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Commentationes Mathematicae Universitatis Carolinae, Vol. 38 (1997), No. 2, 255--262

Persistent URL: http://dml.cz/dmlcz/118923

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On the position of the space of representable operators in the space of linear operators\(^1\)

G. Emmanuele

Abstract. We show results about the existence and the nonexistence of a projection from the space \(L(L^1(\lambda), X)\) of all linear and bounded operators from \(L^1(\lambda)\) into \(X\) onto the subspace \(R(L^1(\lambda), X)\) of all representable operators.

Keywords: representable operators, vector measures, L-projections, copies of \(c_0\)

Classification: 46B28, 46G10, 47B99, 46B25

Introduction

The problem of the complementability of some space \(\mathcal{H}\) of operators in the space \(L(X, Y)\) of all linear operators from a Banach space \(X\) into a Banach space \(Y\) has received much attention since the early sixties (see [Th], [AW], [To], [TW], [Ku], [Ka], [Fa], [Fe1], [Fe2], [E2], [E3], [E5], [EJ], [J], [CC], [BDLR]); particularly studied it has been the case of \(\mathcal{H} = \) compact operators, for which the best result known (see [E3], [J]) states that if a copy of \(c_0\) lives in this space, then there does not exist a projection onto it (in passing we observe that in the only two cases known in which \(c_0\) does not embed into \(K(X, Y) \neq L(X, Y)\) ([E3], [E6]) there is no hope of finding a norm one projection as proved in the recent paper [EJ]). We observe that the presence of copies of \(c_0\) in the smaller space \(\mathcal{H}\) plays an important role for the nonexistence of a projection onto \(\mathcal{H}\) even in other cases ([BDLR], [CC], [DD], [E2], [E5]). In particular, from all of the results in the papers quoted above it follows that when \(X\) and \(Y\) are classical Banach spaces, i.e. spaces with dual isometric to some \(L^p\) space, no projection from the bigger space \(L(X, Y)\) onto a smaller one exists, but in the following case (see [Fa]): Let \((\Omega, \Sigma, \mu)\) be a finite measure space; then the space \(R(L^1[0, 1], L^1(\mu))\) of all representable operators is norm one complemented in the space \(L(L^1[0, 1], L^1(\mu))\).

In this short note we wish to improve this last result by showing that also for other Banach spaces \(X\) the space \(R(L^1(\lambda), X), (S, \mathcal{F}, \lambda)\) a finite measure space, is norm one complemented in \(L(L^1(\lambda), X)\); we also observe that if \(R(L^1(\lambda), X)\) is complemented in \(L(L^1(\lambda), X)\), then clearly \(R(L^1(\lambda), Y)\) is complemented in \(L(L^1(\lambda), Y)\) for any subspace \(Y\) complemented in \(X\).

\(^1\) Work partially supported by M.U.R.S.T. of Italy (40%, 1994)
It is known (see [DU], Chapter III, especially p.62 and 84) that there is a 1-1 correspondence between the space $L(L^1(\lambda), X)$ (resp. $R(L^1(\lambda), X)$) and a subspace of the space $cabv(\lambda, X)$ (resp. $L^1(\lambda, X)$) of all countably additive vector measures $G$ with bounded variation equipped with variation norm $\|G\|(S)$ (resp. countably additive vector measures with bounded variation having a Bochner density), precisely the subspace of those measures for which there is a constant $C > 0$ such that

