1 + |p|),$$

(ii)  $A_i^\alpha(p)$  are differentiable functions on  $R^{nN}$  with the bounded and continuous derivatives, i.e.

$$(3) \quad \left| \frac{\partial A_i^\alpha}{\partial p_\beta^j}(p) \right| \leq M \quad \text{for every } p \in R^{nN},$$

(iii) the strong ellipticity condition, i.e. there exists  $\nu > 0$  such that for every  $p, \xi \in R^{nN}$

$$(4) \quad \frac{\partial A_i^\alpha}{\partial p_\beta^j}(p) \xi_\alpha^i \xi_\beta^j \geq \nu |\xi|^2.$$

From (ii) it follows (see [3, p.169]) the existence of a real function  $\omega(s)$  defined on  $[0, \infty)$ , which is nonnegative, bounded, nondecreasing, concave,  $\omega(0) = 0$  (moreover,  $\omega$  is right continuous at 0 for uniformly continuous  $\partial A_i^\alpha / \partial p_\beta^j$ ) and such that for all  $p, q \in R^{nN}$

$$(5) \quad \left| \frac{\partial A_i^\alpha}{\partial p_\beta^j}(p) - \frac{\partial A_i^\alpha}{\partial p_\beta^j}(q) \right| \leq \omega(|p - q|^2).$$

By a weak solution of (1) we mean a function  $u \in H^{1,2}(\Omega, R^N)$  satisfying

$$(6) \quad \int_{\Omega} A_i^\alpha(Du) D_\alpha \varphi^i dx = 0$$

for every  $\varphi \in H_0^{1,2}(\Omega, R^N)$ .

We will also consider the pair of complementary Young functions

$$(7) \quad \Phi(t) = t \ln_+ at \quad \text{for } t \geq 0, \quad \Psi(t) = \begin{cases} t/a & \text{for } 0 \leq t < 1, \\ e^{t-1}/a & \text{for } t \geq 1, \end{cases}$$

where  $a > 0$  is a constant,  $\ln_+ at = 0$  for  $0 \leq t < 1/a$  and  $\ln_+ at = \ln at$  for  $t \geq 1/a$ . Recall Young's inequality

$$ts \leq \Phi(t) + \Psi(s), \quad t, s \geq 0.$$

For our consideration we also need to introduce quasiconvex functions.

**Definition 3** ([5, p.4]). A function  $G: [0, \infty) \rightarrow R$  is said to be quasiconvex (quasiconcave) on  $[0, \infty)$  if there exist a convex (concave) function  $g(\tilde{g})$  and a constant  $c > 0$  ( $\tilde{c} > 0$ ) such that

$$g(t) \leq G(t) \leq cg(ct), \quad (\tilde{g}(t) \leq G(t) \leq \tilde{c}\tilde{g}(\tilde{c}t)) \quad \text{for } t \geq 0.$$

Next, we will need the following properties of quasiconvex functions:

**Lemma 1** ([5, p. 4]). *Let us consider three statements:*

- (a)  $G(t)$  is quasiconvex (quasiconcave) on  $[0, \infty)$ ;  
 (b) for all  $t_1, t_2 \in [0, \infty)$  and all  $\lambda \in (0, 1)$

$$G(\lambda t_1 + (1 - \lambda)t_2) \leq k_1(\lambda G(k_1 t_1) + (1 - \lambda)G(k_1 t_2))$$

$$\left( \lambda G(t_1) + (1 - \lambda)G(t_2) \leq l_1 G(l_1(\lambda t_1 + (1 - \lambda)t_2)) \right);$$

- (c) there exists a constant  $k_2$  ( $l_2$ ) such that for all  $u \in L^2_{loc}(\Omega, R^N)$  and all balls  $B(x, r) \subset \Omega$

$$\begin{aligned} G\left(\int_{B(x,r)} |u|^2 dy\right) &\leq k_2 \int_{B(x,r)} G(k_2 |u|^2) dy, \\ \left(\int_{B(x,r)} G(|u|^2) dy \leq l_2 G\left(l_2 \int_{B(x,r)} |u|^2 dy\right)\right). \end{aligned}$$

Then (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c).

