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Decaying positive solutions of some quasilinear differential equations


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Decaying positive solutions of some quasilinear differential equations

TADIE

Abstract. The existence of decaying positive solutions in \( \mathbb{R}_+ \) of the equations \((E_\lambda)\) and \((E_1^\lambda)\) displayed below is considered. From the existence of such solutions for the subhomogeneous cases (i.e. \( t^{1-p}F(r,tU,t|U'|) \to 0 \) as \( t \to \infty \)), a super-sub-solutions method (see §2.2) enables us to obtain existence theorems for more general cases.

Keywords: quasilinear elliptic, integral operators, fixed points theory
Classification: 35J70, 35J65, 34C10

1. Introduction

Let \( F \in C([0,\infty)^3; \mathbb{R}_+) \) and \( F_0 \in C([0,\infty)^2; \mathbb{R}_+) \) be such that

\[
\begin{cases}
F(r,T,S) \leq f(r)T^\gamma(1+S^q); \\
F_0(r,T) \leq f(r)T^\gamma \\
\text{where } \gamma, q \geq 0; \ f(r) \simeq r^\theta \text{ at } \infty, \ \theta \in \mathbb{R}.
\end{cases}
\]

For \( a > 1 \) and \( p \in (1, a+1) \), we investigate the existence of \((u, \lambda) \in C^1([0,\infty)) \times (0,\infty) \) which satisfy for \( r \geq 0 \) the equations

\[
\begin{align*}
(E_\lambda) & \quad D_a u + \lambda r^a F^u(r) := (r^a |u'|^{p-2}u')' + \lambda r^a F(r,u,|u'|) = 0 \\
(E_1^\lambda) & \quad \text{and } D_a u + \lambda r^a F_0(r,u) = 0,
\end{align*}
\]

where \( u \) is positive and decaying element of

\[
C_{ap}^1 := \{ u \in C^1([0,\infty)) \mid r^a |u'|^{p-2}u' \in C^1([0,\infty)) \}.
\]

For \( a = n-1, n \in \mathbb{N} \) such \( u \) is a radial solution in \( \mathbb{R}^n \) of the p-Laplacian equations

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda F(|x|,u,|\nabla u|) &= 0 \text{ and} \\
\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda F_0(|x|,u) &= 0,
\end{align*}
\]

respectively. We show that for \( \gamma_0 + q_0 < p - 1 \)

(i) such solution \( U \) exists for

\[
(E^0) \quad D_a U + r^a f(r)U^{\gamma_0}(1+|U'|^{q_0}) = 0, \quad r \geq 0;
\]
(ii) there is $\lambda_0 \equiv \lambda(f,p) > 0$ such that

$$ (E_{\lambda_0}^0) \quad Da u + \lambda_0 r^a f(r) u^{\gamma_0} (1 + |u'|^{q_0}) = 0, \quad r \geq 0 $$

has such a solution $u_0$, say, with $|u_0|_\infty, |u'_0|_\infty \in (0,1]$.
Using $u_0$ as a supersolution for $(E_{\lambda})$, we extend the result to more general cases where $\gamma \geq \gamma_0$, $q \geq q_0$ and $\lambda \in (0,\lambda_0)$.
We will also consider for $\sigma > 0$ and $\theta, \gamma, q \geq 0$ the equation

$$ (F_{\sigma}) \quad Da V + \frac{\sigma r^a}{(1+r)^\theta} V^{\gamma} \{1 + |V'|^q\} = 0, \quad r \geq 0 $$

in the goal to investigate the existence of solutions in $C^1_{ap}$ for $(F_{\sigma})$ where $F$ satisfies

$$ (f_{\theta}) \quad 0 \leq F(r,T,S) \leq (1+r)^{-\theta} T^\gamma (1+S^q). $$

It is important to note that the usual condition $F(r,u,0) \neq 0$ found in the literature for the decaying solutions ([7], [8]) is not required here as the use of a sub-super-solutions method enables us to circumvent that condition.

