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The G_δ -topology and incompactness of ω^α

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Abstract. We establish a relation between covering properties (e.g. Lindelöf degree) of two standard topological spaces (Lemmas 4 and 5). Some cardinal inequalities follow as easy corollaries.

Keywords: Lindelöf degree, G_δ topology, cardinal functions

Classification: 54D20, 45B10

The present note is a contribution into the study of the Lindelöf degree in powers of topological spaces. It answers a question of W.A.R. Weiss.

In what follows, $\kappa^* \subset \beta\kappa$ is the space of all free ultrafilters over a discrete set of κ points, $\mu\kappa \subset \beta\kappa$ is the space of all *uniform* ultrafilters over κ , ω^α denotes the α -th power of the discrete set of integers, $(\mu\kappa, G_\delta)$ is $\mu\kappa$ with the finer G_δ topology. $L(X)$ denotes the Lindelöf degree of X , and $e(X) := \sup\{A \subset X : A \text{ is closed and discrete}\}$ — its *extent*.

Kenneth Kunen ([1]) proves $L(\mu(2^\kappa)^+, G_\delta) \geq \kappa^+$, for the same κ 's as in our Corollary 6. Our Corollary 6 gives here

$$L(\mu(2^\kappa)^+, G_\delta) \geq (2^\kappa)^+.$$

J. Mycielski proved ([2]), by inductive “stepping up”, that, for α less than the 1st weakly inaccessible cardinal,

$$e(\omega^\alpha) = \alpha.$$

Our Corollary 9 is a weaker statement for a larger class of cardinals. This Corollary was obtained first by Łoś [3] in 1959 using group-theoretic methods. See also Juhász [4].

Stevo Todorčević ([5]) proves, assuming the combinatorial statement \square_κ ,

$$L(\omega^\kappa) = \kappa.$$

1. If $\mathcal{A} = \{A_n : n < \omega\}$ is a countable disjoint partition of the cardinal κ , then

$$\begin{aligned} \mu\kappa &= \left(\bigcup_{n < \omega} S_n^{\mathcal{A}} \right) \dot{\cup} (\tilde{S}^{\mathcal{A}}), \text{ where} \\ S_n^{\mathcal{A}} &= \{u \in \mu\kappa : A_n \in u\} \text{ and} \\ \tilde{S}^{\mathcal{A}} &= \{u \in \mu\kappa : \{\bigcup_{n \geq i} A_n : i < \omega\} \subset u\}. \end{aligned}$$

Note that $\tilde{S}^{\mathcal{A}}$ is a G_δ set in $\mu\kappa$.

2. We say that a cover of κ^* or of $\mu\kappa$ is a *proper G_δ -cover* if every set in it is of the form $\tilde{S}^{\mathcal{A}}$ for some countable partition \mathcal{A} of κ .

3. **Lemma.** *If κ is a regular cardinal and $\mu\kappa$ has a proper G_δ -cover of size α , then ω^α has a subset of size κ without a CAP (complete accumulation point).*

PROOF: Suppose $\mu\kappa = \bigcup \{\tilde{S}^{\mathcal{A}^\gamma} : \gamma < \alpha\}$ for some collection $\mathcal{C} = \{\mathcal{A}^\gamma : \gamma < \alpha\}$ of countable partitions $\mathcal{A}^\gamma = \{A_n^\gamma : n < \omega\}$ of κ . For every point $p \in \kappa$ define its history in \mathcal{C} $\bar{p} : \alpha \rightarrow \omega$ by setting $\bar{p}(\gamma) := n$ such that $p \in A_n^\gamma$. Let $P = \{\bar{p} : p < \kappa\} \subset \omega^\alpha$.

Claim 1. $|P| = \kappa$, moreover, for every $p \in \kappa$, $K_p := \{q \in \kappa : \bar{q} = \bar{p}\}$ has size $|K_p| < \kappa$. Indeed, if not, then no $v \ni K_p$ is covered:

$$\forall \gamma < \alpha \quad v \notin \tilde{S}^{\mathcal{A}^\gamma},$$

because

$$v \in S_{\bar{p}(\gamma)}^{\mathcal{A}^\gamma}.$$

And $|P| = \kappa$ follows from the regularity of κ . □

Claim 2. P has no CAP in ω^α . If not, let $\varphi \in \omega^\alpha$ be a CAP of P . Then for every finite $F \subset \alpha$

$$|\{p < \kappa : \bar{p} \upharpoonright F = \varphi \upharpoonright F\}| = \kappa,$$

by Claim 1.