**Proposition 2.** *For all  $u, v \in L^2_{loc}(\Omega, R^N)$ , all balls  $B(x, r) \subset \Omega$  and an arbitrary nondecreasing quasiconvex function  $G$  on  $[0, \infty)$  we have*

(a)

$$\int_{B(x,r)} G(|u + v|^2) dy \leq \frac{k_1}{2} \left( \int_{B(x,r)} G(4k_1 |u|^2) dy + \int_{B(x,r)} G(4k_1 |v|^2) dy \right),$$

(b)

$$\int_{B(x,r)} G(|u - u_{x,r}|^2) dy \leq c_1 \int_{B(x,r)} G(c_2 |u - c|^2) dy,$$

where  $c_1 = \max\{k_1/2, k_2\}$ ,  $c_2 = \max\{4k_1, 4k_1 k_2\}$  and  $c \in R$  is arbitrary.

PROOF: (a) It follows from Lemma 1 (b).

(b) From (a) we get

$$\int_{B_r} G(|u - u_{x,r}|^2) dy \leq \frac{k_1}{2} \left( \int_{B_r} G(4k_1 |u - c|^2) dy + \int_{B_r} G(4k_1 |c - u_{x,r}|^2) dy \right).$$

Now, by means of Hölder's inequality and Lemma 1 (c)

$$\begin{aligned} \int_{B_r} G(4k_1 |c - u_{x,r}|^2) dy &= \mu(B_r) G(4k_1 |c - u_{x,r}|^2) \\ &= \mu(B_r) G\left(4k_1 \left|c - \int_{B_r} u(y) dy\right|^2\right) = \mu(B_r) G\left(\frac{4k_1}{\mu^2(B_r)} \left|\int_{B_r} (u(y) - c) dy\right|^2\right) \\ &\leq \mu(B_r) G\left(\int_{B_r} 4k_1 |u(y) - c|^2 dy\right) \leq k_2 \int_{B_r} G(4k_1 k_2 |u(y) - c|^2) dy \end{aligned}$$

and the result follows easily. □

**Lemma 2** ([9, p. 37]). *Let  $\varphi: [0, \infty] \rightarrow [0, \infty]$  be a monotone function which is absolutely continuous on every closed interval of finite length. If  $v \geq 0$  is measurable and  $E(t) = \{x \in R^n : v(x) > t\}$ , then*

$$\int_{R^n} \varphi \circ v dx = \int_0^\infty \mu(E(t)) \varphi'(t) dt.$$

**Proposition 3.** *Let  $v \in L_{loc}^2(\Omega, R^m)$ ,  $B(x, \sigma) \subset \Omega$ ,  $a > 0$  and  $s \in [1, \infty)$  be arbitrary. If the inequality*

$$\int_{B(x, \tau\sigma)} |v - v_{x, \tau\sigma}|^2 dy \leq \int_{B(x, \sigma)} |v - v_{x, \sigma}|^2 dy$$

holds for some  $\tau \in (0, 1)$ , then there exists a constant  $b$  such that

$$\int_{B(x, \tau\sigma)} \ln_+^s (a|v - v_{x, \tau\sigma}|^2) dy \leq b \int_{B(x, \sigma)} \ln_+^s (a|v - v_{x, \sigma}|^2) dy.$$

For the constant  $b$  we have the following estimate

$$b \leq h \left( \int_{B(x, \sigma)} |v - v_{x, \sigma}|^2 dy \right) \left( \int_{B(x, \sigma)} \ln_+^s (a|v - v_{x, \sigma}|^2) dy \right)^{-1},$$

where  $h(t) = (s/e(s-1))^{s/(s-1)} at$ ,  $t \in [0, e^{s/(s-1)}/a]$  and  $\ln^{s/(s-1)}(at)$ ,  $t \in (e^{s/(s-1)}/a, \infty)$ .

PROOF: We set  $E_{\tau\sigma}(t) = \{y \in B(x, \tau\sigma) : |v - v_{x,\tau\sigma}|^2 > t\}$  for  $t \geq 0$  and  $0 < \tau \leq 1$ . From Lemma 2 and by means of integration by parts we get

$$\begin{aligned} \int_{B_{\tau\sigma}} \ln_+^s(a|v - v_{\tau\sigma}|^2) dy &= \frac{s}{\mu(B_{\tau\sigma})} \int_{1/a}^\infty \mu(E_{\tau\sigma}(t)) \frac{\ln^{s-1}(at)}{t} dt \\ &= \frac{s}{\mu(B_{\tau\sigma})} \left[ \frac{\ln^{s-1}(at)}{t} \int_0^t \mu(E_{\tau\sigma}(\lambda)) d\lambda \right]_{1/a}^\infty \\ &\quad + \frac{s}{\mu(B_{\tau\sigma})} \int_{1/a}^\infty \left( \int_0^t \mu(E_{\tau\sigma}(\lambda)) d\lambda \right) \frac{\ln^{s-1}(at) - (s-1)\ln^{s-2}(at)}{t^2} dt. \end{aligned}$$

