Washek Frank Pfeffer
On variations of functions of one real variable


Persistent URL: http://dml.cz/dmlcz/118902

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1997

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must
contain these Terms of use.
On variations of functions of one real variable

Washek F. Pfeffer

Abstract. We discuss variations of functions that provide conceptually similar descriptive definitions of the Lebesgue and Denjoy-Perron integrals.

Keywords: Lebesgue integral, Denjoy-Perron integral, variational measure

Classification: Primary 26A39, 26A45; Secondary 26A42

The conceptual affinity between the Denjoy-Perron and Lebesgue integrals was established vis-à-vis their Riemannian definitions more than twenty years ago in the works of Henstock [6], Kurzweil [8], and McShane [10]. Yet, until recently, the descriptive definitions of these integrals have little in common. Modifying the variational measures of Thomson [15] and elaborating on a new result of Bon-giorno, Di Piazza, and Skvortsov [2], we shall elucidate the similarities between the contemporary descriptive definitions of the Lebesgue integral, Denjoy-Perron integral, and $F$-integral of [12, Chapter 11].

Our ambient space is the real line $\mathbb{R}$. The interior, diameter, and the Lebesgue measure of a set $E \subset \mathbb{R}$ are denoted by $\text{int} E$, $d(E)$, and $|E|$, respectively. A set $E \subset \mathbb{R}$ with $|E| = 0$ is called negligible. The terms “almost everywhere” and “absolutely continuous” always refer to the Lebesgue measure in $\mathbb{R}$. For $x \in \mathbb{R}$ and $\varepsilon \geq 0$, we let $U(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$.

A cell is a compact nondegenerate subinterval of $\mathbb{R}$, and a figure is a finite (possibly empty) union of cells. We say figures $A$ and $B$ overlap if their interiors meet. With each nonempty figure $A$, we associate two numbers: the perimeter $\|A\|$ equal to twice the number of connected components of $A$, and the regularity

$$r(A) = \frac{|A|}{d(A)\|A\|}.$$ 

For completeness, we let $\|A\| = r(A) = 0$ whenever $A$ is the empty figure. Note that a figure $A$ is a cell whenever $r(A) \geq 1/4$, in which case $r(A) = 1/2$.

Unless specified otherwise, all functions we shall consider are real-valued. If $F$ is a function defined on a cell $A$ and $B$ is a subfigure of $A$ whose connected components are the cells $[a_1, b_1], \ldots, [a_n, b_n]$, we let

$$F(B) = \sum_{i=1}^n [F(b_i) - F(a_i)].$$
Clearly, $F(B \cup C) = F(B) + F(C)$ whenever $B$ and $C$ are nonoverlapping subfigures of $A$. Denoting by the same symbol both the function of points and the associated function of figures will lead to no confusion.

A nonnegative function $\delta$ on a set $E \subset \mathbb{R}$ is called a gage on $E$ whenever its null set $N_\delta = \{ x \in E : \delta(x) = 0 \}$ is countable. A partition is a collection (possibly empty) $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ such that $A_1, \ldots, A_p$ are nonoverlapping figures, and $x_i \in A_i$ for $i = 1, \ldots, p$. Given $\varepsilon > 0$, $E \subset \mathbb{R}^m$, and a gage $\delta$ on $E$, we say that $P$ is

1. cellular if each $A_i$ is a cell;
2. $\varepsilon$-regular if $r(A_i) > \varepsilon$ for $i = 1, \ldots, p$;
3. in $E$ if $\bigcup_{i=1}^p A_i \subset E$;
4. anchored in $E$ if $\{x_1, \ldots, x_p\} \subset E$;
5. $\delta$-fine if it is anchored in $E$ and $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$.

Given a positive gage $\delta$ on $A$, a collection $Q = \{(B_1, y_1), \ldots, (B_q, y_q)\}$ is called a $\delta$-fine McShane partition in $A$ if $B_1, \ldots, B_q$ are nonoverlapping subcells of $A$, each $y_i$ is a point in $A$, and $d(B_i \cup \{y_i\}) < \delta(y_i)$ for $i = 1, \ldots, q$. If each $y_i$ belongs to a set $E \subset A$, we say $Q$ is anchored in $E$.