$$\|G(E)\| \leq C \lambda(E) \quad \forall E \in \mathcal{F}.$$  

In order to study our problem, it appears thus natural to use some results (see [C], [FRP], [E7], [R]) about the existence of a projection from the space $cabv(\lambda, X)$ onto the subspace $L^1(\lambda, X)$. Since under that correspondence the norms in $L(L^1(\lambda), X)$ and in $cabv(\lambda, X)$ are not equivalent, the mere complementability of $L^1(\lambda, X)$ in $cabv(\lambda, X)$ does not seem sufficient to guarantee the existence of the required projection of $L(L^1(\lambda), X)$ onto $R(L^1(\lambda), X)$; indeed, if $G$ is the representing measure of some $T \in L(L^1(\lambda), X)$ and $P$ is the projection of $cabv(\lambda, X)$ onto $L^1(\lambda, X)$, then $PG$ does not necessarily determine an element in $R(L^1(\lambda), X)$, since if $G$ satisfies (1) it seems that there is no reason why even $PG$ must satisfy a similar condition. However, we shall see that if the space $L^1(\lambda, X)$ is an L-summand (we refer to [HWW] for this well known definition) in the space $cabv(\lambda, X)$, then it is possible to construct simply a norm one projection from $L(L^1(\lambda), X)$ onto $R(L^1(\lambda), X)$. We also observe that if $X$ is a Banach lattice not containing $c_0$, then $L^1(\lambda, X)$ is complemented in $cabv(\lambda, X)$, as proved in [C], [E7] and [FRP]; even in this case we are able to show a complementability result about $R(L^1(\lambda), X)$ and $L(L^1(\lambda), X)$ is true, actually getting that $R(L^1(\lambda), X)$ is a projection band in $L(L^1(\lambda), X)$. The influence of the results about $cabv(\lambda, X)$ and $L^1(\lambda, X)$ does not stop here; indeed, similarly to the case of the spaces $cabv(\lambda, X)$ and $L^1(\lambda, X)$ (see [DE]), we shall be also able to prove that if the range space $X$ contains a copy of $c_0$, there is no projection as required. The existing results also show that to get the noncomplementability it is not enough to suppose the mere existence of a copy of $c_0$ inside $R(L^1(\lambda), X)$, differently from the case, quoted above, of the space of compact operators.

Results

First of all, we introduce the notion of a representable operator.

**Definition.** Let $(S, \mathcal{F}, \lambda)$ be a finite measure space and $X$ be a Banach space. A bounded linear operator $T : L^1(\lambda) \to X$ is representable if there exists $g \in L^\infty(\lambda)$ such that

$$T(f) = \int_S f(s)g(s)d\lambda \quad \forall f \in L^1(\lambda).$$

Now we prove a first complementability result, as announced in the introduction, from the proof of which the relevant role of the existence of an L-projection of $cabv(\lambda, X)$ onto $L^1(\lambda, X)$ appears clear (as underlined in the Introduction).
Theorem 1. Let $(S, \mathcal{F}, \lambda)$ be a finite measure space and $X$ be a Banach space such that the space $L^1(\lambda, X)$ is complemented in the space $cabv(\lambda, X)$ by an $L$-projection $L$. Then $R(L^1(\lambda), X)$ is norm one complemented in $L(L^1(\lambda), X)$.

Proof: Let $T$ be an element of $L(L^1(\lambda), X)$ and $G$ the representing vector measure of $T$; it is well known that $\|G(E)\| \leq \|T\|\lambda(E)$ for all $E \in \mathcal{F}$. We want to show first that $\|LG(E)\| \leq \|T\|\lambda(E)$ for all $E \in \mathcal{F}$. Given $E \in \mathcal{F}$ we have

\[
\|G\|(S) = \|G\|(E) + \|G\|(E^c) \leq \|LG\|(E) + \|G - LG\|(E^c) + \|LG\|(E^c) = \|LG\|(S) + \|G - LG\|(S) = \|G\|(S)
\]

from which it follows easily that $\|LG(E)\| \leq \|LG\|(E) \leq \|G\|(E) \leq \|T\|\lambda(E)$ for all $E \in \mathcal{F}$.

Hence, the measure $LG$ gives rise to an operator $LT$ from $L^1(\lambda)$ into $X$ that is representable ([DU, p. 84]). Furthermore, since it is easily seen that $\|LT\| = \sup\{\|LG(E)\|/\lambda(E) : E \in \mathcal{F}, \lambda(E) \neq 0\}$ (see [DU, p. 84]) we also get $\|LT\| \leq \|T\|$. We are done. □

We observe that the projection $L$ constructed above cannot be an $L$-projection, since $L(L^1(\lambda), X)$ always has nontrivial $M$-summands (see [HWW]).