For the sake of simplicity we put  $V_r = \int_{B(x,r)} |v - v_{x,r}|^2 dy$ . The first integral is zero and on the second integral we can use the mean value theorem for the integrals and we have for some  $1/a < \xi_{\tau\sigma}$ ,  $\xi_\sigma < \infty$ ,

$$\begin{aligned} \int_{B_{\tau\sigma}} \ln_+^s(a|v - v_{\tau\sigma}|^2) dy &= sV_{\tau\sigma} \int_{\xi_{\tau\sigma}}^\infty \frac{\ln^{s-1}(at) - (s-1)\ln^{s-2}(at)}{t^2} dt \\ &= \frac{s \ln^{s-1}(a\xi_{\tau\sigma})}{\xi_{\tau\sigma}} V_{\tau\sigma} = \frac{\xi_\sigma \ln^{s-1}(a\xi_{\tau\sigma})}{\xi_{\tau\sigma} \ln^{s-1}(a\xi_\sigma)} \frac{V_{\tau\sigma}}{V_\sigma} \int_{B_\sigma} \ln_+^s(a|v - v_{x,\sigma}|^2) dy \\ &= b(\tau) \int_{B_\sigma} \ln_+^s(a|v - v_{x,\sigma}|^2) dy. \end{aligned}$$

Now the result follows from Lemma 1 (c). □

### 3. The result

For  $x \in \Omega$ ,  $r > 0$  we set  $U_r = U(x, r) = \int_{\Omega(x,r)} |Du - (Du)_{x,r}|^2 dy$ ,  $d_x = \text{dist}(x, \partial\Omega)$  and  $\alpha_n = \mu(B(0, 1))$ . We define  $S_0 = \{x \in \Omega : \overline{\lim}_{r \rightarrow 0+} U(x, r) > 0\}$ .

*Remark 2.* Let  $u$  be a solution of (1). It is well known (see [9, pp. 75, 122]) that  $\lim_{r \rightarrow 0+} U(x, r) = 0$  for all  $x \in \Omega \setminus E$  where  $n - 2 + \beta$  dimensional Hausdorff measure  $H^{n-2+\beta}(E) = 0$  for every  $\beta > 0$ .

Now we can formulate the main theorem.

**Theorem.** Let  $u \in H^{1,2}(\Omega, R^N)$  be a weak solution to the nonlinear system (1) under the hypotheses (i), (ii), (iii). Let  $x \in \mathcal{S}_0$  be arbitrary and suppose that there exists  $d \in (0, d_x/2)$  such that

$$(8) \quad \frac{Kl_2\omega^2}{\nu^2} \left( b \int_{B(x,2d)} \ln_+^{q/(q-1)} \left( \frac{4l_2\omega^2 |Du - (Du)_{x,2d}|^2}{CU_{2d}} \right) dy \right)^{1-1/q} < \frac{1}{4} \tau^n,$$

where  $K = c(n, N, q) (M/\nu)^8$ ,  $\tau = (2^{n+5}A)^{-1/2}$ ,  $l_2, A$  are the constants from Lemma 1 (c), Lemma 3,  $\omega = \omega(2^n l_2 U_{2d})$ ,  $\omega$  is from (5),  $C = 2^{n-8} \nu^2 \tau^n / \alpha_n A$  and  $b$  is the constant from Proposition 3 for the case  $a = 1/CU_{2d}$ ,  $\sigma = 2d$ ,  $v = 2\sqrt{l_2}\omega Du$ ,  $s = q/(q - 1)$  where  $q \in (1, n/(n - 2)]$ . Then there exists a ball  $B(x, r_x) \subset \Omega$  such that  $Du \in \mathcal{L}^{2,n}(B(x, r_x), R^{nN})$  and

$$(9) \quad [Du]_{\mathcal{L}^{2,n}(B(x,r_x), R^{nN})}^2 \leq \max\{2^n(4A\tau^{-n} + 1)U_{2d}, \mu^{-1}(B_{2d}) \int_{\Omega} |Du - (Du)_{\Omega}|^2 dx\}.$$

