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On the product of a compact space with an hereditarily absolutely countably compact space

Maddalena Bonanzinga

Abstract. We show that the product of a compact, sequential $T_2$ space with an hereditarily absolutely countably compact $T_3$ space is hereditarily absolutely countably compact, and further that the product of a compact $T_2$ space of countable tightness with an hereditarily absolutely countably compact $\omega$-bounded $T_3$ space is hereditarily absolutely countably compact.

Keywords: compact, countably compact, absolutely countably compact, hereditarily absolutely countably compact, $\omega$-bounded, countable tightness, sequential space

Classification: 54D20, 54B10, 54D55

Introduction and preliminary

Recently Matveev in [Mat1], [Mat2] introduced a new property called absolute countable compactness (acc) which is stronger than countable compactness. He also introduced the related property hereditarily absolutely countably compact (hacc). Matveev ([Mat1]) proved that the acc and hacc properties are not necessarily preserved by products with compact space (see [Mat1, Example 2.2 and Remark 5.3]). Matveev proved however that if $Y$ is compact and first countable, and $X$ is an acc or hacc $T_2$ space, then $X \times Y$ is acc or hacc (see [Mat1, Theorem 2.3 or 5.4]), and he raised the following two questions

Question 1 [Mat1]. Is $X \times Y$ acc provided $Y$ is a compact space with countable tightness and $X$ is an acc space?

Question 2 [Mat1]. Is $X \times Y$ hacc provided $Y$ is a compact space with countable tightness and $X$ is an hacc space?

Vaughan ([Vau2]) proved that the product of a compact sequential $T_2$ space with an acc $T_3$ space is acc. With the previous result and the following well known Balogh’s theorem, Vaughan gave an affirmative answer to Question 1 in models of the proper forcing axiom [PFA] (Question 1 remains open in ZFC).

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Theorem 1 [Bal]. [PFA] Every compact Hausdorff space of countable tightness is sequential.

In this paper we augment Vaughan’s proof ([Vau2]) to show that the product of a compact sequential $T_2$ space with an hacc $T_3$ space is hacc. Then, with this result and Balogh’s theorem, we can give an affirmative answer to Question 2 provided we assume the proper forcing axiom (Question 2 remains open in ZFC).

We also consider further conditions under which the product of a compact space with an hacc space is hacc. Vaughan ([Vau2]) proved that the product of a compact $T_2$ space of countable tightness with an acc $\omega$-bounded $T_3$ space is acc and that the product of a compact $T_2$ space of countable density-tightness (defined below) with an acc $T_3$ space of countable density-tightness is acc. In this paper we consider “analogs” of the previous results for hacc spaces, that is we prove that the product of a compact $T_2$ space of countable tightness with an hacc $\omega$-bounded $T_3$ space is hacc and that the product of a compact $T_2$ space of countable tightness with an acc $T_3$ space of countable tightness is hacc. Further, we prove that the product of a compact $T_2$ space of countable density-tightness with an hacc $T_3$ space of countable density-tightness need not be hacc.

Concerning Question 2, we note that if in some model of set theory there is a counterexample $X \times Y$ to Question 2, then $Y$ must be a compact non-sequential space with countable tightness, and $X$ must be an hacc space which is not $\omega$-bounded and does not have countable tightness or, equivalently, there exists a closed subset of $X$ which does not have countable density-tightness.

Recall the following definitions:

Definition 1 [Eng]. A space $X$ is called countably compact provided every countable open cover of $X$ has a finite subcover.

Note that a characterization of countable compactness (see [Eng, 3.12.22(d)]) states that a $T_2$ space $X$ is countable compact iff for every open cover $U$ of $X$ there exists a finite set $F \subset X$ such that $St(F, U) = \bigcup \{U \in U : U \cap F \neq \emptyset\} = X$.

Definition 2 [Mat1]. A space $X$ is said to be absolutely countably compact (acc) provided for every open cover $U$ of $X$ and every dense $D \subset X$, there exists a finite set $F \subset D$ such that $St(F, U) = \bigcup \{U \in U : U \cap F \neq \emptyset\} = X$.

Matveev ([Mat1]) noted that every compact space is acc and every acc $T_2$ space is countably compact; he proved that every countably compact space with countable density-tightness (defined below) is acc. Further Vaughan ([Vau2]) proved that every countably compact, orthocompact space (defined below) is acc.

Matveev ([Mat1]) demonstrated that acc is not hereditary with respect to closed subsets, even with respect to regular closed subsets. Then he introduced the following definition.

Definition 3 [Mat1]. A space $X$ is said to be hereditarily absolutely countably compact (hacc) if all closed subspaces of $X$ are acc.
We also recall some other definitions. A space $X$ has countable tightness provided that whenever $A \subset X$ and $x \in \overline{A}$ there exists a countable $C \subset A$ such that $x \in \overline{C}$. Using different terminology, Matveev introduced ([Mat1]) the notion of countable density-tightness.

