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On the cardinality of functionally Hausdorff spaces

ALESSANDRO FEDELI

Abstract. In this paper two new cardinal functions are introduced and investigated. In particular the following two theorems are proved:
(i) If $X$ is a functionally Hausdorff space then $|X| \leq 2^{fs(X)\psi\tau(X)}$;
(ii) Let $X$ be a functionally Hausdorff space with $fs(X) \leq \kappa$. Then there is a subset $S$ of $X$ such that $|S| \leq 2^\kappa$ and $X = \bigcup\{cl_{\tau\theta}(A) : A \in [S]^\leq\kappa\}$.

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A space $X$ is said to be functionally Hausdorff if whenever $x \neq y$ in $X$ there is a continuous real valued function $f$ defined on $X$ such that $f(x) = 0$ and $f(y) = 1$.

In the last years many results involving cardinal functions related to $s$ (spread) have been obtained by several authors (see e.g. [8], [9], [10], [12]).

In this paper we give a result on the bound of the cardinality of functionally Hausdorff spaces using two new cardinal functions $fs$ and $\psi\tau$ related to $s$ and $\psi$ respectively. Moreover we prove, for functionally Hausdorff spaces, a variant of a well-known result on spread due to Shapirovskii ([11, Theorem 3], [4, Theorem 5.1]).

We refer the reader to [1], [4] and [7] for notations and definitions not explicitly given. $\chi(X)$, $s(X)$ and $\psi(X)$ denote respectively the character, the spread and the pseudocharacter of a space $X$.

Let $A$ be a subset of a space $X$:
(i) ([5], [6]) The $\tau$-closure of $A$, denoted by $cl_\tau(A)$, is the set of all points $x \in X$ such that any cozero-set neighbourhood of $x$ intersects $A$.
(ii) ([2]) The $\tau\theta$-closure of $A$, denoted by $cl_{\tau\theta}(A)$, is the set of all points $x \in X$ such that $cl_\tau(V) \cap A \neq \emptyset$ for every open neighbourhood $V$ of $x$.

For every $X$ and every $A \subset X$ we have $\overline{A} \subset cl_{\tau\theta}(A) \subset cl_\tau(A)$. It is clear that if $X$ is completely regular then $\overline{A} = cl_{\tau\theta}(A) = cl_\tau(A)$ for every $A \subset X$.

Definition 1. Let $X$ be a space. The functional spread of $X$, denoted by $fs(X)$, is the smallest infinite cardinal number $\kappa$ such that for every open family $U$ of $X$ and every $A \subset \bigcup U$ there exist a $V \in [U]^\leq\kappa$ and a $B \in [A]^\leq\kappa$ such that $A \subset cl_{\tau\theta}(B) \cup \bigcup\{cl_\tau(V) : V \in V\}$. 
Remark 2. Let $\mathcal{U}$ be an open cover of a space $X$, let $s(X) \leq \kappa$. By a well-known result of Shapirovskii it follows that there are a $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ and an $A \in [X]^{\leq \kappa}$ such that $X = \overline{A} \cup \bigcup \mathcal{V}$. Since $s(Y) \leq s(X)$ for every subspace $Y$ of $X$ it easily follows that $fs(X) \leq s(X)$. However the above inequality can be proper as the following example shows. For every $x \in R$ let $\mathcal{B}_x = \{\{x\} \cup B(x, \frac{1}{n}) \cap \emptyset : n \in \mathbb{N}\}$ and let $X$ be $R$ with the topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in R}$. Then $X$ is a functionally Hausdorff space such that $fs(X) = \omega < s(X)$.

Definition 3. Let $X$ be a functionally Hausdorff space and let $x \in X$. A family of open neighbourhoods of $x$ is said to be a $\tau$-pseudobase for $x$ if $\bigcap\{\text{cl}_\tau(U) : U \in \mathcal{U}\} = \{x\}$. Let $\psi_\tau(x, X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a } \tau\text{-pseudobase for } x\} + \omega$, the $\tau$-pseudocharacter of $X$ is defined as follows: $\psi_\tau(X) = \sup\{\psi_\tau(x, X) : x \in X\}$.

Remark 4. It is obvious that for every Tychonoff space $X$ we have $\psi_\tau(X) = \chi(X)$. Moreover $\psi_\tau(X) \leq \chi(X)$ for every functionally Hausdorff space $X$. In fact let $x \in X$ and let $\mathcal{B}_x$ be a local base at $x$, we claim that $\bigcap\{\text{cl}_\tau(B) : B \in \mathcal{B}_x\} = \{x\}$. Let us consider a point $y \in X \setminus \{x\}$, since $X$ is functionally Hausdorff there is a continuous mapping $f : X \to \mathbb{I}$ such that $f(x) = 0$ and $f(y) = 1$. Let $B \in \mathcal{B}_x$ such that $B \subset f^{-1}([0, \frac{1}{2}))$, then $\text{cl}_\tau(B) \subset f^{-1}([0, \frac{1}{2}])$. Hence $y \notin \text{cl}_\tau(B)$.