**Proposition 1.** A function $f$ on a cell $A$ is Lebesgue integrable in $A$ if and only if there is a function $F$ on $A$ satisfying the following condition: given $\varepsilon > 0$, we can find a positive gage $\delta$ on $A$ so that

$$\sum_{i=1}^p |f(x_i)|A_i| - F(A_i)| < \varepsilon$$

for each $\delta$-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$. The function $F$ is the indefinite Lebesgue integral of $f$ in $A$; in particular, $F$ is continuous.

**Proof:** The continuity of $F$ at $x \in A$ is easily established by choosing a sufficiently small positive gage $\delta$ on $A$ and considering a $\delta$-fine partition

$$\{(A \cap [x - \eta, x + \eta], x)\}$$

(see [12, Corollary 2.3.2] for details).

Suppose the condition of the proposition is satisfied, and select a $\delta$-fine McShane partition $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ in $A$. Denote by $x_1, \ldots, x_p$ the distinct points among $y_1, \ldots, y_q$, and let $C_i = \bigcup\{B_j : y_j = x_i\}$. As $F$ is continuous, there is a $\delta$-fine cellular partition $\{(D_1, x_1), \ldots, (D_p, x_p)\}$ in $A$ such that

$$\sum_{i=1}^p \left[ |f(x_i)| \cdot |D_i| + |F(D_i)| \right] < \varepsilon$$

and

$$\sum_{i,k=1}^p \left[ |f(x_i)| \cdot |C_i \cap D_k| + |F(C_i \cap D_k)| \right] < \varepsilon.$$
If \( A_i = D_i \cup (C_i - \bigcup_{k=1}^{p} D_k) \), then \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) is a \( \delta \)-fine partition in \( A \), and we have

\[
\varepsilon > \sum_{i=1}^{p} [f(x_i)|A_i| - F(A_i)] = \sum_{i=1}^{p} [f(x_i)|D_i| - F(D_i)]
\]

\[
+ \sum_{i=1}^{p} [f(x_i)|C_i| - F(C_i)] - \sum_{i,k=1}^{p} [f(x_i)|C_i \cap D_k| - F(C_i \cap D_k)]
\]

\[
> \sum_{j=1}^{q} [f(y_j)|B_j| - F(B_j)] - 2\varepsilon.
\]

From this inequality we deduce \( \sum_{j=1}^{q} |f(y_j)|B_j| - F(B_j)| < 6\varepsilon \).

Conversely, suppose we can find a positive gage \( \delta \) on \( A \) so that

\[
\sum_{j=1}^{q} |f(y_j)|B_j| - F(B_j)| < \varepsilon
\]

for each \( \delta \)-fine McShane partition in \( A \), and select a \( \delta \)-fine partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). If \( A_{i,1}, \ldots, A_{i,n_i} \) are the connected components of \( A_i \), then

\[
\{(A_{i,j}, x_i) : j = 1, \ldots, n_i \text{ and } i = 1, \ldots, p\}
\]

is a \( \delta \)-fine McShane partition in \( A \), and we have

\[
\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i} |f(x_i)|A_{i,j}| - F(A_{i,j})| < \varepsilon.
\]

Thus the condition of the theorem is equivalent to \( f \) being McShane integrable in \( A \), and the proposition follows from \([5, \text{Theorem 10.9}]\). \( \square \)

In Proposition 1, a positive gage is needed to assure the continuity of \( F \). If \( F \) is assumed continuous and a positive gage is replaced by an arbitrary gage, the condition of Proposition 1 defines an integral that is closed with respect to the formation of improper integrals, and thus slightly more general than the Lebesgue integral.

**Proposition 2.** A function \( f \) on a cell \( A \) is Denjoy-Perron integrable in \( A \) if and only if there is a continuous function \( F \) on \( A \) satisfying the following condition: given \( \varepsilon > 0 \), we can find a gage \( \delta \) on \( A \) so that

\[
\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon
\]
for each \( \delta \)-fine cellular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). The function \( F \) is the indefinite Denjoy-Perron integral of \( f \) in \( A \).