**Definition 4** [Vau2]. The **density-tightness** of a space $X$, denoted $d_t(X)$, is the smallest infinite cardinal $\kappa$ such that for every dense subset $D \subset X$ and every $x \in X$ there exists a subset $E$ of $D$ such that $|E| \leq \kappa$ and $x \in \overline{E}$.

Further $X$ is called **orthocompact** provided for every open cover $\mathcal{U}$ there exists an open refinement $\mathcal{V}$ such that for every $V' \subset V$, we have $\bigcap\{V \in \mathcal{V} : x \in V\}$ is open for each $x \in X$. A set $A \subset X$ is called **sequentially closed** if and only if $A$ contains all limits of all sequences from $A$; $X$ is called a **sequential** space provided every sequentially closed set is closed (every sequential space has countable tightness, see [Eng, 1.7.13(c)]). $X$ is called **$\omega$-bounded** if every countable subset is contained in a compact set.

Further we will use the following standard notation: for a set $D$, $[D]^{<\omega}$ denotes the set of all finite subsets of $D$ and $[D]^{\omega}$ the set of all finite or countable subsets of $D$. If $D$ is a subset of a topological space $X$, the $\aleph_0$-**closure** of $D$ (see [Arh]) is the set $[D]_{\aleph_0} = \bigcup\{M : M \in [D]^{\omega}\}$.

1. **The product of a compact sequential $T_2$ space with an hacc $T_3$ space is hacc.**

**Theorem 1.1.** If $Y$ is a compact sequential $T_2$ space, and $X$ is an hacc $T_3$ space, then $X \times Y$ is hacc.

**Proof:** We proceed similarly to the corresponding proof of Vaughan ([Vau2, Theorem 1.2]). By contradiction, suppose there exists a closed non acc subset $F$ of $X \times Y$, i.e., there exist a closed $F \subset X \times Y$, an open cover $\mathcal{U}$ of $F$ and a dense subset $D$ of $F$ such that for all $B \in [D]^{<\omega}$ we have that $St(B, \mathcal{U}) \not\supset F$. Proceeding as in [Vau2, Theorem 1.2], we conclude that the closed sets $F_B = \pi_Y(F \setminus St(B, \mathcal{U}))$ form a filter base on $Y$, where $\pi_Y$ is the projection on $Y$. Hence by compactness there exists

$$y \in \cap\{F_B : B \in [D]^{\omega}\}.$$  

Since $X \times \{y\}$ is homeomorphic to the hacc space $X$, and $F$ is closed in $X \times Y$, then $(X \times \{y\}) \cap F$ is acc.

As in [Vau2, Theorem 1.2], there exists an open set $V \subset X$ such that $(V \times \{y\}) \cap F \neq \emptyset$ and

$$(1) \quad (V \times \{y\}) \cap [D]_{\aleph_0} = \emptyset.$$  

Let $Z = \pi_Y((V \times Y) \cap [D]_{\aleph_0})$. Again as in [Vau2, Theorem 1.2], $Z$ is sequentially closed in $Y$ and, since $Y$ is sequential, $Z$ is closed in $Y$. Note that $\pi_Y((V \times Y) \cap D) \subset \pi_Y((V \times Y) \cap [D]_{\aleph_0}) = Z \subset \pi_Y((V \times Y) \cap F)$. Since $D$ is dense in $F$ and
\((V \times Y) \cap F\) is a nonempty open subset of \(F\), we have that \((V \times Y) \cap D\) is dense in \((V \times Y) \cap F\). As \(\pi_Y\) is a continuous mapping, we have that \(\pi_Y((V \times Y) \cap D)\) is dense in \(\pi_Y((V \times Y) \cap F)\), then \(\pi_Y((V \times Y) \cap D) \subseteq \pi_Y((\overline{V \times Y}) \cap [D]_{\aleph_0}) = \overline{Z} = Z\).

Since \(y \in \pi_Y((V \times \{y\}) \cap F)\), we have that \(y \in Z = \pi_Y((\overline{V \times Y}) \cap [D]_{\aleph_0})\). Then \((\overline{V \times \{y\}}) \cap [D]_{\aleph_0} \neq \emptyset\), but this contradicts (1), and completes the proof. \(\square\)

**Corollary 1.1.** \([PFA]\) If \(Y\) is a compact \(T_2\) space with countable tightness, and \(X\) is an hacc \(T_3\) space, then \(X \times Y\) is not hacc.

### 2. The product of a compact \(T_2\) space of countable tightness with an hacc \(\omega\)-bounded \(T_3\) space is hacc.

**Theorem 2.1.** \(X \times Y\) is hacc provided \(Y\) is a compact \(T_2\) space of countable tightness and \(X\) is an hacc, \(\omega\)-bounded \(T_3\) space.