In the sequel the following notations and conventions will be used:

$\mu := 1/(p-1)$; \quad $t_* := \max\{1,t\}$; \quad $\int \phi := \int \phi(s) \, ds$;

$$ (1.0) \quad \begin{cases} w(t) := (1+t)^{-m}, & m = \mu b, \quad b \in (0,a+1-p] \\ \forall R > 0, & |u|_R := |u|_{C([0,R])} \quad \text{and} \quad \psi(t) := w(t)^\gamma f(t). \end{cases} $$

$C$ or $c$ will denote generic positive constants.
The main results are the following:

**Theorem 1.** Suppose that $(\gamma_0 + q_0) < p - 1$ and that

$$ (1.1) \quad \int_0^\infty s^{b+p-1} \psi(s) < \infty \quad \text{or} \quad \gamma_0 < (p-1) \left\{ \frac{b+p+\theta}{b} \right\}. $$

(1) Then $(E^0)$ has a decaying positive solution $U \in C^1_{ap}$ such that at $\infty$,

$$ (1.2) \quad U(r) \leq C r^{-m} \quad (U(r) \simeq r^{-m} \text{ if } b = a + 1 - p). $$

Moreover $\exists \lambda_0 \equiv \lambda(f,p) > 0$ such that $(E_{\lambda_0}^0)$ has a similar solution $u_0$, say, with $|u_0|_\infty, |u'_0|_\infty \in (0,1]$.

(2) For $\lambda \in (0,\lambda_0)$, $\gamma \geq \gamma_0$ and $q \geq q_0$, $(E_{\lambda})$ has a decaying positive solution $u \in C^1_{ap}$ which satisfies $(1.2)$.
Theorem 2. Suppose that \( \theta \in [0, p] \). If
\[
\gamma > \frac{(p-1)(a+1-\theta)}{a+1-p},
\]
then \( \forall q \geq 0 \), \((F_\sigma)\) has a decaying positive solution \( V \in C^1_{ap} \) and for \( \tau > 1 \) such that \( \gamma = (p-1)[a+1-p+\tau(\theta - \gamma)]/(a+1-p) \), at \( \infty \)
\[
V(r) \leq Cr^{-(a+1-p)/\gamma},
\]
provided that \( \sigma \) is small enough e.g.
\[
0 < \sigma < \left\{ \max(1, \frac{a+1-p}{\tau(p-1)}) \right\}^{\gamma+1-p} \left( \frac{a+1-p}{\tau} \right)^p (p-1)^{1-p}(\tau-1).
\]
In particular if
\[
\gamma \geq \gamma_1 := \{p^2 + (p-1)(a+1-p-\theta)/(a+1-p),
\]
then \( \forall q \geq 0 \) and \( 0 < 2\sigma < \gamma_1 := (a+1)\{\frac{a+1-p}{a+1-p}\}^{\gamma_1}, \)
\((F_\sigma)\) has such a solution \( V \) with \( V(r) \simeq r^{-(a+1-p)/(p-1)} \) at \( \infty \).

Theorem 3. (1) If \( \gamma_0(p-1) < 1 \) and (1.1) holds, then \( \forall \lambda > 0 \) and \( \gamma = \gamma_0 \),
\((E^1_\lambda)\) has a decaying positive solution \( u_\lambda \in C^1_{ap} \) which satisfies (1.2).
There is \( \lambda_0 \equiv \lambda(f, p) > 0 \) such that \((E^1_{\lambda_0})\) has such a solution \( u \) with \( |u|_\infty, |u'|_\infty \in (0, 1] \).
For \( \lambda \in (0, \lambda_0) \) and \( \gamma \geq \gamma_0 \), \((E^1_\lambda)\) has a decaying solution in \( C^1_{ap} \) which satisfies (1.2).

(2) Let \( \theta \in [0, p] \); for \( \gamma > (p-1)(a+1-\theta)/(a+1-p) \) and \( \tau > 1 \) such that
\[
\gamma = (p-1)\frac{a+1-p+\tau(\theta - \gamma)}{a+1-p},
\]
and \( 0 < \lambda \leq \{\frac{a+1-p}{a+1-p}\}^p (p-1)^{1-p}(\tau-1) \),
\((E^1_\lambda)\) has a decaying positive solution \( u \in C^1_{ap} \) which satisfies (1.4). In particular
if \( 0 \leq F_0(r, u) \leq u^{\gamma}/(1+r)^{\theta} \), \( \lambda \leq (a+1-p)/\gamma \{p-1\}^p p \) and \( \gamma \geq \gamma_1 \),
it has such a solution \( u \) such that \( u(r) \simeq r^{-(a+1-p)/(p-1)} \) at \( \infty \).

Remarks 4. (1) In Theorem 1, when \( p \geq 2 \), \( \theta \) has to be less than \( -p \) and even
for this case the existence of solutions for \( \gamma > p-1 \) is an extension of the known results ([7], [8]).