**Proposition 4.** Set  $\omega_{\infty} = \lim_{t \rightarrow \infty} \omega(t)$ ,  $V_1 = c_1 (M/\nu)^{3n+8} (\omega_{\infty}/\nu)^2$  and  $V_2 = c_2 (M/\nu)^{3n+6} (\omega_{\infty}/\nu)^2$ . If

$$(10) \quad V_2 \leq e^q \ \& \ q^{q-1} V_1 V_2^{1-1/q} < 1 \quad \text{or} \quad V_2 > e^q \ \& \ V_1 \ln^{q-1} V_2 < 1,$$

then condition (8) holds for every  $x \in \mathcal{S}_0$ . Here  $q \in (1, n/(n - 2)]$ ,  $c_1 = c_1(n, N, q)$  and  $c_2 = c_2(n, N)$ .

PROOF: Let  $x \in \mathcal{S}_0$  and  $d \in (0, d_x/2)$  be arbitrary such that  $U(x, 2d) > 0$ . From Proposition 3 it follows that the left hand side of (8) is equal or less than  $Kl_2\omega_{\infty}^2 h^{1-1/q}(4\omega_{\infty}^2 U_{2d})/\nu^2$ . From the definition of the function  $h(t)$  and assumption (10) it follows that (8) is satisfied. □

**Example.** We can consider the system (1) for  $n = 3, N = 2$  where  $A_i^{\alpha}(p) = (a \delta_{ij} \delta_{\alpha\beta} + b \delta_{i\alpha} \delta_{j\beta} \arctan |p|^2 / 2\pi) p_{\beta}^j$ ,  $a, b$  are constants,  $0 < b/6 < a$ . We have

$$\frac{\partial A_i^{\alpha}}{\partial p_{\beta}^j}(p) \xi_{\alpha}^i \xi_{\beta}^j \geq (a - b/6) |\xi|^2, \quad \forall \xi, p \in R^6,$$

$\omega_{\infty} \leq b$  and  $|\partial A_i^{\alpha} / \partial p_{\beta}^j(p)| \leq M = a + b$ . Setting  $P = b/a$  we get that  $V_1 < 4c_1 P^2 (1 + P)^{3n+8} / (1 - P/6)^{3n+10}$ ,  $V_2 < 4c_2 P^2 (1 + P)^{3n+6} / (1 - P/6)^{3n+8}$  and it is not difficult to see that (10) is satisfied for  $P$  sufficiently small.



**Corollary 1.** *Let  $\Omega_0 \Subset \Omega$  be arbitrary and of type  $\mathcal{A}$  and the assumptions of Theorem be satisfied for every  $x \in \overline{\Omega}_0 \cap \mathcal{S}_0$ . Then there are constants  $\mathcal{U}, d_0, r_0 > 0$  such that  $Du \in \mathcal{L}^{2,n}(\Omega_0, R^{nN})$  and the following estimate*

$$(11) \quad \begin{aligned} [Du]_{\mathcal{L}^{2,n}(\Omega_0, R^{nN})}^2 &\leq \max\{2^n(4A\tau^{-n} + 1)\mathcal{U}, \\ &\mu^{-1}(B_{2d_0}) \int_{\Omega} |Du - (Du)_{\Omega}|^2 dx, \\ &(Ar_0^n)^{-1} \int_{\Omega_0} |Du - (Du)_{\Omega_0}|^2 dx \} \end{aligned}$$

holds.

PROOF: From Remark 2, Theorem and the definition of the set  $\mathcal{S}_0$  it follows that for every  $x \in \overline{\Omega}_0$  there exists  $B(x, r_x) \subset \Omega$  such that  $Du \in \mathcal{L}^{2,n}(B(x, r_x), R^{nN})$ . As  $\overline{\Omega}_0$  is the compact set and the system balls  $\{B(x, r_x)\}$  covers of  $\overline{\Omega}_0$  we can choose a finite subcover  $\{B(x_j, r_{x_j})\}_{j=1}^m$ . If we set  $\mathcal{U} = \max\{U(x_j, 2d_{x_j}) : 1 \leq j \leq m\}$ ,  $r_0 = \min\{r_{x_j} : 1 \leq j \leq m\}$  and  $d_0 = \min\{d_{x_j} : 1 \leq j \leq m\}$ , then the estimate follows from Remark 1.  $\square$

**Corollary 2.** *Let the assumptions of Corollary 1 be satisfied. Then  $u \in A^1(\overline{\Omega}_0, R^N)$ .*

PROOF: It follows from Proposition 1 (d), Poincaré’s inequality and Corollary 1.  $\square$

#### 4. Lemmas

The statement of the following lemma is well known (see e.g. [1], [3], [7], [8]).

