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Persistent URL: http://dml.cz/dmlcz/118922

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Antiproximinal sets in the Banach space $c(X)$

S. COBZAȘ

Abstract. If $X$ is a Banach space then the Banach space $c(X)$ of all $X$-valued convergent sequences contains a nonvoid bounded closed convex body $V$ such that no point in $C(X) \setminus V$ has a nearest point in $V$.

Keywords: antiproximinal sets, best approximation

Classification: 41A65

The distance from an element $x$ of a normed space $X$ to a nonvoid subset $M$ of $X$ is defined by $d(x, M) = \inf \{\|x - y\| : y \in M\}$. An element $y \in M$ such that $\|x - y\| = d(x, M)$ is called a nearest point to $x$ in $M$ and the set of all nearest points to $x$ in $M$ is denoted by $P_M(x)$. The set $M$ is called proximinal if $P_M(x) \neq \emptyset$ for all $x \in X$, and antiproximinal if $P_M(x) = \emptyset$ for all $x \in X \setminus M$. (Observe that $P_M(y) = \{y\}$ for all $y \in M$.)

Let $X^*$ be the conjugate space to $X$ and let $M$ be a nonvoid convex subset of $X$. A functional $f \in X^*$ is said to support $M$ (at $x$) if there exists $x \in M$ such that $f(x) = \inf f(M)$ or $f(x) = \sup f(M)$. Obviously $f \in X^*$ supports the closed unit ball $B_X$ of $X$ if and only if there exists $x \in B_X$ such that $f(x) = \|f\|$. If $f \neq 0$ then every $x \in B_X$ verifying this equality must be of norm one, i.e. $\|x\| = 1$. We shall denote by $S(M)$ the set of all support functionals of the set $M$.

V. Klee [13] called a Banach space $X$ of type $N_1$ if it contains a nonvoid closed convex antiproximinal set and of type $N_2$ if it contains a nonvoid bounded closed convex antiproximinal set. A hyperplane $\{x \in X : f(x) = a\}$ with $f \in X^*, f \neq 0,$ and $a \in \mathbb{R}$, is proximinal if $f \in S(B_X)$ and antiproximinal if $f \notin S(B_X)$. Since, by James theorem, a Banach space $X$ is reflexive if and only if $S(B_X) = X^*$, it follows that a Banach space is of type $N_1$ if and only if it is non-reflexive.

The first example of a Banach space of type $N_2$ was exhibited by M. Edelstein and A.C. Thompson [9] — the Banach space $c_0$ contains a bounded symmetric closed antiproximinal convex body. By a convex body we mean a convex set with nonvoid interior. A bounded symmetric closed convex body is called a convex cell. In [4] it was shown that the space $c$ also contains an antiproximinal convex cell and this property is shared by any Banach space of continuous functions isomorphic to $c$ ([5]). The existence of antiproximinal convex cells in more general spaces of continuous functions was proved by V.P. Fonf [10] (see also [11]).

The aim of the present note is to prove the existence of an antiproximinal convex cell in the Banach space $c(X)$ of all $X$-valued convergent sequences, where $X$ is a
non-trivial Banach space. The proof is simpler than the proof in the scalar case given in [4]. The case of the space $c_0(X)$ was considered in [6]. The notation is standard and all spaces will be considered over $\mathbb{R}$.

Let $\omega$ be the first infinite ordinal. Then $\mathbb{N} = [1, \omega]$ and $[1, \omega]$ is a compact Hausdorff space with respect to the interval topology (called also ordinal topology). If $X \neq \{0\}$ is a Banach space then $c(X)$ can be identified with the Banach space $C([1, \omega], X)$ of all continuous functions from $[1, \omega]$ to $X$, equipped with the usual sup-norm. An element $x \in c(X)$ will be denoted by $x = (x(i) : 1 \leq i \leq \omega)$ and sometimes by $(x(\omega)|x(1), x(2), \ldots)$. The conjugate of $c(X)$ is the space $l^1(X^*) = l^1([1, \omega], X^*)$ of all sequences $f = (f_i : 1 \leq i \leq \omega)$ such that $\|f\| := \sum_{1 \leq i \leq \omega} \|f_i\| < \infty$, the duality between $c(X)$ and $l^1(X^*)$ being given by the formula

$$f(x) = \sum_{1 \leq i \leq \omega} f_i(x(i))$$

for $f \in l^1(X^*)$ and $x \in c(X)$. Again the alternate notation $(f_\omega|f_1, f_2, \ldots)$ will be used to designate an element of $l^1(X^*)$.

The main result of this paper is:

**Theorem 1.** The Banach space $c(X)$ contains a bounded closed antiproximinal convex body.

The proof will be based on the following characterization of antiproximinal sets.