(2) As concerned \((E^1_\lambda)\) with \( F_0 \) in (f) and \( a = n-1 \), radial solutions in \( C^1([0, \infty)) \cap C^2((0, \infty)) \) are known to exist ([3]) for
\[
\gamma \geq \frac{(p-1)n+p}{n-p} \quad \text{if} \quad \theta = 0; \quad \gamma > \frac{(p-1)n+p(1+\theta)}{n-p} \quad \text{if} \quad \theta \in (-p, 0);
\]
\[
p-1 < \gamma < \frac{(p-1)n+p}{n-p} \quad \text{if} \quad \theta < -p;
\]
\[
\gamma < p-1 \quad \text{with} \quad \theta < -p \quad ([6]).
\]
So, the existence of solutions of \((E^1_\lambda)\) in \( C^1_{ap} \) for \( \gamma > \frac{(p-1)(n-\theta)}{n-p} \) and \( \theta \in [0, p] \)
provided by Theorem 3 seems to be new.
2. Preliminaries

2.1. Properties of some integrals.

Define

\[ J(t) := \int_t^\infty \left( \int_0^r \left( \frac{s}{r} \right)^a \psi(s) \right)^\mu \]  \quad \text{and} \quad K(t) := J(t)/w(t); \tag{2.1} \]

\[ \nu := \begin{cases} 0 & \text{if } b = a + 1 - p \\ a + 1 - p - b & \text{if } b \in (0, a + 1 - p) \end{cases} \]

\[ \Psi_0 := \begin{cases} \frac{1}{m} \left( \int_0^1 (s^a \psi)^\mu \right) & \text{if } b = a + 1 - p \\ \frac{p-1}{a+1-p} \left( \int_0^1 s^a \psi(s) \right)^\mu & \text{if } b < a + 1 - p \end{cases} \]

\[ \Psi_1 := 2^m \left\{ \int_0^1 (\int_0^r \psi)^\mu + \frac{1}{m} \left( \int_0^\infty s^{b+p-1} \psi \right)^\mu \right\}. \tag{2.2} \]

**Lemma 2.1.** If

\[ \int_0^\infty s^{b+p-1} \psi(s) < \infty \text{ or } \gamma > (p-1) \frac{(b+p+\theta)}{b}, \tag{2.3} \]

where \( b \in (0, a + 1 - p] \), then \( \forall t \geq 0 \)

\[ |J(t)| \leq \left( \int_0^\infty (1 + s^{b+p-1})^\mu \right)^{\mu} t_*^{-m-1} := \Psi_1 t_*^{-m-1}. \tag{2.5} \]

**Proof:** \( J(t) = \int_t^\infty r^{-m-1} \left\{ r^{-a+b+p-1} \int_0^r s^a \psi \right\}^\mu \leq \int_t^\infty r^{-m-1} \left( \int_0^\infty s^{b+p-1} \psi \right)^\mu \) on one hand and

\[ J(t) \leq \int_0^1 (\int_0^r \psi)^\mu + \int_1^\infty \left( \int_0^\infty s^{b+p-1} \psi \right)^\mu \] on the other hand; the right hand side of (2.4) then follows from the fact that \( (1+t)^m t_*^{-m} \leq 2^m \).

\[ 0 \leq -J(t)' \leq t^{-m-1} \left( \int_0^\infty s^{b+p-1} \psi \right)^\mu \] on one hand and

\[ |J(t)'| \leq \left( \int_0^\infty \psi \right)^\mu \] on the other hand; (2.5) is obtained.

\[ J(t) \geq \Psi_0 t_*^{-\gamma/(p-1)} \] whence \( K(t) \geq \Psi_0 t_*^{-\nu/(p-1)}. \)

The left hand side of (2.4) is then obtained. \( \square \)

For \( B > A > 0 \) define for \( C^1 := C^1([0, \infty)) \)

\[ E := E(A, B) = \begin{cases} \{ v \in C^1; A \leq v \leq B; |(wv)'| \leq Bt_*^{-m-1} \} & \text{if } b = a + 1 - p, \\ \{ v \in C^1; 0 \leq v \leq B; V \geq A \text{ in } [0,1]; |(wv)'| \leq Bt_*^{-m-1} \} & \text{otherwise}. \end{cases} \tag{2.6} \]

Define the operator \( G \) on \( E \) by

\[ G\phi(t) := (1+t)^m \int_t^\infty \left\{ r^{-a} \int_0^r s^a \psi(s) \phi(s)^\gamma (1 + |(w \phi)'|^q) \right\}^\mu. \tag{2.7} \]
Lemma 2.2. If (2.3) holds, then $G : E \rightarrow C^1$ is continuous and $GE$ is equicontinuous in $C^1$.