**Proof:** In view of [5, Chapter 11], it suffices to show that if the condition of the proposition holds, it holds already for a positive gage \( \delta^+ \). To this end, enumerate the null set \( N_\delta \) of \( \delta \) as \( z_1, z_2, \ldots \), and find \( \theta_n > 0 \) so that

\[
|f(z_n)| \cdot |C| + |F(C)| < 2^{-n} \varepsilon
\]

for each cell \( C \subset U(z_n, \theta_n) \) and \( n = 1, 2, \ldots \). Now let

\[
\delta^+(x) = \begin{cases} 
\theta_n & \text{if } x = z_n \text{ for an integer } n \geq 1, \\
\delta(x) & \text{if } x \in A - N_\delta.
\end{cases}
\]

Given a \( \delta^+ \)-fine cellular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \), observe that

\[
\sum_{i=1}^{p} |f(x_i)|A_i - F(A_i) < \sum_{\delta(x_i) > 0} |f(x_i)|A_i - F(A_i) + \sum_{n=1}^{\infty} 2^{-n} \varepsilon < 2\varepsilon,
\]

which establishes the proposition. \( \Box \)

According to [5, Chapter 11], a gage in Proposition 2 can be replaced by a positive gage, in which case the continuity of \( F \) can be deduced as in Proposition 1. However, a slight modification of [12, Example 12.3.5] shows that Proposition 2 is false when cellular partitions, which are \((1/4)\)-regular partitions, are replaced by \( \alpha \)-regular partitions with \( \alpha < 1/4 \).

Propositions 1 and 2 lead to the definition of the \( \mathcal{F} \)-integral, which lies properly in between the Lebesgue and Denjoy-Perron integrals. It was introduced in [13] as a coordinate free multidimensional integral that integrates partial derivatives of differentiable functions (cf. [11]).

**Definition 3.** A function \( f \) on a cell \( A \) is called \( \mathcal{F} \)-integrable in \( A \) whenever there is a continuous function \( F \) on \( A \) satisfying the following condition: given \( \varepsilon > 0 \), we can find a gage \( \delta \) on \( A \) so that

\[
\sum_{i=1}^{p} |f(x_i)|A_i - F(A_i) < \varepsilon
\]

for each \( \delta \)-fine \( \varepsilon \)-regular partition \( \{(A_1, x_1), \ldots, (A_p, x_p)\} \) in \( A \). The function \( F \), uniquely determined by \( f \), is called the indefinite \( \mathcal{F} \)-integral of \( f \) in \( A \).

We note that the additivity properties of the \( \mathcal{F} \)-integral depend on the use of arbitrary, not necessarily positive, gages.
Remark 4. One may also consider the integrals defined by means of $\alpha$-regular partitions, where $0 < \alpha < 1/4$ is a fixed number. Whether different $\alpha$’s produce different integrals is unclear, however, the work of Jarník and Kurzweil [9] suggests this may be the case. We do not study these integrals, since they may not be invariant with respect to diffeomorphisms (a diffeomorphic image of an $\alpha$-regular figure need not be $\alpha$-regular).

Let $F$ be a function defined on a cell $A$, and let $E \subset A$ be an arbitrary set. Elaborating on the ideas of B.S. Thomson [15, Chapter 3], we define variations of $F$ corresponding to the integrals discussed earlier.

**Lebesgue variation:**

$$V^L F(E) = \inf_{\delta} \sup_P \sum_{i=1}^{p} \left| F(A_i) \right|$$

where $\delta$ is a positive gage on $E$ and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine partition in $A$ anchored in $E$.

**Denjoy-Perron variation:**

$$V^{DP} F(E) = \inf_{\delta} \sup_P \sum_{i=1}^{p} \left| F(A_i) \right|$$

where $\delta$ is a gage on $E$ and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine cellular partition in $A$ anchored in $E$.

**$F$-variation:**

$$V^F F(E) = \sup_{\alpha} \inf_{\delta} \sup_P \sum_{i=1}^{p} \left| F(A_i) \right|$$

where $\alpha > 0$, $\delta$ is a gage on $E$, and $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine $\alpha$-regular partition in $A$ anchored in $E$.