**Lemma 2 ([9]).** A nonvoid closed convex subset $M$ of a Banach space $X$ is antiproximinal if and only if

$$S(M) \cap S(B_X) = \{0\},$$

where $B_X$ denotes the closed unit ball of $X$.

The following lemma gives some information about the support functionals of the unit ball of $c(X)$. The characterization of support functionals of the unit ball of $C(T)$, for a compact Hausdorff space $T$, was given by S.I. Zakhovickij [19] in the scalar case and by V.L. Chakalov [1] for vector-valued functions. For characterization of support functionals of the unit balls in other concrete Banach spaces, see [7], [14] and [15].

**Lemma 3.** Let $B_c$ be the closed unit ball of $c(X)$ and let $f = (f_i : 1 \leq i \leq \omega)$, $f \neq 0$, be an element in $l^1(X^*)$.

(a) If $f = (f_i : 1 \leq i \leq \omega) \in S(B_c) \setminus \{0\}$ and $x = (x(i) : 1 \leq i \leq \omega) \in B_c$ is such that $f(x) = \|f\|$, then $f_i(x(i)) = \|f_i\|$ for all $i \in [1, \omega]$ and $\|x(i)\| = 1$ for all $i \in [1, \omega]$ such that $f_i \neq 0$.

(b) Let $\mathbb{N} = [1, \omega]$ and let $\sigma_i : \mathbb{N} \to \mathbb{N}$, $i = 1, 2$, be two strictly increasing functions such that $\sigma_1(\mathbb{N}) \cap \sigma_2(\mathbb{N}) = \emptyset$. Let $h \in X^*$, $h \neq 0$, and $\alpha_j, \beta_j > 0$, $j \in \mathbb{N}$. 
If \( f = (f_i : 1 \leq i \leq \omega) \in l^1(X^*) \) is such that \( f_{\sigma_1(j)} = \alpha_j h \) and \( f_{\sigma_2(j)} = -\beta_j h \) for all \( j \in \mathbb{N} \), then \( f \notin S(B_c) \).

**Proof:**

(a) Let \( f \in S(B_c) \setminus \{0\} \) and let \( x \in B_c \) be such that \( f(x) = \|f\| \). Since \( f_i(x(i)) \leq \|f_i\| \cdot \|x(i)\| \), for all \( i \in [1, \omega] \), it follows that

\[
\sum_{1 \leq i \leq \omega} \|f_i\| = \|f\| = f(x) = \sum_{1 \leq i \leq \omega} f_i(x(i)) \leq \sum_{1 \leq i \leq \omega} \|f_i\| \cdot \|x(i)\| \leq \sum_{1 \leq i \leq \omega} \|f_i\|,
\]

implying \( f_i(x(i)) = \|f_i\| \), for all \( i \in [1, \omega] \), and \( \|x(i)\| = 1 \) for all \( i \in [1, \omega] \) such that \( f_i \neq 0 \).

(b) Let \( h \in X^*, h \neq 0, \alpha_j, \beta_j, \sigma_1, \sigma_2 \) and \( f \in l^1(X^*) \) fulfill the hypotheses of the lemma and suppose, on the contrary, that there exists an element \( x = (x(i) : 1 \leq i \leq \omega) \in B_c \) such that \( f(x) = \|f\| \). Taking into account the first point of the lemma it follows that

\[
\alpha_j \|h\| = \|f_{\sigma_1(j)}\| = \alpha_j h(x(\sigma_1(j)))
\]

and

\[
\beta_j \|h\| = \|f_{\sigma_2(j)}\| = -\beta_j h(x(\sigma_2(j)))
\]

implying \( h(x(\sigma_1(j))) = \|h\| \) and \( h(x(\sigma_2(j))) = -\|h\| \), for all \( j \in \mathbb{N} \). Since \( \sigma_k(j) \to \omega \) for \( j \to \omega, k = 1, 2 \), and the functions \( x \) and \( h \) are continuous, the above equalities yield, for \( j \to \omega \), the contradiction \( h(x(\omega)) = \|h\| > 0 \) and \( h(x(\omega)) = -\|h\| < 0 \).