The above inequality can be proper. Let $\tau$ be the euclidean topology on $R$ and let $X$ be $R$ with the topology $\sigma = \{V \setminus C : V \in \tau, C \subset R \text{ and } |C| \leq \omega\}$. Then $X$ is a functionally Hausdorff space such that $\psi_\tau(X) = \omega < \chi(X)$.

A relation between $\psi_\tau$ and $fs$ is given in the following

Proposition 5. If $X$ is a functionally Hausdorff space then $\psi_\tau(X) \leq 2^{fs(X)}$.

Proof: Let $fs(X) = \kappa$ and $x \in X$. Since $X$ is functionally Hausdorff then for every $y \in X \setminus \{x\}$ there are open sets $U_y$ and $V_y$ such that $x \in U_y$, $y \in V_y$ and $\text{cl}_\tau(U_y) \cap \text{cl}_\tau(V_y) = \emptyset$. Since $fs(X) = \kappa$ we can find $A, B \in [X \setminus \{x\}]^{\leq \kappa}$ such that $X \setminus \{x\} \subset \text{cl}_\tau(A) \cup \bigcup\{\text{cl}_\tau(V_y) : y \in B\}$.

Let $C = \{C \subset A : x \notin \text{cl}_\tau(C)\}$, for every $C \in C$ take a cozero-set $G(C)$ such that $x \in G(C)$ and $\text{cl}_\tau(G(C)) \subset X \setminus \text{cl}_\tau(C)$. Set $A = \{G(C) : C \in C\}$, $B = \{U_y : y \in B\}$ and $\mathcal{U} = A \cup B$. Clearly $|\mathcal{U}| \leq 2^\kappa$. We claim that the family $\mathcal{U}$ of open neighbourhoods of $x$ is a $\tau$-pseudobase for $x$. Let us take $z \in X \setminus \{x\}$. If $z \in \bigcup\{\text{cl}_\tau(V_y) : y \in B\}$ then there is an $y \in B$ such that $z \notin \text{cl}_\tau(U_y) \supset \bigcap\{\text{cl}_\tau(U) : U \in \mathcal{U}\}$. If $z \in \text{cl}_\tau(A)$ let $B_z = \{B_\lambda : \lambda \in \Lambda\}$ be the family of all open neighbourhoods of $z$, choose a point $x_\lambda \in \text{cl}_\tau(B_\lambda \cap V_z) \cap A$ for every $B_\lambda \in B_z$ and set $C = \{x_\lambda : \lambda \in \Lambda\}$. Clearly $C \subset A$, $z \in \text{cl}_\tau(A)$ and $x \notin \text{cl}_\tau(A)$. Therefore $C \in C$ and $z \notin \text{cl}_\tau(G(C)) \supset \bigcap\{\text{cl}_\tau(U) : U \in \mathcal{U}\}$. Hence $\bigcap\{\text{cl}_\tau(U) : U \in \mathcal{U}\} = \{x\}$.

Theorem 6. If $X$ is a functionally Hausdorff space then $|X| \leq 2^{fs(X)}$.

Proof: Let $fs(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a $\tau$-pseudobase for $x$ with $|\mathcal{V}_x| \leq \kappa$. Construct a sequence $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of $X$ and a sequence of open collections $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$ such that:
(i) \(|A_\alpha| \leq 2^\kappa\) for every \(\alpha < \kappa^+\);
(ii) \(V_\alpha = \{V : V \in V_x, x \in \bigcup_{\beta < \alpha} A_\beta\}\), \(0 < \alpha < \kappa^+\);
(iii) If \(W\) is a family of \(\leq \kappa\) elements of \(V_\alpha\) and \(K_\lambda, \lambda < \kappa\), are subsets of \(\bigcup_{\beta < \alpha} A_\beta\) with \(|K_\lambda| \leq \kappa\) and \(X \neq \bigcup_{\lambda < \kappa} \text{cl}_{\tau_\theta}(K_\lambda) \cup \bigcup\{\text{cl}_{\tau}(W) : W \in W\}\), then \(A_\alpha \setminus (\bigcup_{\lambda < \kappa} \text{cl}_{\tau_\theta}(K_\lambda) \cup \bigcup\{\text{cl}_{\tau}(W) : W \in W\}) \neq \emptyset\).