It now becomes important to find examples of spaces $X$ for which $L^1(\lambda, X)$ is an $L$-summand in $cabv(\lambda, X)$. We can (partially) answer this question with the following

Proposition 2. Let $X$ be a Banach space such that $L^1(\lambda, X)$ is an $L$-summand in the bidual. Then $L^1(\lambda, X)$ is an $L$-summand in $cabv(\lambda, X)$.

Proof: In the recent papers [E7], [R] it is remarked that $cabv(\lambda, X)$ is isometric to a closed subspace of $(L^1(\lambda, X))^\ast\ast$; hence the restriction of the $L$-projection of $(L^1(\lambda, X))^\ast\ast$ onto $L^1(\lambda, X)$ works well to reach our target.

So if

(i) $X = L^1(\mu)$,

(ii) $X$ is a predual of a $W^\ast$-algebra,

(iii) $X$ is a nicely placed subspace of $L^1(\mu),$

(iv) $X$ is isometric to a quotient space $Y/Z$ where both $L^1(\lambda, Y)$ and $L^1(\lambda, Z)$ are $L$-summands in their respective biduals (for instance we can choose $Y = L^1(\mu)$ and $Z$ a reflexive subspace of it),

then we can apply our Theorem 1 as a consequence of results in [E7], [HWW], [R]. We note that the case (i) gives the old result by Fakhouri ([Fa]) quoted in the Introduction, but our proof seems to be simpler. □

In the next result we present some other cases in which Theorem 1 is applicable even if we do not know if the considered quotient space satisfies or not the hypothesis of Proposition 2.
Proposition 3. Let $X$ be a Banach space such that $L^1(\lambda, X)$ is an $L$-summand in the space $cabv(\lambda, X)$ and $Z$ be a closed subspace of $X$ having the Radon-Nikodym Property such that the map $\tilde{Q}$ from $cabv(\lambda, X)$ into $cabv(\lambda, X/Z)$ defined by $[\tilde{Q}(\nu)](E) = Q[\nu(E)]$, $E \in \mathcal{F}$ ($Q$ denotes the quotient map of $X$ onto $X/Z$), is also a quotient map (see [E7] for results implying the validity of this assumption). Then, $L^1(\lambda, X/Z)$ is an $L$-summand in $cabv(\lambda, X/Z)$.

Proof: Denote by $L$ the existing L-projection of $cabv(\lambda, X)$ onto $L^1(\lambda, X)$. In [E7] it is proved that the map $\tilde{L} : cabv(\lambda, X/Z) \to L^1(\lambda, X/Z)$ defined by

$$\tilde{L}(\tilde{\nu}) = \tilde{Q}[L(\nu)] \quad \forall \tilde{\nu} \in cabv(\lambda, X/Z), \nu \in cabv(\lambda, X), \tilde{Q}(\nu) = \tilde{\nu}$$

actually is a projection onto $L^1(\lambda, X/Z)$. Now, we show it is an L-projection. Let us suppose there is $\tilde{\nu}_0 \in cabv(\lambda, X/Z)$ for which

$$h = \|\tilde{L}(\tilde{\nu}_0)(S) + \|\tilde{\nu}_0 - \tilde{L}(\tilde{\nu}_0)||S - \|\tilde{\nu}_0||S > 0.$$ 

Choose $\nu_0 \in cabv(\lambda, X)$ such that $\tilde{Q}(\nu_0) = \tilde{\nu}_0$ and $\|\nu_0||S < \|\tilde{\nu}_0||S + h$. We get

$$\|\nu_0||S < \|\tilde{\nu}_0||S + h = \|\tilde{L}(\tilde{\nu}_0)||S + \|\tilde{\nu}_0 - \tilde{L}(\tilde{\nu}_0)||S \leq \|L(\nu_0)||S + \|\nu_0 - L(\nu_0)||S = \|\nu_0||S$$

from which our claim follows. \[\square\]

For instance, Proposition 3 can be applied with $X = L^1$ and $Z = H_0^1$.