Other result we need for the proof of the Theorem 1 is the following one, emphasizing the behaviour of support functionals under linear isomorphisms. If \( X, Y \) are Banach spaces and \( A : X \to Y \) is an isomorphism then its conjugate \( A^* : Y^* \to X^* \) is an isomorphism too and \( (A^*)^{-1} = (A^{-1})^* \) ([8, Lemma VI 3.7]). The support functionals of a set \( M \subseteq X \) and of the set \( A(M) \subseteq Y \) are related as follows:

**Lemma 4** ([9, Lemma 1]). Let \( X, Y \) be Banach spaces, \( M \) a nonvoid closed convex subset of \( X \) and \( A : X \to Y \) an isomorphism. Then

\[
S(M) = A^*(S(A(M))).
\]

More exactly

\[
g \in S(A(M)) \Leftrightarrow A^*g \in S(M).
\]

Now we are in position to pass to:
Proof of Theorem 1: First we construct an isomorphism $A : c(X) \to c(X)$ in the following way. For an element $x = (x(i) : 1 \leq i \leq \omega) \in c(X)$ define $Ax : [1, \omega] \to X$ by

\[
Ax(\omega) = x(\omega) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} x(2j - 1)
\]

and

\[
Ax(i) = x(i) + \sum_{1 \leq j \leq 2^i} (-1)^j 2^{-j-2} x(2j - 1) + 2^{i-1} \sum_{1 \leq j < \omega} (-1)^j 2^{-j} x(2^i(2j - 1))
\]

for $1 \leq i < \omega$. Since the series in the right hand sides of the equalities (5) and (6) are norm convergent and $X$ is a Banach space, it follows that the definition of $Ax$ makes sense. Since

\[
\|Ax(\omega) - Ax(i)\| \leq \|x(\omega) - x(i)\| + 2^{i-1} \sum_{1 \leq j < \omega} 2^{-j} \|x\| = \|x(\omega) - x(i)\| + 2^{i-1} \|x\|,
\]

and $\lim_{i \to \omega} x(i) = x(\omega)$, it follows that $\lim_{i \to \omega} Ax(i) = Ax(\omega)$, i.e. $Ax$ is an element of $c(X)$. Obviously the operator $A : c(X) \to c(X)$ is linear. By (5) and (6) we have

\[
\|Ax(\omega)\| \leq \|x\| + 2^{-2} \|x\| = (5/4) \|x\|
\]

and, respectively,

\[
\|Ax(i)\| \leq \|x\| + 2^{-2} \|x\| + 2^{-i-1} \|x\| \leq (3/2) \|x\|
\]

for $1 \leq i < \omega$, implying

\[
(7) \quad \|Ax\| \leq (3/2) \|x\|,
\]

for all $x \in c(X)$, which is equivalent to the continuity of the operator $A$.

Now let $x \in c(X)$, $x \neq 0$, and let $i_0 \in [1, \omega]$ be such that $\|x(i_0)\| = \|x\| := \sup\{\|x(i)\| : 1 \leq i \leq \omega\}$. If $i_0 = \omega$, then, by (5), $\|Ax\| \geq \|Ax(\omega)\| \geq \|x(\omega)\| - 2^{-2} \|x\| = (3/4) \|x\|.$

If $1 \leq i_0 < \omega$, then by (6)

\[
\|Ax\| \geq \|Ax(i_0)\| \geq \|x(i_0)\| - (2^{-2} + 2^{-i_0-1}) \|x\| \geq (1/2) \|x\|.
\]

It follows that

\[
(8) \quad \|Ax\| \geq (1/2) \|x\|,
\]
for all \( x \in c(X) \). The inequalities (7) and (8) show that \( A \) is an isomorphism of 
\( c(X) \) onto \( c(X) \). Its conjugate \( A^* \) will be an isomorphism of \( l^1(X^*) \) onto \( l^1(X^*) \) 
acting by the formula

\[
A^* f(x) = f(Ax) = \sum_{1 \leq i \leq \omega} f_i(Ax(i)),
\]

for \( f \in l^1(X^*) \) and \( x \in c(X) \). Taking into account the formulae (5) and (6), 
defining the operator \( A \), one obtains

\[
f_\omega(Ax(\omega)) = f_\omega(x(\omega)) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} f_\omega(x(2j - 1))
\]

and

\[
f_i(Ax(i)) = f_i(x(i)) + \sum_{1 \leq j \leq 2^i} (-1)^j 2^{-j-2} f_i(x(2j - 1)) +
\]

\[+ 2^{-i-1} \sum_{1 \leq j < \omega} (-1)^j 2^{-j} f_i(x(2^i(2j - 1))).\]