**Lemma 3.** *Let  $v \in H^{1,2}(\Omega, R^N)$  be a weak solution to the system (1) satisfying (i), (ii) and (iii), where  $\partial A_i^\alpha / \partial p_j^\beta$  are the constants. Then there exists a constant  $A = c(n, N)(M/\nu)^6$  such that for every  $x \in \Omega$  and  $0 < \sigma \leq R \leq \text{dist}(x, \partial\Omega)$  the following estimate holds*

$$\int_{B(x,\sigma)} |Dv(y) - (Dv)_{x,\sigma}|^2 dy \leq A \left(\frac{\sigma}{R}\right)^{n+2} \int_{B(x,R)} |Dv(y) - (Dv)_{x,R}|^2 dy.$$

The following lemma is possible to derive by the difference quotient method (see e.g. [1], [3], [7], [8]).

**Lemma 4.** *Let  $u \in H^{1,2}(\Omega, R^N)$  be a weak solution to the system (1) satisfying (i), (ii) and (iii). Then  $u \in H_{loc}^{2,2}(\Omega, R^N)$  and for all  $x \in \Omega$ ,  $0 < \sigma < \varrho \leq \text{dist}(x, \partial\Omega)$  we have*

$$\int_{B(x,\sigma)} |D^2u|^2 dy \leq \frac{6n(M/\nu)^2}{(\varrho - \sigma)^2} \int_{B(x,\varrho)} |Du - (Du)_{x,\varrho}|^2 dy.$$

**Lemma 5** ([6]). *Let  $1 \leq p, q < \infty, p^{-1} - q^{-1} \leq n^{-1}, R > 0, x \in \mathbb{R}^n$ . Then for  $u \in H^{1,p}(B(x, R), \mathbb{R}^N)$  we have*

$$\left( \int_{B(x,R)} |u(y)|^q dy \right)^{1/q} \leq cR^{1+n/q-n/p} \left( R^{-p} \int_{B(x,R)} |u(y)|^p dy + \int_{B(x,R)} |Du(y)|^p dy \right)^{1/p},$$

where  $c = c(n, N, p, q)$  is a constant independent of  $x, R$  and  $u$ .

**Lemma 6.** *Let  $u \in H^{1,2}(\Omega, \mathbb{R}^N)$  be a weak solution to (1) satisfying (i), (ii) and (iii). Then for every ball  $B(x, 2R) \subset \Omega$  and an arbitrary constant  $a > 0$  we have*

$$\begin{aligned} & \int_{B(x,R)} |Du - (Du)_{x,R}|^2 \ln_+(a |Du - (Du)_{x,R}|^2) dy \\ & \leq c \left( \frac{M}{\nu} \right)^2 \left( \int_{B(x,2R)} \ln_+^{q/(q-1)}(4a |Du - (Du)_{x,2R}|^2) dy \right)^{1-1/q} \\ & \qquad \qquad \qquad \int_{B(x,2R)} |Du - (Du)_{x,2R}|^2 dy, \end{aligned}$$

where  $1 < q \leq n/(n - 2)$  and  $c = c(n, N, q)$ .

PROOF: Let  $x \in \Omega$  and  $0 \leq R \leq \frac{1}{4} \text{dist}(x, \partial\Omega)$ . We denote  $B_R = B(x, R)$  for simplicity. From Lemma 4 it follows that  $Du \in H_{loc}^{1,2}(\Omega, \mathbb{R}^N)$ . By means of Sobolev's imbedding theorem  $H^{1,2}(B_R, \mathbb{R}^N) \hookrightarrow L^s(B_R, \mathbb{R}^N)$ , where  $B_R \subset \Omega$  be arbitrary and  $1 \leq s \leq 2n/(n - 2)$ . From this we obtain by Proposition 2 (b) and Lemma 5

$$\begin{aligned} & \int_{B_R} |Du - (Du)_R|^2 \ln_+(a |Du - (Du)_R|^2) dy \\ & \leq 4 \int_{B_R} |Du - (Du)_{2R}|^2 \ln_+(4a |Du - (Du)_{2R}|^2) dy \\ & \leq 4 \left( \int_{B_R} |Du - (Du)_{2R}|^{2q} dy \right)^{1/q} \left( \int_{B_R} \ln_+^{q/(q-1)}(4a |Du - (Du)_{2R}|^2) dy \right)^{1-1/q} \\ & \leq cR^{n(1/q-1)+2} \left( R^{-2} \int_{B_R} |Du - (Du)_{2R}|^2 + \int_{B_R} |D^2u|^2 dy \right) \times \end{aligned}$$