Let \(A = \bigcup_{\alpha < \kappa^+} A_\alpha\). It is enough to show that \(A = X\). Suppose not and let \(z \in X \setminus A\). Let \(V_z = \{V_\lambda : \lambda \in \Lambda\}\), \(|\Lambda| \leq \kappa\), since \(\{z\} = \bigcap\{\text{cl}_{\tau}(V_\lambda) : \lambda \in \Lambda\}\) it follows that \(X \setminus \{z\} = \bigcup\{X \setminus \text{cl}_{\tau}(V) : \lambda \in \Lambda\}\).

For every \(\lambda \in \Lambda\) let \(S_\lambda = A \cap (X \setminus \text{cl}_{\tau}(V_\lambda))\), and for every \(y \in S_\lambda\) let \(U_y \in V_y\) such that \(z \notin \text{cl}_{\tau}(U_y)\). Since \(fs(X) \leq \kappa\) there are \(B_\lambda, C_\lambda \in [S_\lambda]^{\leq \kappa}\) such that \(S_\lambda \subset \text{cl}_{\tau_\theta}(C_\lambda) \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B_\lambda\}\).

Let \(B = \bigcup\{B_\lambda : \lambda \in \Lambda\}\), hence \(A = \bigcup\{S_\lambda : \lambda \in \Lambda\} \subset \bigcup\{\text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}\) and \(z \notin \bigcup\{\text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}\) (clearly \(z \notin \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}\), moreover for every \(\lambda \in \Lambda\) \(V_\lambda\) is an open neighbourhood of \(z\) such that \(\text{cl}_{\tau}(V_\lambda) \cap C_\lambda = \emptyset\), so \(z \notin \bigcup\{\text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\}\).

Choose \(\alpha \in \kappa^+\) such that \(B \cup \bigcup\{C_\lambda : \lambda \in \Lambda\} \subset \bigcup\{A_\beta : \beta < \alpha\}\). Now \(X \neq \bigcup\{\text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}\) by (iii) \(A_\alpha \setminus (\bigcup_{\lambda < \kappa} \text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}) \neq \emptyset\). Since \(A \subset \bigcup\{\text{cl}_{\tau_\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{\text{cl}_{\tau}(U_y) : y \in B\}\) we have a contradiction. \(\square\)

**Remark 7.** The above theorem can be proved using elementary submodels (our approach follows that of [13, 14, 3]). Let \(\kappa = fs(X)\psi_\tau(X)\) and let \(\tau\) and \(G\) be the topology on \(X\) and the family of all cozero sets of \(X\) respectively. For every \(x \in X\) let \(B_x\) be a \(\tau\)-pseudobase for \(x\) such that \(|B_x| \leq \kappa\) and let \(\psi : X \to \mathcal{P}(\tau)\) be the map defined by \(\psi(x) = B_x\) for every \(x \in X\). Let \(M\) be an elementary submodel such that \(|M| = 2^\kappa\), \(X, \tau, G, \psi \in M\) and \(M\) is closed under \(\kappa\)-sequences.

Observe that for every \(x \in X \cap M\) it follows that \(B_x \subset M\). We claim that \(X \subset M\) (and hence \(|X| \leq 2^\kappa\)). Suppose not, choose a point \(z \in X \setminus M\) and let \(B_z = \{B_\lambda : \lambda \in \Lambda\}\), \(|\Lambda| \leq \kappa\). Since \(\{z\} = \bigcap\{\text{cl}_{\tau}(B_\lambda) : \lambda \in \Lambda\}\) it follows that \(|X \setminus \{z\} = \bigcup\{X \setminus \text{cl}_{\tau}(B_\lambda) : \lambda \in \Lambda\}\). Let \(S_\lambda = X \cap \Lambda \cap (X \setminus \text{cl}_{\tau}(B_\lambda))\) for every \(\lambda \in \Lambda\). For every \(y \in S_\lambda\) let \(U_y \in M\) such that \(y \in U_y\) and \(z \notin \text{cl}_{\tau}(U_y)\). \(\{U_y : y \in S_\lambda\}\) is a family of open subsets of \(X\) such that \(S_\lambda \subset \bigcup\{U_y : y \in S_\lambda\}\). Since \(fs(X) \leq \kappa\) there are \(A_\lambda \in [S_\lambda]^{\leq \kappa}\) and \(Y_\lambda \in [U_\lambda]^{\leq \kappa}\) such that \(S_\lambda \subset \text{cl}_{\tau_\theta}(A_\lambda) \cup \bigcup\{\text{cl}_{\tau}(V) : V \in V_\lambda\}\). Let \(V_\lambda = \bigcup\{\text{cl}_{\tau}(V) : V \in V_\lambda\}\), observe that \(V = \{V_\lambda : \lambda \in \Lambda\}, A = \{\text{cl}_{\tau_\theta}(A_\lambda) : \lambda \in \Lambda\} \subset M\) and \(M\) is closed under \(\kappa\)-sequences so \(V, A \in M\). Set \(V = \bigcup V\) and \(A = \bigcup A\), by elementarity it follows that \(A \cup V \in M\). Now \(z \in X \setminus (A \cup V)\) so by elementarity there is some \(x \in X \cap M\) such that \(x \notin A \cup V\). Since \(X \cap M \subset A \cup V\) we have a contradiction.