Let \( c_0(X) \) denote the Banach space of all \( X \)-valued sequences converging to 
zero. It follows that \( c_0(X) = \{ x \in C([1, \omega], X) : x(\omega) = 0 \} \). The spaces \( c(X) \) 
and \( c_0(X) \) are isomorphic, an isomorphism \( H : c(X) \rightarrow c_0(X) \) being given by the formula

\[
H(x) = (0|x(\omega), x(1) - x(\omega), x(2) - x(\omega), \ldots)
\]

for \( x = (x(\omega)|x(1), x(2), \ldots) \in c(X) \) (see [20, p.55]). Its conjugate \( H^* \) will be 
an isomorphism of \( c_0(X)^* \) onto \( c(X)^* \). The conjugate \( c_0(X)^* \) of \( c_0(X) \) can be 
identified with the space

\[
W := \{ f \in l^1([1, \omega], X^*) : f = (f_i : 1 \leq i \leq \omega), f_\omega = 0 \},
\]

or equivalently

\[
W = \{ f \in l^1([1, \omega], X^*) : f = (0|f_1, f_2, \ldots) \},
\]

normed by \( \| f \| = \sum_{1 \leq i < \omega} \| f_i \| \). The duality between \( c_0(X) \) and \( W \) is given by the formula

\[
f(y) = \sum_{1 \leq i < \omega} f_i(y(i)) ,
\]

for \( f = (0|f_1, f_2, \ldots) \in W \) and \( y = (0|y(1), y(2), \ldots) \in c_0(X) \). Since for \( x = (x(\omega)|x(1), x(2), \ldots) \in c(X) \) and \( f = (0|f_1, f_2, \ldots) \in W \) we have

\[H^* f(x) = f(Hx) = f((0|x(\omega), x(1) - x(\omega), x(2) - x(\omega), \ldots))\]
it follows that
\[ H^*f = (f_1 - \sum_{2 \leq j < \omega} f_j f_2, f_3, \ldots). \]

Denote by \( B_c \) and \( B_{c_0} \) the closed unit balls of \( c(X) \) and \( c_0(X) \) respectively, and put
\[ V = (HA)^{-1}(B_{c_0}). \]

Since \( A \) and \( H \) are isomorphisms, it follows that \( V \) is a bounded symmetric closed convex body in \( c(X) \). We shall show that the set \( V \) is antiproximinal in \( c(X) \). To this end, by Lemma 2, it suffices to show that
\[ S(V) \cap S(B_c) = \{0\}. \]

Since, by (16), \( B_{c_0} = HA(V) \) we have
\[ S(B_{c_0}) = S(HA(V)). \]

By Lemma 4, \( S(V) = \{(HA)^*f : f \in S(HA(V))\} \) and therefore
\[ S(V) = \{(HA)^*f : f \in S(B_{c_0})\}. \]

It follows that the relation (17) will be a consequence of the implication
\[ f \in S(B_{c_0}) \setminus \{0\} \Rightarrow (HA)^*f \notin S(B_c). \]

In order to prove (20) observe that \( f = (0|f_1, f_2, \ldots) \in c_0(X)^* \), \( f \neq 0 \), supports the unit ball \( B_{c_0} \) of \( c_0(X) \) if and only if there exists \( n \in [1, \omega[ \) such that \( f_i = 0 \) for \( i > n \) and \( f_i \in S(B_X) \), for \( 1 \leq i \leq n \), where \( B_X \) denotes the closed unit ball of the space \( X \).

Now let \( f = (0|f_1, \ldots, f_n, 0, \ldots) \), \( f_n \neq 0 \), be a support functional of \( B_{c_0} \) and let us show that \((HA)^*f \notin S(B_c)\).

First suppose \( n = 1 \), i.e. \( f = (0|f_1, 0, \ldots) \) with \( f_1 \in S(B_X) \), \( f_1 \neq 0 \). By (15), \( H^*f = (f_1|0, \ldots) \) so that, denoting \( g = A^*H^*f = (HA)^*f \), formula (10) gives
\[ g(x) = f_1(x(\omega)) + \sum_{1 \leq j < \omega} (-1)^j 2^{-j-2} f_1(x(2j-1)) \]
for all \( x \in c(X) \). For \( j = 2k \) and \( j = 2k - 1 \), \( 1 \leq k < \omega \), one obtains \( g_{4k-1} = 2^{-2k-2} f_1 \) and \( g_{4k-3} = -2^{2k-3} f_1 \), respectively, so that, by Lemma 3(b), \( g \notin S(B_c) \).

If \( n \geq 2 \) then
\[ h := H^*f = (f_1 - \sum_{2 \leq i \leq n} f_i f_2, \ldots, f_n, 0, \ldots). \]

Taking into account formula (11) it follows that \( g = A^*h \) verifies \( g_{2n-1(4k-3)} = -2^{-2k+1-n} f_n \) and \( g_{2n-1(4k-1)} = 2^{-2k-n} f_n \) for all \( k \in [1, \omega[ \).

Appealing again to Lemma 3(b) it follows that \( g = A^*H^*f \notin S(B_c) \).

Theorem 1 is completely proved. \( \square \)
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References


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(Received May 8, 1996, revised November 11, 1996)