Arguments analogous to those of [15, Theorems 3.7 and 3.15] reveal that the extended real-valued functions $V^L F$, $V^{DP} F$, and $V^F F$ are Borel regular measures in $A$ (cf. [12, Lemma 3.3.14] and [3, Lemma 4.6]). We shall use this important fact in the proof of Proposition 6 below. The inequalities

$$V^{DP} F \leq V^F F \leq V^L F$$

(1)

follow directly from the definitions.

Remark 5. Let $F$ be a continuous function on a cell $A$. Employing ideas which proved Proposition 1, it is easy to show that in defining $V^L F(E)$ we can use $\delta$-fine McShane partitions. Similarly, $V^{DP} F(E)$ can be defined by positive gages (cf. [2, Proposition 6] and the proof of Proposition 2).

If $F$ is a function on a cell $A$, we denote by $VF(B)$ the usual variation of $F$ over a figure $B \subset A$ [5, Chapter 4].
Proposition 6. If $F$ is a continuous function in a cell $A$, then
\begin{equation}
V^{\text{DP}} F(B) = V^{\text{F}} F(B) = VF(B)
\end{equation}
for each figure $B \subset A$, and $V^L F(A) = VF(A)$. Moreover, $V^{\text{DP}} F = V^{\text{F}} F$ whenever $V^{\text{F}} F$ is $\sigma$-finite, and $V^{\text{F}} F = V^L F$ whenever $V^L F$ is $\sigma$-finite.

**Proof:** Equality (2), which is an easy consequence of generalized Cousin’s lemma [7, Lemma 6], was established in [1, Proposition 4.8].

If $V^{\text{F}} F$ is $\sigma$-finite, then $V^{\text{DP}} F$ and $V^{\text{F}} F$ vanish on all but countably many singletons. Thus it is not difficult to deduce from (2) that $V^{\text{DP}} F(U) = V^{\text{F}} F(U)$ for each relatively open set $U \subset A$ (see [12, Lemma 3.4.4] for details). As $V^{\text{DP}} F$ and $V^{\text{F}} F$ are $\sigma$-finite Borel regular measures in $A$, they coincide.

Let $B$ be a subfigure of $A$, and let $\text{int}_A B$ be the relative interior of $B$ in $A$. Choose a positive gage $\delta$ on $\text{int}_A B$ so that $A \cap U(x, \delta(x)) \subset B$ for each $x \in \text{int}_A B$, and let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a $\delta$-fine partition in $A$ anchored in $\text{int}_A B$. By the choice of $\delta$, each $A_i$ is contained in $B$, and so if $A_{i,1}, \ldots, A_{i,k_i}$ are the connected components of $A_i$, then
\begin{equation}
\sum_{i=1}^p |F(A_i)| \leq \sum_{i=1}^p \sum_{j=1}^{k_i} |F(A_{i,j})| \leq VF(B).
\end{equation}

From this and (1), we obtain
\begin{equation}
V^{\text{F}} F(\text{int}_A B) \leq V^L F(\text{int}_A B) \leq VF(B);
\end{equation}
in particular, $V^L F(A) = VF(A)$ by (2). Using (3), the proof is completed by the argument employed in the previous paragraph. \hfill \square

Lemma 7. Let $F$ be a function on a cell $A$. If $VF^L(\{x\}) = 0$ for each $x \in A$, then $VF^L(A) < +\infty$.