$$\begin{aligned} & \times \left( \int_{B_R} \ln_+^{q/(q-1)} (4a |Du - (Du)_{2R}|^2) dy \right)^{1-1/q} \\ & \leq c \left( \frac{M}{\nu} \right)^2 R^{-n(1-1/q)} \int_{B_{2R}} |Du - (Du)_{2R}|^2 dy \times \\ & \quad \times \left( \int_{B_R} \ln_+^{q/(q-1)} (4a |Du - (Du)_{2R}|^2) dy \right)^{1-1/q} \end{aligned}$$

and we finally obtain the result. □

**5. Proof of Theorem**

Set  $A_{ij}^{\alpha\beta}(\zeta) = \partial A_i^\alpha / \partial p_\beta^j(\zeta)$ ,  $A_{ij,0}^{\alpha\beta} = A_{ij}^{\alpha\beta}((Du)_R)$ ,

$$\tilde{A}_{ij}^{\alpha\beta} = \int_0^1 A_{ij}^{\alpha\beta}((Du)_R + t(Du - (Du)_R)) dt,$$

$B_R = B(x, R)$  and  $U_R = U(x, R)$  for simplicity. Then the system (1) can be rewritten as

$$-D_\alpha \left( A_{ij,0}^{\alpha\beta} D_\beta u^j \right) = -D_\alpha \left( \left( A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta} \right) \left( D_\beta u^j - (D_\beta u^j)_R \right) \right).$$

Split  $u$  as  $v + w$  where  $v$  is the solution of the Dirichlet problem

$$\begin{cases} -D_\alpha \left( A_{ij,0}^{\alpha\beta} D_\beta v^j \right) = 0 & \text{in } B_R \\ v - u \in H_0^{1,2} (B_R, R^N). \end{cases}$$

For every  $0 < \sigma \leq R$  from Lemma 3 it follows

$$\int_{B_\sigma} |Dv - (Dv)_\sigma|^2 dy \leq A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_R} |Dv - (Dv)_R|^2 dy,$$

hence

$$(12) \quad \int_{B_\sigma} |Du - (Du)_\sigma|^2 dy \leq 2A \left( \frac{\sigma}{R} \right)^{n+2} \int_{B_R} |Dv - (Dv)_R|^2 dy + 2 \int_{B_R} |Dw|^2 dy.$$

Now  $w \in H_0^{1,2} (B_R, R^N)$  satisfies

$$\begin{aligned} & \int_{B_R} A_{ij,0}^{\alpha\beta} D_\beta w^j D_\alpha \varphi^i dy \leq \int_{B_R} \left| A_{ij,0}^{\alpha\beta} - \tilde{A}_{ij}^{\alpha\beta} \right| \left| D_\beta w^j - (D_\beta w^j)_R \right| \left| D_\alpha \varphi^i \right| dy \\ & \leq \left( \int_{B_R} \omega^2 \left( |Du - (Du)_R|^2 \right) |Du - (Du)_R|^2 dy \right)^{1/2} \left( \int_{B_R} |D\varphi|^2 dy \right)^{1/2} \end{aligned}$$

for any  $\varphi \in H_0^{1,2}(B_R, R^N)$ , where  $\omega$  is from (5). Hence, choosing  $\varphi = w$ , we get

$$\nu^2 \int_{B_R} |Dw|^2 dy \leq \int_{B_R} \omega^2 (|Du - (Du)_R|^2) |Du - (Du)_R|^2 dy.$$

Now applying the Young inequality (with the complementary functions (7)) on the right-hand side, we obtain for every  $\varepsilon > 0$

$$(13) \quad \nu^2 \int_{B_R} |Dw|^2 dy \leq \varepsilon \int_{B_R} |Du - (Du)_R|^2 \ln_+ (a\varepsilon |Du - (Du)_R|^2) dy + \frac{2}{a} \int_{B_R} e^{\omega_R^2/\varepsilon-1} dy,$$

where  $\omega_R^2 = \omega^2(|Du - (Du)_R|^2)$ .