Therefore, the family $\mathcal{F} := \{A_{\varphi(\gamma)}^\gamma : \gamma < \alpha\}$ has the uniform finite intersection property (i.e. $\forall \mathcal{F}_0 \in [\mathcal{F}]^{< \aleph_0} \quad |\cap \mathcal{F}_0| = \kappa$). [By $\bar{p} \upharpoonright F = \varphi \upharpoonright F \iff p \in \bigcap_{\gamma \in F} A_{\varphi(\gamma)}^\gamma$].

Pick a $u \in \mu\kappa$ extending \mathcal{F} .

Then $u \notin \bigcup_{\gamma < \alpha} \tilde{S}^{\mathcal{A}^\gamma} = \mu\kappa$. Contradiction. Hence $P \subset \omega^\alpha$ has no CAP in ω^α , so it is as required. □

4. Lemma. *If κ^* has a proper G_δ -cover of size α , then ω^α has a closed discrete subset of size κ .*

PROOF: Here we, similarly, study the family P of the histories of points $p \in \kappa$ in the family of partitions of κ defining our proper G_δ -cover, in this case of κ^* . Only finitely many points $p \in \kappa$ may have the same history, so $|P| = \kappa$, and, arguing as in Claim 2 of the previous lemma, $P \subset \omega^\alpha$ has no limit points in ω^α whatsoever. \square

5. Theorem. *If κ is a regular not Ulam measurable cardinal, then*

$$L(\omega^{L(\mu\kappa, G_\delta)}) \geq \kappa.$$

6. Corollary. *$L(\mu\kappa, G_\delta) \geq \kappa$, for the same κ 's.*

PROOF OF THEOREM 5 AND COROLLARY 6: Since every ultrafilter over κ is countably incomplete, there is a proper G_δ cover of $\mu\kappa$, and so $L(\mu\kappa, G_\delta) = \alpha \Rightarrow$ there is a proper G_δ -cover of size $\alpha \Rightarrow$ (By Lemma 3) ω^α has a subset of size κ without a CAP \Rightarrow

- (a) $L(\omega^\alpha) \geq \kappa$, and
- (b) $\alpha \geq \kappa$ (because $\alpha \geq L(\omega^\alpha)$).

\square

7. Corollary. *$L(\omega^{2^\kappa}) \geq \kappa$, for the same κ 's as in Theorem 5.*

PROOF: $(\mu\kappa, G_\delta)$ has a base of size $(2^\kappa)^\omega = 2^\kappa$. Hence $L(\mu\kappa, G_\delta) \leq 2^\kappa$ and so $L(\omega^{2^\kappa}) \geq L(\omega^{L(\mu\kappa, G_\delta)}) \geq \kappa$. \square

8. Theorem. *If κ is not Ulam measurable, then*

$$L(\kappa^*, G_\delta) \geq L(\omega^{L(\kappa^*, G_\delta)}) \geq e(\omega^{L(\kappa^*, G_\delta)}) \geq \kappa.$$

PROOF: Immediate from Lemma 4. \square

9. Corollary. *If $\kappa < 1^{st}$ measurable cardinal, then*

$$e(\omega^{2^\kappa}) \geq \kappa,$$

i.e. ω^{2^κ} has a closed discrete subspace of size κ .

PROOF: Same as of Corollary 7. \square

10. Corollary. *Let λ be a strong limit cardinal \leq the 1^{st} measurable cardinal. Then the set $\{e(\omega^\alpha) : \alpha < \lambda\}$ is cofinal in λ . Hence, if $cf(\lambda) > \omega$, the set $\{\alpha < \lambda : e(\omega^\alpha) = \alpha\}$ is closed and unbounded in λ .*

Remark. Murray Bell observed that the converses of Lemmas 3 and 4 are also true.

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