**Proof:** Observe first $F$ is continuous at $x \in A$ whenever $V^L F(\{x\}) = 0$. According to Remark 5, for each $y \in A$, there is an $\eta_y > 0$ such that $\sum_{j=1}^q |F(B_j)| < 1$ for every $\eta_y$-fine McShane partition $\{(B_1, y_1), \ldots, (B_q, y_q)\}$ in $A$ anchored in $\{y\}$, i.e., with $y_1 = \cdots = y_q = y$. Since $A$ is compact, we can find points $z_1, \ldots, z_n$ in $A$ so that $A$ is covered by $U(z_1, \eta_{z_1}), \ldots, U(z_n, \eta_{z_n})$. Define a positive gage $\delta$ on $A$ as follows: given $x \in A$, select a $\delta(x) > 0$ so that $U(x, \delta(x))$ is contained in some $U(z_k, \eta_{z_k})$. Now each $\delta$-fine McShane partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$ is the disjoint union of families $P_1, \ldots, P_n$ such that $A_i \subset U(z_k, \eta_{z_k})$ whenever $(A_i, x_i) \in P_k$. It follows that $\{(A_i, z_k) : (A_i, x_i) \in P_k\}$ is an $\eta_{z_k}$-fine McShane partition in $A$ anchored in $\{z_k\}$, and so
\begin{equation}
\sum_{i=1}^p |F(A_i)| = \sum_{k=1}^n \sum_{(A_i, x_i) \in P_k} |F(A_i)| < n.
\end{equation}
In view of this and Remark 5, we have $VF^L(A) \leq n$. \hfill \square
Proposition 8. A function $F$ in a cell $A$ is absolutely continuous if and only if $V^LF$ is absolutely continuous.

Proof: Let $F$ be absolutely continuous, and choose an $\eta > 0$ and a negligible set $E \subset A$. There is a $\delta > 0$ such that $\sum_{j=1}^{n}|F(B_j)| < \varepsilon$ for each collection $B_1, \ldots, B_n$ of nonoverlapping subcells of $A$ with $\sum_{j=1}^{n}|B_j| < \eta$. Find an open set $U$ containing $E$ so that $|U| < \eta$, and select a positive gage $\delta$ on $E$ such that $U(x, \delta(x)) \subset U$ for each $x \in E$. Now if $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ is a $\delta$-fine partition in $A$ anchored in $E$, then it is a partition in $U$. If $A_{i,1}, \ldots, A_{i,n_i}$ are the connected components of $A_i$, then

$$\sum_{i=1}^{p}|F(A_i)| \leq \sum_{i=1}^{p} \sum_{j=1}^{n_i}|F(A_{i,j})| < \varepsilon,$$

and $V^LF(E) = 0$ by the arbitrariness of $\varepsilon$.

Conversely, assume that $V^LF$ is absolutely continuous, and choose an $\varepsilon > 0$. In view of Lemma 7, there is an $\eta > 0$ such that $V^LF(E) < \varepsilon$ whenever $E \subset A$ and $|E| < \eta$ [14, Theorem 6.11]. If $B \subset A$ is the union of nonoverlapping cells $B_1, \ldots, B_n$ and $|B| < \eta$, then Proposition 6 implies

$$\sum_{j=1}^{n}|F(B_j)| \leq \sum_{j=1}^{n}|V^LF(B_j)| = V^LF(B) = V^{DP}F(B) \leq V^LF(B) < \varepsilon,$$

establishing the absolutely continuous of $F$.

We shall use the expression “$F$ is the indefinite integral of its derivative,” which has the following usual meaning: the function $F$ is differentiable almost everywhere in its domain, and it is the indefinite integral of $F'$ extended arbitrarily to the domain of $F$.

Theorem 9. A function $F$ on a cell $A$ is the indefinite Lebesgue integral of its derivative if and only if $V^LF$ is absolutely continuous.

Proof: The theorem follows from Proposition 8 and [5, Theorem 4.15].

Corollary 10. A function $F$ on a cell $A$ is the indefinite Lebesgue integral of its derivative whenever $V^{DP}F$ is absolutely continuous and $V^LF$ is $\sigma$-finite.

Proof: If $V^LF$ is $\sigma$-finite, then $V^LF = V^{DP}F$ by Proposition 6, and the corollary follows from Theorem 9.

Proposition 11. Let $F$ be a continuous function on a cell $A$. If $V^{DP}F$ is absolutely continuous it is $\sigma$-finite.

Proof: In a roundabout way the proposition was proved in [2, Theorem 5]. We present a direct proof, which is virtually identical to that of [2, Theorem 1].
Suppose $V^{DP} F$ is absolutely continuous but not $\sigma$-finite, and denote by $U_\circ$ the union of all open sets $U$ with $V^{DP} F(A \cap U) < +\infty$. Since $U_\circ$ is Lindelöf, the $V^{DP} F$ measure of $A \cap U_\circ$ is $\sigma$-finite. The set $K = A - U_\circ$ is compact, and it is easy to verify that $V^{DP} F(K \cap U) = +\infty$ for each open set $U$ which meets $K$. As $V^{DP} F(\{x\}) = 0$ for every $x \in A$, the set $K$ is perfect.