**Remark 8.** The \(w\)-compactness degree of a space \(X\), denoted by \(wcd(X)\), is the smallest infinite cardinal \(\kappa\) such that for every open cover \(U\) of \(X\) there is a \(V \in [U]^{\leq \kappa}\) such that \(X = \bigcup\{\text{cl}_{\tau}(V) : V \in V\}\). In [2] it is shown that \(|X| \leq 2^{wcd(X)}\chi(X)\) for every functionally Hausdorff space \(X\). It is worth noting that Theorem 6 can give a better bound than the above result. The space \(X\)
in Remark 4 is a functionally Hausdorff space such that $|X| = 2^{fs(X)\psi_r(X)} < 2^{wed(X)_\chi(X)}$.

A fundamental result of Shapirovskii on spread says that if $X$ is a Hausdorff space with $s(X) \leq \kappa$ then there is a subset $S$ of $X$ such that $|S| \leq 2^\kappa$ and $X = \bigcup \{A : A \in [S]^{\leq \kappa}\}$.

We conclude this paper with the following

**Theorem 9.** Let $X$ be a functionally Hausdorff space with $fs(X) \leq \kappa$. Then there is a subset $S$ of $X$ such that $|S| \leq 2^\kappa$ and $X = \bigcup \{\text{cl}_{\tau_0}(A) : A \in [S]^{\leq \kappa}\}$.

**Proof:** By Proposition 5 it follows that $\psi_r(X) \leq 2^\kappa$, so for every $x \in X$ there is a $\tau$-pseudobase $B_x$ for $x$ such that $B_x \leq 2^\kappa$. Let $\tau$ and $\mathcal{G}$ be the topology on $X$ and the family of all cozero-sets of $X$ respectively. Moreover let $\psi : X \to \mathcal{P}(\tau)$ be the map defined by $\psi(x) = B_x$ for every $x \in X$. Take an elementary submodel $\mathcal{M}$ of cardinality $2^\kappa$ such that $X, \tau, \mathcal{G}, \psi \in \mathcal{M}$ and which is closed under $\kappa$-sequences. $X \cap \mathcal{M}$ is a subset of $X$ with the required properties. Let $x \in X$, we may assume that $x \notin X \cap \mathcal{M}$. We claim that there is a subset $A$ of $X$ such that $|A| \leq \kappa$ and $x \in \text{cl}_{\tau_0}(A)$. Observe that $B_y \subseteq \mathcal{M}$ for every $y \in X \cap \mathcal{M}$. Now for every $y \in X \cap \mathcal{M}$ take a $B_y \subseteq B_y$ (so $B_y \subseteq \mathcal{M}$) such that $x \notin \text{cl}_{\tau}(B_y)$. Since $fs(X) \leq \kappa$ it follows that there are $A, B \subseteq [X \cap \mathcal{M}]^{\leq \kappa}$ such that $X \cap \mathcal{M} \subseteq \text{cl}_{\tau_0}(A) \cup \{\text{cl}_{\tau}(B_y) : y \in B\}$. Since $A \subseteq [\mathcal{M}]^{\leq \kappa}$ and $\mathcal{M}$ is closed under $\kappa$-sequences it follows that $A \in \mathcal{M}$ and hence $\text{cl}_{\tau_0}(A) \in \mathcal{M}$. Moreover $\{\text{cl}_{\tau}(B_y) : y \in B\} \subseteq [\mathcal{M}]^{\leq \kappa}$ and again $\{\text{cl}_{\tau}(B_y) : y \in B\} \in \mathcal{M}$. Therefore $\text{cl}_{\tau_0}(A) \cup \{\text{cl}_{\tau}(B_y) : y \in B\} \in \mathcal{M}$, hence $X = \text{cl}_{\tau_0}(A) \cup \{\text{cl}_{\tau}(B_y) : y \in B\}$ and $x \in \text{cl}_{\tau_0}(A)$. □

**References**


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