From (12) and (13) it follows

$$(14) \quad \int_{B_\sigma} |Du - (Du)_\sigma|^2 dy \leq 4A \left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_R|^2 dy + \frac{2(2A+1)}{\nu^2} \left( \varepsilon \int_{B_R} |Du - (Du)_R|^2 \ln_+ (a\varepsilon |Du - (Du)_R|^2) dy + \frac{2}{a} \int_{B_R} e^{\omega_R^2/\varepsilon-1} dy \right),$$

We can estimate the right-hand side by means of Lemma 1 (c) (for the quasiconcave case), Lemma 6 and we get

$$\int_{B_\sigma} |Du - (Du)_\sigma|^2 dy \leq 4A \left(\frac{\sigma}{R}\right)^{n+2} \int_{B_R} |Du - (Du)_R|^2 dy + \frac{2(2A+1)}{\nu^2} \left[ \varepsilon c \left(\frac{M}{\nu}\right)^2 \left( \int_{B_{2R}} \ln_+^{q/(q-1)} (4a\varepsilon |Du - (Du)_{2R}|^2) dy \right)^{1-1/q} \times \int_{B_{2R}} |Du - (Du)_{2R}|^2 dy + \frac{2\alpha_n R^n}{a} e^{l_2\omega^2(l_2U_R)/\varepsilon-1} \right].$$

Setting

$$\phi(t) = \int_{B_t} |Du - (Du)_t|^2 dy,$$

$$F_\varepsilon(t) = \left( \int_{B_t} \ln_+^{q/(q-1)} (4a\varepsilon |Du - (Du)_t|^2) dy \right)^{1-1/q},$$

we can rewrite the previous inequality as follows:

$$(15) \quad \phi(\sigma) \leq 4A \left(\frac{\sigma}{R}\right)^{n+2} \phi(R) + \frac{K\varepsilon}{\nu^2} F_\varepsilon(2R)\phi(2R) + \frac{2^4\alpha_n A}{a\nu^2} e^{l_2\omega^2(2^n l_2 U_{2R})/\varepsilon-1} R^n,$$

where  $K = c(n, N, q) (M/\nu)^8$ . From the assumptions of Theorem it follows that there exists  $d \in (0, d_x/2)$  such that (8) holds. Now we are going to prove that

$$(16) \quad \phi\left(2\tau^k d\right) \leq \tau^{kn} \phi(2d)$$

for every natural number  $k$  and  $\tau = (2^{n+5}A)^{-1/2}$ . Let  $k = 1$ . If we put in (15)  $a = 1/CU_{2d}$ ,  $\varepsilon = l_2\omega^2(2^n l_2 U_{2d})$ ,  $\sigma = 2\tau d$  and  $R = d$  we get

$$\begin{aligned} \phi(2\tau d) &\leq 2^{n+4} A\tau^{n+2} \phi(d) + \frac{Kl_2\omega^2}{\nu^2} F_\varepsilon(2d)\phi(2d) + \frac{2^4\alpha_n A}{\nu^2} CU_{2d}d^n \\ &\leq 2^{n+4} A\tau^{n+2} \phi(2d) + \frac{Kl_2\omega^2}{\nu^2} b^{1-1/q} F_\varepsilon(2d)\phi(2d) + \frac{1}{4}\tau^n \phi(2d) \\ &\leq \left(2^{n+4} A\tau^2 + \frac{1}{4} + \frac{1}{4}\right) \tau^n \phi(2d) = \tau^n \phi(2d). \end{aligned}$$

Thus (16) holds for  $k = 1$ . Consequently  $U_{2\tau d} \leq U_{2d}$  and by means of Proposition 3 we have  $F_\varepsilon(2\tau d) \leq b^{1-1/q} F_\varepsilon(2d)$ .

Let us suppose that (16) holds for  $k \geq 1$ . Similarly to consideration above we have  $U_{2\tau^k d} \leq U_{2d}$  and  $F_\varepsilon(2\tau^k d) \leq b^{1-1/q} F_\varepsilon(2d)$ . We will show that (16) holds for  $k + 1$ . Setting  $a = 1/CU_{2d}$ ,  $\varepsilon = l_2\omega^2(2^n l_2 U_{2d})$ ,  $\sigma = 2\tau^{k+1}d$  and  $R = \tau^k d$  in (15) we obtain

