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A note on Möbius inversion over power set lattices

Klaus Dohmen

Abstract. In this paper, we establish a theorem on Möbius inversion over power set lattices which strongly generalizes an early result of Whitney on graph colouring.

Keywords: Möbius inversion, power set lattices, graphs, hypergraphs, colourings

Classification: 05C15, 05C65, 05A15, 06A07, 06E99

1. Introduction

An important technique in combinatorics is the principle of Möbius inversion over partially ordered sets (see [3, Chapter 25]). For power set lattices, the principle of Möbius inversion states the following:

Proposition. Let $S$ be a finite set, $f$ and $g$ mappings from the power set of $S$ into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$, where $[X,S]$ denotes the interval $\{Y \mid X \subseteq Y \subseteq S\}$. Then, for any $X \subseteq S$,

\begin{equation}
    f(X) = \sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} g(Y).
\end{equation}

Proof: By the asserted relation between $f$ and $g$, the sum in (1) equals

\begin{align*}
    \sum_{Y \in [X,S]} (-1)^{|Y \setminus X|} \sum_{Z \in [Y,S]} f(Z) &= \sum_{Z \in [X,S]} f(Z) \sum_{Y \in [X,Z]} (-1)^{|Y \setminus X|},
\end{align*}

and this is $f(X)$ since the inner sum on the right is zero unless $X = Z$. \hfill \Box

2. A modified inversion formula

The following theorem states that under certain conditions not all terms have to be considered when evaluating the sum in (1). It can be thought of as a generalization of Whitney’s Broken-Circuits-Theorem on graph colouring.

Theorem. Let $S$ be a poset and $f, g$ mappings from the power set of $S$ into an additive group such that $g(X) = \sum_{Y \in [X,S]} f(Y)$ for any $X \subseteq S$. For fixed $X \subseteq S$, let $C$ be a set of non-empty subsets of $S$ such that each $C \in C$ is bounded
from below by an element \( C \in S \setminus (C \cup X) \) and \( f(Y) = 0 \) for all \( Y \) including \( C \cup X \) and not containing \( C \). Then

\[
(2) \quad f(X) = \sum_{Y \in [X,S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} g(Y),
\]

where

\[
(3) \quad \mathcal{Y}_0 := \{ Y \subseteq S | Y \not\supseteq C \text{ for all } C \in \mathcal{C} \}.
\]

**Proof:** Let \( \leq \) denote the partial ordering relation on \( S \) and \( \leq^* \) one of its linear extensions. For each subset \( Y \) of \( S \), \( \min^* Y \) denotes the minimum of \( Y \) with respect to \( \leq^* \). Consider an enumeration \( C_1, \ldots, C_n \) of \( \mathcal{C} \) such that \( \min^* C_1 \leq^* \ldots \leq^* \min^* C_n \), and define \( \mathcal{Y}_m := \{ Y \subseteq S | C_m \subseteq Y, C_{m+1} \not\subseteq Y, \ldots, C_n \not\subseteq Y \} \) for \( m = 1, \ldots, n \). Obviously, the power set of \( S \) is the disjoint union of \( \mathcal{Y}_0, \ldots, \mathcal{Y}_n \).

The proposition gives

\[
f(X) = \sum_{m=0}^{n} \sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y).
\]

We claim that the inner sum on the right-hand side is zero for \( m = 1, \ldots, n \). The assertions force \( C_m < c \) and hence \( C_m <^* c \) for every \( c \in C_m \). From the latter we conclude \( C_m <^* \min^* C_m \leq^* \min^* C_k \) and therefore \( C_m \not\subseteq C_k \) for \( k = m, \ldots, n \).

For such \( k, C_k \subseteq Y \) if and only if \( C_k \subseteq Y_m \) where \( Y_m := \{ Y \setminus \{C_m\} \} \cup \{\{C_m\} \setminus Y\} \).