In the papers [C], [E7] and [FRP] it is shown that $L^1(\lambda, X)$ is a projection band inside $cabv(\lambda, X)$, when $X$ is a Banach lattice not containing copies of $c_0$ (in $cabv(\lambda, X)$, the positive elements are those measures taking $\mathcal{F}$ into the positive cone of $X$); we are also able to use this result to get one more complementability result of $R(L^1(\lambda, X))$ inside $L(L^1(\lambda, X)$) that surely is not a consequence of the previous ones, because the projection of $cabv(\lambda, X)$ onto $L^1(\lambda, X)$ constructed in [C],[E7] and [FRP] is not an L-projection. Actually, we shall prove more than the mere complementability (and in this way we make more precise the result from the paper [Fa] quoted at the beginning).

Theorem 4. Let $X$ be a Banach lattice not containing copies of $c_0$. Then $R(L^1(\lambda, X)$ is a projection band in $L(L^1(\lambda, X)$.

Proof: We fix some notation: if $T \in L(L^1(\lambda, X)$), then $G_T$ will denote its representing measure (possessing a Radon-Nikodym derivative $g_T$ in case $T \in R(L^1(\lambda, X))$ and if $G \in cabv(\lambda, X)$ is a representing measure of some operator in $L(L^1(\lambda, X)$ we shall denote it by $T_G$. First of all we observe that, under our hypotheses, $L(L^1(\lambda, X)$ is a Banach lattice (see [MN, Theorem 1.5.11]) under the natural order, i.e. $T \leq H$ if and only if $H - T \geq 0$; hence $T \leq H$ if and only
if \( G_T \leq G_H \). We start showing that if \( T \) is in \( R(L^1(\lambda), X) \), then even \( T^+, T^- \), \(|T|\) are in \( R(L^1(\lambda), X) \); to this aim we observe that, for \( E \in \mathcal{F} \)

\[
G_{T^+}(E) = T^+(\chi_E) = \sup\{T(f) : 0 \leq f \leq \chi_E\} = \sup \left\{ \int_S g_T(s)f(s)d\lambda : 0 \leq f \leq \chi_E \right\} \leq \sup \left\{ \int_S |g_T(s)|f(s)d\lambda : 0 \leq f \leq \chi_E \right\} = \int_E |g_T(s)|d\lambda,
\]

where we used the fact that \( L^1(\lambda, X) \) is a Banach lattice and that the integral is a positive operator. Since the measure \( \int |g_T(s)|d\lambda \) is in \( L^1(\lambda, X) \) and \( L^1(\lambda, X) \) is an ideal in \( cabv(\lambda, X) \) (see the paper [C] for instance), we get that \( T^+ \in R(L^1(\lambda), X) \); similarly we can prove that \( T^- \in R(L^1(\lambda), X) \) and so also \(|T| \in R(L^1(\lambda), X)\). Once we have got this, it is very simple to prove that \( R(L^1(\lambda), X) \) is an ideal in \( L(L^1(\lambda), X) \). Let now \( T \) be an element in the band generated by \( R(L^1(\lambda), X) \); there is a net \((T_\alpha) \subset R(L^1(\lambda), X) \) and a decreasing net \((H_\alpha) \subset L(L^1(\lambda), X) \) such that

\[
|T_\alpha - T| \leq H_\alpha, \quad H_\alpha \downarrow 0.
\]

We have the following chain of inequalities, valid for all \( E \in \Sigma \),

\[
(G_{T_\alpha} - G_T)^+(E) = \sup \{ (G_{T_\alpha} - G_T)(B) : B \in \mathcal{F}, B \subset E \} = \sup \{ (T_\alpha - T)(\chi_B) : B \in \mathcal{F}, B \subset E \} \leq \sup \{ (T_\alpha - T)(f) : 0 \leq f \leq \chi_E \} = (T_\alpha - T)^+(\chi_E) \leq |T_\alpha - T|(\chi_E) \leq H_\alpha(\chi_E) = G_{H_\alpha}(E).
\]

Similarly, we can get that, for all \( E \in \Sigma \),

\[
(G_{T_\alpha} - G_T)^-(E) \leq G_{H_\alpha}(E).
\]