Claim. If $U$ is an open set which meets $K$, then $A \cap U$ contains a disjoint collection $A_1, \ldots, A_p$ of at least two cells such that the interior of each $A_i$ meets $K$, and 

$$ (4) \quad \sum_{i=1}^p |F(A_i)| > 1. $$

Proof: Select a gage $\delta$ on $K \cap U$ so that $U(x, \delta(x)) \subset U$ for each $x \in K \cap U$. There is a $\delta$-fine cellular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$ anchored in $K \cap U$ such that (4) holds. By the choice of $\delta$, each $A_i$ is contained in $A \cap U$. Since $F$ is continuous and $K$ is perfect, we can modify the cells $A_i$ so that they become disjoint, their interiors meet $K$, and they are still contained in $A \cap U$ and satisfy (4). If $p = 1$ and $A_1 = [a, b]$, find points $c$ and $d$ so that $a < c < d < b$ and both $(a, c)$ and $(d, b)$ meet $K$. As $F$ is continuous and 

$$ 1 < |F(A_1)| \leq |F([a, c])| + |F([c, d])| + |F([d, b])|, $$

the points $c$ and $d$ can be selected so that $1 < |F([a, c])| + |F([d, b])|$. Thus we may assume $p \geq 2$, and the claim is established.

Using the claim, construct inductively disjoint families $\{A_{k,1}, \ldots, A_{k,p_k}\}$ of subcells of $A$ so that the following conditions are satisfied for $k = 1, 2, \ldots$.

1. $K \cap \text{int } A_{k,i} \neq \emptyset$ for $i = 1, \ldots, p_k$.
2. Each $A_{k+1,j}$ is contained in some $A_{k,i}$.
3. Each $A_{k,i}$ contains at least two cells $A_{k+1,i}$.
4. $|\bigcup_{i=1}^{p_k} A_{k,i}| < 1/k$.
5. $\sum_{A_{k+1,j} \subset A_{k,i}} |F(A_{k+1,j})| > 1$ for $i = 1, \ldots, p_k$.

It follows from conditions 3 and 4 that $N = \bigcap_{k=1}^\infty \bigcup_{i=1}^{p_k} A_{k,i}$ is a negligible perfect subset of $A$. We obtain a contradiction by showing that $V^{DP} F(N) \geq 1$.

To this end, choose a gage $\delta$ on $N$, and for $k = 1, 2, \ldots$, let 

$$ N_k = \{x \in N : \delta(x) > 1/k\}. $$

Since the set $\bigcup_{k=1}^\infty N_k = N - N_\delta$ is completely metrizable [4, Theorem 4.3.23]. By the Baire category theorem some $N_\delta$ is dense in $(N - N_\delta) \cap U$, where $U$ is an open set which meets $N - N_\delta$. There is an integer $k > s$ such that some $A_{k-1,i}$ is contained in $U$. Condition 4 implies that $d(A_{k,i}) < 1/s$ for $i = 1, \ldots, p_k$. Hence choosing $x_i \in A_{k,i} \cap N_s$, we obtain a $\delta$-fine cellular partition $\{(A_{k,1}, x_1), \ldots, (A_{k,p_k}, x_{p_k})\}$ in $A$ anchored in $N$. The desired contradiction follows from condition 5. □
Theorem 12. A continuous function $F$ on a cell $A$ is the indefinite Denjoy-Perron integral of its derivative if and only if $V^{DP} F$ is absolutely continuous.

Proof: The theorem follows from Proposition 11 and [1, Theorem 4.4], which asserts that $F$ is the indefinite Denjoy-Perron integral of its derivative if and only if $V^{DP} F$ is absolutely continuous and $\sigma$-finite. \[ \square \]

Theorem 13. A continuous function $F$ on a cell $A$ is the indefinite $\mathcal{F}$-integral of its derivative if and only if $V^F F$ is absolutely continuous.