By this, \( Y \in \mathcal{Y}_m \) if and only if \( Y_m \in \mathcal{Y}_m \). In addition, \( X \subseteq Y \) if and only if \( X \subseteq Y_m \). Hence,

\[
\sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|} g(Y) = \frac{1}{2} \sum_{Y \in [X,S] \cap \mathcal{Y}_m} \left( (-1)^{|Y \setminus X|} g(Y) + (-1)^{|Y_m \setminus X|} g(Y_m) \right).
\]

Since \( |Y \setminus X| \equiv |Y_m \setminus X|(\text{mod } 2) \), it suffices to check \( g(Y) = g(Y_m) \) for all \( Y \in [X,S] \cap \mathcal{Y}_m \). By the asserted relation between \( f \) and \( g \),

\[
g(Y) = \sum_{Z \in [Y,S], C_m \not\subseteq Z} f(Z) + \sum_{Z \in [Y,S], C_m \subseteq Z} f(Z).
\]

It is easy to see that the right sum remains unchanged when \( Y \) is replaced by \( Y_m \). The same holds for the left sum which, by the assertions of the theorem, equals zero. \( \square \)
Remark. To compare the number of terms in (1) and (2), we define \( \chi := |\mathcal{Y}_0 \cap [X,S]|/|[X,S]| \). Obviously, \( 0 \leq \chi \leq 1 \). By the well-known principle of inclusion and exclusion (which is a particular case of the next corollary),

\[
\chi = \sum_{C' \subseteq C} (-1)^{|C'|}2^{|X|}-|X \cup \bigcup_{C \in C'} C|.
\]

Hence, if \( C \) contains \( n \) pairwise disjoint sets of cardinality \( m \) \((n \in \mathbb{N}_0, m \in \mathbb{N})\) all of them being disjoint with \( X \), then \( \chi \leq (1 - 2^{-m})^n \), and this tends to zero as \( n \to \infty \).

**Corollary.** Let \( \mathcal{A} \) be a boolean algebra of sets, \( P \) a mapping from \( \mathcal{A} \) into an additive group such that \( P(\emptyset) = 0 \) and \( P(A \cup B) = P(A) + P(B) \) for all disjoint pairs \( A, B \in \mathcal{A} \), \( S \) a finite poset, \( \{A_s\}_{s \in S} \subseteq \mathcal{A} \), \( X \subseteq S \) and \( C \) a set of non-empty subsets of \( S \) such that each \( C \in \mathcal{C} \) is bounded from below by an element \( \underline{\mathcal{C}} \in S \setminus (C \cup X) \) and \( \bigcap_{c \in C} A_c \subseteq A_{\mathcal{C}} \). Then

\[
P \left( \bigcap_{x \in X} A_x \cap \bigcap_{s \in S \setminus X} \overline{C}A_s \right) = \sum_{Y \in [X,S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} P \left( \bigcap_{y \in Y} A_y \right),
\]

where \( \mathcal{Y}_0 \) is defined as in (3) and \( \overline{C}A_s \) denotes the complement of \( A_s \) in \( \mathcal{A} \).

**Proof:** For \( Y \subseteq S \) define \( f(Y) := P(\bigcap_{y \in Y} A_y \cap \bigcap_{s \in S \setminus Y} \overline{C}A_s) \), \( g(Y) := P(\bigcap_{y \in Y} A_y) \). For \( Y \) including \( C \) and not containing \( \underline{\mathcal{C}} \) there is some \( B \in \mathcal{A} \) such that \( f(Y) = P(\bigcap_{c \in C} A_c \cap \overline{C}A_{\mathcal{C}} \cap B) \), and hence \( f(Y) = 0 \). Therefore, the theorem can be applied.