$$\begin{aligned} \phi(2\tau^{k+1}d) &\leq 2^{n+4} A\tau^{n+2} \phi\left(\tau^k d\right) + \frac{Kl_2\omega^2}{\nu^2} F_\varepsilon(2\tau^k d)\phi(2\tau^k d) \\ &\quad + \frac{2^4\alpha_n A}{\nu^2} e^{\omega^2(2^n l_2 U_{2\tau^k d})/\omega^2(2^n l_2 U_{2d})-1} \tau^{kn} CU_{2d}d^n \\ &\leq 2^{n+4} A\tau^{n+2} \phi\left(2\tau^k d\right) + \frac{Kl_2\omega^2}{\nu^2} F_\varepsilon(2\tau^k d)\phi(2\tau^k d) + \frac{1}{4}\tau^{(k+1)n} \phi(2d) \\ &\leq 2^{n+4} A\tau^{n+2} \tau^{kn} \phi(2d) + \frac{Kl_2\omega^2}{\nu^2} b^{1-1/q} F_\varepsilon(2d)\tau^{kn} \phi(2d) + \frac{1}{4}\tau^{(k+1)n} \phi(2d) \\ &\leq \left(2^{n+4} A\tau^2 + \frac{1}{4} + \frac{1}{4}\right) \tau^{(k+1)n} \phi(2d) = \tau^{(k+1)n} \phi(2d). \end{aligned}$$

Let us consider the two possibilities:

(a) if  $\tau \leq t < 1$ , then  $t^{-n}\phi(td) \leq \tau^{-n}\phi(td) \leq \tau^{-n} \sup_{t \in [\tau, 1]} \phi(td)$  and also

$$(17) \quad \phi(td) \leq \left( \tau^{-n} \sup_{t \in [\tau, 1]} \phi(td) \right) t^n,$$

(b) if  $0 < t < \tau$ , then there exists natural  $k \geq 1$  such that  $\tau^{k+1} \leq t < \tau^k$ . From Proposition 3, (8), (16) and (15) with  $a = 1/CU_{2d}$ ,  $\varepsilon = l_2\omega^2(2^n l_2 U_{2d})$ ,  $\sigma = td$  and  $R = \tau^k d$  we have

$$(18) \quad \begin{aligned} \phi(td) &= \phi\left(\frac{t}{\tau^k}(\tau^k d)\right) \\ &\leq 4A \left(\frac{t}{\tau^k}\right)^{n+2} \phi(\tau^k d) + \frac{K\varepsilon}{\nu^2} F_\varepsilon(2\tau^k d) \phi(2\tau^k d) \\ &\quad + \frac{2^4 \alpha_n A}{a\nu^2} e^{l_2\omega^2(2^n l_2 U_{2\tau^k d})/\varepsilon - 1} \tau^{kn} d^n \\ &\leq 4A \left(\frac{t}{\tau^k}\right)^{n+2} \tau^{kn} \phi(2d) + \frac{Kl_2\omega^2}{\nu^2} b^{1-1/q} F_\varepsilon(2d) \tau^{kn} \phi(2d) \\ &\quad + \frac{2^4 \alpha_n A}{\nu^2} CU_{2d} \tau^{kn} d^n \\ &\leq \left( 4A \left(\frac{t}{\tau^k}\right)^{n+2} \tau^{kn} + \tau^{(k+1)n} \right) \phi(2d) \\ &\leq \left( 4A\tau^{-n} \left(\frac{t}{\tau^k}\right)^{n+2} + 1 \right) \tau^{(k+1)n} \phi(2d) < (4A\tau^{-n} + 1) t^n \phi(2d). \end{aligned}$$

In both cases (17) and (18) we obtain

$$t^{-n}\phi(td) \leq c, \quad t \in (0, 1],$$

where  $c = \max\{\tau^{-n} \sup_{t \in [\tau, 1]} \phi(td), (4A\tau^{-n} + 1)\phi(2d)\} = (4A\tau^{-n} + 1)\phi(2d)$ . Let  $0 < r < \text{dist}(B(x, r_x), \partial\Omega)$ . Hence  $U(y, r)$  is uniformly continuous for fixed  $r$  in  $\overline{B(x, r_x)} \subset \Omega$ . According to Proposition 3, the expression

$$\frac{Kl_2\omega^2}{\nu^2} \left( b \int_{B(y,r)} \ln_+^{q/(q-1)} \left( \frac{4l_2\omega^2 |Du - (Du)_{y,r}|^2}{CU(y,r)} \right) dz \right)^{1-1/q}$$

is also uniformly continuous with respect to  $y$  in  $\overline{B(x, r_x)}$  and we arrive at the conclusion. □

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