Proof: As the converse follows from [3, Theorem 5.3], assume $V^F F$ is absolutely continuous. Then $V^{DP} F$ is absolutely continuous by (1), and Theorem 12 implies that $F$ is differentiable at each $x \in A - N$, where $N$ is a negligible subset of $A$. We show that $F$ is the indefinite $\mathcal{F}$-integral of the function $f$ defined by the formula

$$f(x) = \begin{cases} F'(x) & \text{if } x \in A - N, \\ 0 & \text{if } x \in N. \end{cases}$$

To this end, choose an $\varepsilon > 0$, and for each $x \in A - N$, find an $\eta_x > 0$ so that

$$|F'(x)|B - F(B)| < \varepsilon^2 d(B) \|B\|$$

for each figure $B \subset A \cap U(x, \eta_x)$; the existence of $\eta_x$ is a readily verifiable consequence of the differentiability of $F$ at $x$. By our assumption, there is a gage $\beta$ on $N$ such that

$$\sum_{i=1}^P |F(A_i)| < \varepsilon$$

for each $\beta$-fine $\varepsilon$-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$ anchored in $N$. Let

$$\delta(x) = \begin{cases} \eta_x & \text{if } x \in A - N, \\ \beta(x) & \text{if } x \in N, \end{cases}$$

and select a $\delta$-fine $\varepsilon$-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$. Then

$$\sum_{i=1}^P |f(x)|A_i| - F(A_i)| = \sum_{x_i \in N} |F(A_i)| + \varepsilon^2 \sum_{x_i \not\in N} d(B) \|B\|$$

$$< \varepsilon + \varepsilon \sum_{x_i \not\in N} |A_i| \leq \varepsilon (1 + |A|),$$

and the theorem is proved. \[ \square \]

Corollary 14. Let $F$ be a continuous function on a cell $A$. If $V^F F$ is absolutely continuous it is $\sigma$-finite.

Proof: In view of Theorem 13, the function $F$ is the indefinite $\mathcal{F}$-integral of a function $f$ on $A$. Fix an integer $n \geq 1$ and let $E = \{x \in A : |f(x)| < n\}$. Since

$$A = \bigcup_{k=1}^{\infty} \{x \in A : |f(x)| < k\},$$

$$\sum_{i=1}^P |f(x)|A_i| - F(A_i)| = \sum_{x_i \in N} |F(A_i)| + \varepsilon^2 \sum_{x_i \not\in N} d(B) \|B\|$$

$$< \varepsilon + \varepsilon \sum_{x_i \not\in N} |A_i| \leq \varepsilon (1 + |A|),$$

and the theorem is proved. \[ \square \]
it suffices to show that $V^F F(E) < +\infty$. To this end, choose a positive $\varepsilon \leq 1$, and find a gage $\delta$ on $A$ so that
\[
\sum_{i=1}^{p} \left| f(x) |A_i| - F(A_i) \right| < \varepsilon
\]
for each $\delta$-fine $\varepsilon$-regular partition in $A$. If such a partition is anchored in $E$, then
\[
\sum_{i=1}^{p} \left| F(A_i) \right| \leq \sum_{i=1}^{p} \left| f(x) |A_i| - F(A_i) \right| + \sum_{i=1}^{p} \left| f(x) \right| \cdot |A_i| < \varepsilon + n \sum_{i=1}^{p} |A_i| \leq 1 + n|A|,
\]
and we conclude that $V^F F(E) \leq 1 + n|A|$.

\[\square\]

**Corollary 15.** A continuous function $F$ on a cell $A$ is the indefinite $\mathcal{F}$-integral of its derivative whenever $V^{DP} F$ is absolutely continuous and $V^F F$ is $\sigma$-finite.

**Proof:** If $V^F F$ is $\sigma$-finite, then $V^F F = V^{DP} F$ by Proposition 6, and the corollary follows from Theorem 13. \[\square\]

**References**


Department of Mathematics, University of California, Davis, CA 95616, USA

E-mail: wfpfeffer@ucdavis.edu

(Received February 1, 1996)