Proof: With $F^u_1 := u^\gamma(1 + |(wu)'|^q)$, $\forall u, v \in E$, $\Gamma_1(A) := A^\gamma \leq F^u_1 \leq B^\gamma(1 + B^q) := \Gamma_2(B)$ and $|F^u_1 - F^v_1| \leq C(\gamma, q, A, B)|u - v|_{C^1}$; $\Gamma$ standing for $\Gamma_1(A)$ or $\Gamma_2(B)$ according to the sign of $\mu - 1$,

\[(2.8) \quad \left| \left( \int_0^r \left( \frac{s}{r} \right)^\alpha \psi(s)F^u_1(s) \right)^\mu - \left( \int_0^r \left( \frac{s}{r} \right)^\alpha \psi(s)F^v_1(s) \right)^\mu \right| \leq \mu \Gamma \int_0^r \left( \frac{s}{r} \right)^\alpha \psi(s) |F^u_1 - F^v_1| \leq C_1(\mu, C, \Gamma)|u - v|_{C^1} \left\{ \int_0^r \left( \frac{s}{r} \right)^\alpha \psi \right\}^\mu.
\]

From (2.8) simple estimations lead to

\[(2.9) \quad |(Gu - Gv)'(t)| + |(Gu - Gv)(t)| \leq C |u - v|_{C^1} \{ |K(t)'| + K(t) \}
\]

and the continuity is obtained via Lemma 2.1.

(i) $\forall u \in E$,

$$|(Gu(t))'| \leq \Gamma^\mu \{ (1 + t)^m |K(t)| + m(1 + t)^m - 1 K(t) \} \leq C(\Gamma, B, \psi)$$

by Lemma 2.1 whence $GE$ is equicontinuous in $C([0, \infty))$.

(ii) $\forall t > s > 0$ and $u \in E$,

$$|(Gu)'(t) - (Gu)'(s)| \leq \Gamma^\mu \{ (1 + t)^m t^a - (1 + s)^m s^a - (\int_0^s y^a \psi(y)) \} + m(1 + t)^m - 1 (1 + s)^m - 1 |K(t) + m(1 + s)^m - 1 K(t) - K(s)| \} := O(t - s)$$

and $\{(Gu)' | u \in E\}$ is equicontinuous in $C([0, \infty))$. The equicontinuity follows from (i) and (ii).

2.2 A super-sub-solutions method.

Consider for $h \in C([0, \infty)^3; \mathbb{R}_+)$

\[(H) \quad H(v) := D_a v + r^a h^u(r) \equiv (r^a |v'|^{p-1} v')' + r^a h(r, v, |v'|) = 0.
\]

Definition 2.3. (1) Let $v \in C^1([0, \infty))$ be piecewise $C^2$. $v$ will be said to be a supersolution (subsolution) of (H) if

$$H(v) \leq (\geq) 0 \quad \forall \text{ a.e. } r \geq 0.
$$

(2) $w, v \in C^1([0, \infty))$ piecewise $C^2$ will be said to be H-compatible if

$\forall$ a.e. $r \geq 0$ $\quad 0 \leq w(r) \leq v(r); \quad v'(r) \leq w'(r) \leq 0; \quad H(v) \leq 0 \leq H(w)$.

Lemma 2.4. Suppose that $h^u$ is non decreasing in $u$ and $|u'|$. Let $w, v \in C^1([0, \infty))$ be H-compatible with $|v|_{C^1} \equiv |v|_{C^1([0, \infty))} < \infty$. Then

$D_a V + r^a h^w(r) := (r^a |V'|^{p-2} V')' + r^a h^w(r) = 0$ and $D_a W + r^a h^w(r) = 0$

have solutions $V, W \in C^1_{ap}$ such that $\forall r \geq 0$,

\[(2.10) \quad w \leq W \leq V \leq v \quad \text{and} \quad v' \leq V' \leq W' \leq w'.
\]
Proof: The existence of solutions of the equations in the lemma is in no doubt in view of the hypotheses on $v$. We are going to indicate how to construct those which satisfy (2.10). Define the sequences
\[ v_n(r) = \begin{cases} v(n) + I_n v(r) & \text{for } r < n \\ v(r) & \text{otherwise} \end{cases} \]
where $I_n v(r) := \int_r^n (\int_0^r s/t_a h^v) \mu$, $\mu := 1/(p - 1)$.