**Remark.** Let \( X \) be empty and \( S_{\text{min}} \) resp. \( S_{\text{max}} \) denote the set of minimal resp. maximal elements of \( S \). If the mapping \( s \mapsto A_s \) is antitone, then it can be achieved that \( \mathcal{Y}_0 \) is the power set of \( S_{\text{min}} \) (Proof: Set \( \mathcal{C} := \{\{s\} \mid s \in S \setminus S_{\text{min}}\} \), and for each \( C \in \mathcal{C} \) choose a lower bound \( \underline{\mathcal{C}} \in S \setminus C \).) By the duality principle for posets, ‘below’ can be replaced by ‘above’ both in the theorem and in the corollary. By this, if \( s \mapsto A_s \) is isotone, then it can be achieved that \( \mathcal{Y}_0 \) becomes the power set of \( S_{\text{max}} \).

**Example 1.** In (4), \( C \) can be replaced by the set of its minimal elements with respect to set inclusion. This is an immediate consequence of the corollary and the preceding remark since \( C \mapsto [C, S] \) is an antitone mapping.

**Example 2.** A hypergraph is a set \( S \) of non-empty sets whose union \( \bigcup S \) is finite. The elements of \( S \) resp. \( \bigcup S \) are the edges resp. vertices of the hypergraph; their number is denoted by \( m(S) \) resp. \( n(S) \). Define \( m^*(S) := \sum_{s \in S} (|s| - 1) \). For \( k \in \mathbb{N} \), let \( S^{(k)} \) consist of all \( k \)-element edges of \( S \). The edges of \( S^{(1)} \) are called loops. The subsets of \( S \) are called partial hypergraphs of \( S \). A cycle in \( S \) is a sequence \((v_1, s_1, \ldots, v_k, s_k)\) where \( k > 1 \) and \( v_1, \ldots, v_k \) resp. \( s_1, \ldots, s_k \) are...
distinct vertices resp. edges, $v_i, v_{i+1} \in s_i$ for $i = 1, \ldots, k - 1$ and $v_k, v_1 \in s_k$. With respect to a linear ordering relation on $S$, a broken circuit of $S$ is obtained from the edge-set of a cycle in $S$ by removing the smallest edge. For any $\lambda \in \mathbb{N}$, a $\lambda$-colouring of $S$ is a mapping $f : \bigcup S \rightarrow \{1, \ldots, \lambda\}$ (the set of colours). For $X \subseteq S$, $P_{S,X}(\lambda)$ stands for the number of $\lambda$-colourings of $S$ such that $X$ is the set of monochromatic edges. We now establish the following statement:

Let $S$ be a loop-free, linearly ordered hypergraph, and let $X$ be a partial hypergraph of $S$ such that $S^{(2)} \setminus X$ is an initial segment of $S$ and each cycle in $S$ has an edge of $S^{(2)} \setminus X$. Then $P_{S,X}(\lambda) = \sum_{i,j} \rho_{ij} \lambda^{n(S)-i}$ where $\rho_{ij}$ equals $(-1)^{j-|X|}$ times the number of partial hypergraphs $Y$ of $S$ including $X$ but no broken circuits of $S$ and satisfying $m^*(Y) = i$ and $m(Y) = j$.

Proof: For $s \in S$ define $A_s$ as the set of $\lambda$-colourings of $S$ such that $s$ is monochromatic. For any broken circuit $C$ of $S$ let $\overline{C}$ be the unique edge such that $C \cup \{\overline{C}\}$ is the edge-set of a cycle in $S$. The assertions force $C \in S^{(2)} \setminus (C \cup X)$. Obviously, $\overline{C} \in S^{(2)}$ entrains $\bigcap_{c \in C} A_c \subseteq A_{\overline{C}}$. By the corollary, $P_{S,X}(\lambda) = \sum_{Y} (-1)^{|Y\setminus X|} |\bigcap_{y \in Y} A_y|$ where the summation is extended over all partial hypergraphs $Y$ of $S$ including $X$ but no broken circuits of $S$. By [1, Proposition], $|\bigcap_{y \in Y} A_y| = \lambda^{n(S)-m^*(Y)}$. The result now follows.

Note. A particular case of the previous example, namely where $X$ is empty, is published in [2]. For simple graphs and empty $X$, the above statement is due to Whitney (see [4]) and called Whitney’s Broken-Circuits-Theorem.

References


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