If we are able to show that \( G_{H_\alpha} \downarrow 0 \), we shall have that \( G_T \in L^1(\lambda, X) \) because \( L^1(\lambda, X) \) is a band in \( cabv(\lambda, X) \), a fact implying that \( T \in R(L^1(\lambda), X) \). But this is quite clear because \((H_\alpha)\) is decreasing and so \( H_\alpha \leq H_\beta \) for \( \alpha \geq \beta \) from which \( G_{H_\alpha} \leq G_{H_\beta} \) follows. Furthermore, using also a result due to Riesz-Kantorovich (see [AB, Theorem 1.13]) it is easy to see that \( \inf G_{H_\alpha} = 0 \). Hence, \( R(L^1(\lambda), X) \) is a band in \( L(L^1(\lambda), X) \). It remains just to show that \( R(L^1(\lambda), X) \) is a projection band in \( L(L^1(\lambda), X) \). So let us take \( T \geq 0, T \in L(L^1(\lambda), X) \); we consider the set \( Z = [0, T] \cap R(L^1(\lambda), X) \) and we observe that each element of \( Z \) has a representing measure contained in the set \( Y = [0, G_T] \cap L^1(\lambda, X) \); conversely, each element in \( Y \) determines an element in \( Z \), because if \( G \in Y \) we have \( 0 \leq G \leq G_T \) from which clearly follows, for all \( E \in \Sigma \)

\[
\|G(E)\| \leq \|G_T(E)\| \leq \|T\|\lambda(E)
\]
and so $T_G \in Z$. Let $G_0 = \sup Y$; such a supremum must exist since $L^1(\lambda, X)$ is a projection band in $cabv(\lambda, X)$ (hence $G_0 \leq G_T$ and $G_0 \in L^1(\lambda, X)$); it is not difficult to show that $T_{G_0} = \sup Z$, a fact that concludes our proof.

We observe that similar arguments to those used in Theorem 5 in [E7] allow us to show that suitable quotients of spaces $X$’s for which the complementability occurs also enjoy the same property; we do not prove here this result, but we simply state it as

**Proposition 5.** Suppose $X$ is such that $R(L^1(\lambda), X)$ is complemented into $L(L^1(\lambda), X)$. Suppose $Z$ is a closed subspace of $X$ with the Radon-Nikodym property. Define a map $\tilde{Q} : L(L^1(\lambda), X) \to L(L^1(\lambda), X/Z)$ by putting $[\tilde{Q}(T)](f) = Q[T(f)]$ ($Q$ is the quotient map of $X$ onto $X/Z$) for all $f \in L^1(\lambda)$. If $\tilde{Q}$ is a quotient map, thus $R(L^1(\lambda), X/Z)$ is complemented into $L(L^1(\lambda), X/Z)$.

This Proposition 5 allows us to improve the results about the complementability of $R(L^1(\lambda), X)$ inside $L(L^1(\lambda), X)$ that are consequences of Proposition 3; however we must underline that, in our opinion, Proposition 3 is of an independent interest, because of the nature of the projection from $cabv(\lambda, X/Z)$ onto $L^1(\lambda, X/Z)$ obtained there; in particular, we observe that Proposition 3 allows us to present some more occurrence in which the Lebesgue Decomposition Theorem (see [DU]) can be improved (see Remark 2 in [E7]).

The assumption of surjectivity considered in Proposition 5 (see also Proposition 3) cannot be dropped at all, since if $X = l_1$ and $X/Z = c_0$, we have that $R(L^1(\lambda), X) = L(L^1(\lambda), X)$, but that $R(L^1(\lambda), X/Z)$ is not complemented (see the following Theorem 6) in $L(L^1(\lambda), X/Z)$.

As remarked in the Introduction there is some case in which the projection from $L(L^1(\lambda), X)$ onto $R(L^1(\lambda), X)$ cannot be found; this happens, for instance, when $X$ contains a copy of $c_0$ as it happens in the case of the spaces $cabv(\lambda, X)$ and $L^1(\lambda, X)$ (see [DE]).