$D_a v_n + t^a h^v = 0 \quad \text{in } B_n = [0, n]; \quad \forall v \quad \text{for } r \geq n.$

$w_n$ are defined from $w$ in the same way.

In $B_n$, $v_n(r)' = -(\int_0^r (s/r)^a h^v) \mu \leq -(\int_0^r (s/r)^a h^w) \mu = w_n(r)'$.

As $v', (v_n)' \leq 0$ in $B_n$, $\{t^a[|v_n|^{p-1} - |v'|^{p-1}]\}' \leq 0$ whence $v' \leq (v_n)' \leq (w_n)'$ there. Thus $w_n \leq v_n \leq v$ as $v(n) = v_n(n) \geq w(n) = w_n(n)$.

Similarly in $B_n$, $w \leq w_n$ and $(w_n)' \leq w'$. So, $\forall n \in \mathbb{N} \quad w \leq w_n \leq v_n \leq v$ and $v' \leq (v_n)' \leq (w_n)' \leq w' \leq 0$.

So, $\forall M > 0$ and $B_M = [0, M]$,
\[ n > M \implies |v_n|_{C^1(B_M)} \leq |v|_{C^1} \quad \text{and} \quad |v_n|_{C^1(B_M)} \leq |v|_{C^1} \]
whence $(w_n)$ and $(v_n)$ have subsequences $(\tilde{w}_n)$ and $(\tilde{v}_n)$ say, which converge in $C^1(B_M)$ to $W_M$ and $V_M$ say, such that for some $w(M) \leq a_M \leq b_M \leq v(M)$, in $B_M$:
\[ W_M(r) = a_M + I_M w(r) \quad \text{and} \quad V_M(r) = b_M + I_M v(r). \]

In the same way $(\tilde{w}_n)_{n > M}$ and $(\tilde{v}_n)_{n > M}$ have subsequences which converge in $C^1(B_{2M})$ to $W_{2M}$ and $V_{2M}$ say, and $W_{2M} |_{B_M} = W_M$, $V_{2M} |_{B_M} = V_M$.

$W$ and $V$ are obtained as inductive limit of $(W_{kM})_{k \in \mathbb{N}}$ and $(V_{kM})_{k \in \mathbb{N}} ([5])$.

**Theorem 2.5.** (1) Suppose that the hypotheses on $w$ and $v$ in the Lemma 2.4 hold. Then (H) has a solution $\phi \in C_{ap}$ such that $w, \phi \leq v$.

(2) The existence of such a positive and decreasing supersolution for (H) is sufficient for the existence of a non trivial solution $u \in C_{ap}$ of (H) such that $0 \leq u \leq v$.

Proof: (1) Define on $E = \{ \phi \in C^1([0, \infty)) \mid w \leq \phi \leq v \text{ and } v' \leq \phi' \leq w' \}$ the operator $I$ by $I\phi(t) := A + \int_t^\infty (\int_0^t (s/r)^a h^\phi(s)) \mu$ where $A := \lim_{\infty} v(r)$.

(a) Let $\Phi = I\phi$ for $\phi \in E$; $h^w \leq h^\phi \leq h^v$ whence using the same arguments as in Lemma 2.4, $IE \subset E$ as $W \leq \Phi \leq V$ and $V' \leq \Phi' \leq W'$, $W$ and $V$ being those in that lemma.

(b) The continuity of $I : E \rightarrow E$ is easy to verify, following the same steps (with slight modifications) as for Lemma 2.2.

(c) $IE$ is equicontinuous as:
\[
\forall \phi \in E \text{ and } t > s > 0, \quad (2.11) \quad |\Phi'(t) - \Phi'(r)| \\
\leq \begin{cases} \left\{ \frac{t^a - s^a}{t-a} \left( \frac{1}{s} \int_0^s r h^v \right) + \frac{1}{t} \int_s^r r h^v \right\}^\mu & \text{if } \mu \leq 1, \\
\mu \left( \frac{1}{s^a} \int_0^s r h^v \right)^{\mu-1} \left\{ \frac{t^a - s^a}{t-a} \left( \frac{1}{s} \int_0^s r h^v \right) + \frac{1}{t} \int_s^r r h^v \right\} & \text{if } \mu > 1 \end{cases} \]

and \( \{ \Phi' \mid \phi \in E \} \) is equicontinuous as a subset of \( C([0, \infty)) \);

(ii) \( |\Phi'(t)| \leq |v'|_\infty \) whence \( IE \) is equicontinuous as a subset of \( C([0, \infty)) \).