**Proof:** Also in this case, we proceed similarly to the corresponding proof of Vaughan ([Vau2, Theorem 1.4]). The beginning of the proof repeats the first part of the proof of Theorem 1.1, with the modification (as in [Vau2, Theorem 1.4]) that the sets \(F_B\) need not be closed. We get that there exists

\[ y \in \bigcap \{F_B : B \in [D]^{\omega}\} \]

and an open set \(V \subset X\) such that \((V \times \{y\}) \cap F \neq \emptyset\) and

\[ (\overline{V \times \{y\}}) \cap [D]_{\aleph_0} = \emptyset. \]

Now we show that

\[ y \in \pi_Y((V \times Y) \cap D). \]

Since \(D\) is dense in \(F\), we have that \(\pi_Y((V \times Y) \cap D)\) is dense in \(\pi_Y((V \times Y) \cap F)\) and then \(\pi_Y((V \times Y) \cap F) \subset \pi_Y((\overline{V \times Y}) \cap D)\). Hence, as \((V \times \{y\}) \cap F \neq \emptyset\) and \(y \in \pi_Y((V \times \{y\}) \cap F)\), we have that \(y \in \pi_Y((V \times Y) \cap D)\).

Proceeding as in [Vau2, Theorem 1.4], we obtain the desired conclusion. \(\square\)

Vaughan ([Vau2]) obtained that the product of a compact \(T_2\) space of countable tightness with a countably compact \(GO\)-space (generalized ordered spaces, i.e., spaces which are subspaces of linearly ordered topological spaces; see [FL]) is acc. By Theorem 2.1, we have the following result concerning hacc (and then acc).

**Corollary 2.1.** \(X \times Y\) is hacc provided \(Y\) is a compact \(T_2\) space of countable tightness and \(X\) is a countably compact \(GO\)-space.

**Proof:** Since every countably compact \(GO\)-space is \(\omega\)-bounded ([GFW, Theorem 3]), and every \(GO\)-space is orthocompact ([FL, 5.23]) we have that \(X\) is an \(\omega\)-bounded, orthocompact space; further, as every countably compact, orthocompact space is acc ([Vau2]) and orthocompactness and countable compactness are hereditary with respect to closed subspaces, we have that \(X\) is hacc. Then, by Theorem 2.1, \(X \times Y\) is hacc. \(\square\)
3. The product of a compact $T_2$ space of countable tightness with an acc $T_3$ space of countable tightness is hacc.

We have the following result:

**Proposition 3.1.** $X \times Y$ is hacc provided $Y$ is a compact $T_2$ space with countable tightness and $X$ is an acc $T_3$ space with countable tightness.

**Proof:** It is well-known that $X \times Y$ is countably compact. By Malyhin’s Theorem (see [Mal]), $X \times Y$ has countable tightness. Both properties, countable compactness and countable tightness, are hereditary with respect to closed subsets. Thus by Matveev’s Theorem (see [Mat1, Theorem 1.8]), $X \times Y$ is hacc. □

While trying to answer to the following question: *do there exist a space $X$ and a compact space $Y$ such that $X$ is hacc, $X \times Y$ is acc, but $X \times Y$ is not hacc?*, we obtained the following example which answer this question and also demonstrates why some assumptions in Theorems 1.1 and 2.1 cannot be weakened.

**Example 3.1.** The product of an hacc Tychonoff space of countable density-tightness with a compact $T_2$ space of countable density-tightness need not be hacc.

Consider the Franklin-Rajagopalan spaces (see [Eng, 3.12.17(d)]) $X = T \cup Z$ and $Y = T \cup Z'$, where $T$ is homomorphic to $\omega$ with discrete topology, $Z$ is homomorphic to the ordinal $t$ (see [vD], [Vau1] or [Eng, 3.12.17(d)] where it is denoted $\delta$) with order topology and $Z'$ is homomorphic to $t + 1$ with order topology. We have that $d_t(X) = \omega$ and $d_t(Y) = \omega$ because both $X$ and $Y$ contain countable dense subsets of isolated points (so, of course, $d_t(X \times Y) = \omega$ and then $X \times Y$ is acc ([Mat1, Lemma 1.7])). Now we show that $X$ is hacc. Let $F$ be a closed subspace of $X$. Then $F = F_T \cup F_Z$ where $F_T = F \cap T$ and $F_Z = F \setminus F_T$. $F_T$ has countable density-tightness because $T$ is a countable set of isolated points; further, since $Z$ is a linearly ordered topological space, then $F_Z$ is a GO-space. Then, $F_T$ and $F_Z$ are acc ([Mat1, Lemma 1.7] and [Vau2, Corollary 1.7], respectively). Since $T$ is open in $X$ and $F$ is closed in $X$, we have that $F_T$ and $F_Z$ are regular closed in $F$. So, $F$ is written as union of regular closed acc spaces; then, from Proposition 4.3 in [Mat1], it follows that $F$ is acc. So, $X$ is hacc. Further $Y$ is compact, but $X \times Y$ is not hacc because it contains a closed copy of $t \times (t + 1)$ that is not acc (see [Bon, Theorem 1.2]). It is worth mentioning that under the assumption $t = \omega_1$, the space $X$ is even first-countable.

**References**


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