**Theorem 6.** Let $X$ contain a copy of $c_0$. Then $R(L^1(\lambda), X)$ is uncomplemented in $L(L^1(\lambda), X)$.

**Proof:** We first construct a complemented copy of $c_0$ in $R(L^1(\lambda), X)$ following a general procedure described in [E4]. Let us denote by $(x_n)$ the copy of the unit vector basis of $c_0$ in $X$ and by $(r_n)$ the sequence of Rademacher functions in $(L^1(\lambda))^\ast$. The sequence $(r_n \otimes x_n)$ is easily seen to be a copy of the unit vector basis of $c_0$ in $R(L^1(\lambda), X)$ (see for instance [E3]). If $(x_n^\ast)$ is a bounded sequence in $X^\ast$ such that $x_m(x_n^\ast) = \delta_{mn}$, the sequence $(r_n \otimes x_n^\ast)$ is easily seen to be a weak*-null sequence in $(R(L^1(\lambda), X))^\ast$ (here we consider $(r_n)$ as a sequence in $L^1(\lambda)$ and we use its weak convergence to $\theta$ as well as the fact that each representable operator is a Dunford-Pettis operator, [DU]). Hence, we can suppose ([E1], [S]) that $(r_n \otimes x_n)$ spans a complemented copy $K$ of $c_0$ in $R(L^1(\lambda), X)$; it is now a standard fact ([Ka]) that it is possible to construct a linear map from $l^\infty$ into $L(L^1(\lambda), X)$ that is an isomorphism onto some subspace $H$ of $L(L^1(\lambda), X)$, with
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$H$ containing $K$. These facts, as observed in ([Ka], [E2], [E3], [E5], [J]), imply the nonexistence of a projection from $L(L^1(\lambda), X)$ onto $R(L^1(\lambda), X)$. We are done.

\[\square\]

**Remark 1.** We observe that in order to get the conclusion of Theorem 5 it suffices to suppose that $R(L^1(\lambda), X)$ contains a complemented copy of $c_0$; we did that assuming that $X$ contains a copy of $c_0$. This is the only possibility we have; indeed, it is well known that $R(L^1(\lambda), X)$ is isometrically isomorphic to $L^\infty(\lambda, X)$ (see [DU]) and so it is enough to use a result by Diaz ([D]) stating that if $c_0$ embeds complementably in $L^\infty(\lambda, X)$, then $X$ necessarily contains a copy of $c_0$ to show the necessity of our assumption.

**Remark 2.** The proof of Theorem 5 actually shows that any subspace $\mathcal{H}$ of the space of all Dunford-Pettis operators from $L^1(\lambda)$ into $X$ is uncomplemented in $L(L^1(\lambda), X)$, whenever $X$ contains a copy of $c_0$, provided $\mathcal{H}$ contains finite dimensional operators.

**Remark 3.** We observe that on the contrary to the case of the nonexistence of a projection onto the space of compact operators mentioned at the beginning, it is possible for $R(L^1(\lambda), X)$ to contain a copy of $c_0$ and to be complemented in $L(L^1(\lambda), X)$ at the same time; for instance, in the case of $X = L^1(\mu)$ it is well known that $c_0$ lives in $R(L^1(\lambda), X)$ (see [Fe2], [E3]), even if $R(L^1(\lambda), X)$ is complemented in $L(L^1(\lambda), X)$; under this point of view $R(L^1(\lambda), X)$ behaves differently from the space of compact operators.

The present, in some sense surprising, results suggest the following final comment: the problem of the complementability of $R(L^1(\lambda), X)$ in $L(L^1(\lambda), X)$ is quite different from that of the complementability of other spaces of operators in $L(L^1(\lambda), X)$ (even if in both cases the considered norms are the supnorm) and is quite close to (or at least heavily depending upon) the problem of the complementability of $L^1(\lambda, X)$ in $cabv(\lambda, X)$ (even if the considered norms are different). We think it could be interesting to continue to investigate to which extent this dependence is valid.

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(Received August 31, 1996)