As \( E \) is a closed and convex subset of \( C^1 \), the three reasons enable us to apply the Schauder-Tychonoff fixed point theorem to \( I \); \( I \) has a fixed point in \( E \) which is such a solution.

(2) For \( \sigma \geq \mu(2a - p) \) and \( z(r) = r^{-\sigma} \) in \( D = [1, \infty) \), \( D_a z > 0 \) in \( D \). Let \( \rho > 0 \) be such that \( z < v/2 \) and \( v' \leq z' \leq 0 \) for \( r > \rho \). Define \( z_1 \) and \( z_2 \) by

\[
z_1(r) = \begin{cases} z(\rho) & \text{for } r \leq \rho \\ z(r) & \text{for } r > \rho \end{cases} \quad \text{and} \quad z_2(r) = \begin{cases} 0 & \text{for } r \leq \rho \\ |z'(r)| & \text{for } r > \rho. \end{cases}
\]

For \( h_1^z := h(r, z_1, z_2) \), the function \( Z \) constructed from \( v \) as \( W \) in Lemma 2.4 with \( h_1^z \) replacing \( h^v \) is such that \( Z, v \) are \( H \)-compatible and (1) applies.

Without any extra difficulties, Definition 2.3, Lemma 2.4 and Theorem 2.5 apply to \( (H) \) where rather \( h \in C([0, \infty)^2; \mathbb{R}_+) \) and \( h(r, u) \) non decreasing in \( u \geq 0 \).

3. Proofs of the main theorems

3.1. Proof of Theorem 1. Let \( E \) be that in (2.6). \( \forall \phi \in E \),

\[
G\phi(t) = (1 + t)^m \int_t^\infty \{ \int_0^r \frac{s^a}{r} \psi(s)\phi(s)^{\gamma_0}(1 + |(w\phi)'|^{q_0}) \}^\mu \leq B^{\mu \gamma_0}(1 + B^{q_0})^\mu K(t) \\
\leq B^{\mu \gamma_0}(1 + B^{q_0})^\mu \Phi_1 \text{ by (2.4)}.
\]

\[|(w\phi)'(t)| \leq B^{\mu \gamma_0}(1 + B^{q_0})^\mu |J(t)'| \leq \Psi^1 B^{\mu \gamma_0}(1 + B^{q_0})^\mu \text{ by (2.5)}.
\]

For \( t \in [0, 1] \) if \( b < a + 1 - p \),

\[
G\phi(t) \geq \int_1^\infty \{ \int_0^r \frac{s^a}{r} \psi(s)\phi(s)^{\gamma_0} \}^\mu \geq A^{\mu \gamma_0} J(1) \geq A^{\mu \gamma_0} \frac{1}{m} (\int_0^1 s^a \psi)^\mu := N_2 A^{\mu \gamma_0}
\]

and for \( b = a + 1 - p \) similar lower bound is obtained \( \forall t \geq 0 \).

\( GE \subset E \) if we can find \( B > A > 0 \) such that

\[
(3.1) \quad \{ B^{\gamma_0}(1 + B^{q_0}) \}^\mu (\Psi_1 + \Psi_1) \leq B \quad \text{and} \quad N_2 A^{\mu \gamma_0} \geq A.
\]

Because \( \mu(\gamma_0 + q_0) < 1 \), in \( \{(x, y) \mid x > 0, y > 0 \} \) the curve of \( y = x \) lies above that of \( y = \{ x^{\gamma_0}(1 + x^{\gamma_0}) \}^\mu (\Psi_1 + \Psi_1) \) for \( x \geq x_0 \equiv x_0(\Psi_1, \Psi_1, \gamma_0, q_0) \). Also \( N_2 A^{\mu \gamma_0} \geq A \) for \( A \geq A_0 := A_0(N_2) \) as \( \mu \gamma_0 < 1 \).

So, with \( A_1 := \min\{x_0, A_0\} \), \( \forall (A, B) \in (0, A_1) \times [x_0, \infty) \), (3.1) holds and for such \( A \) and \( B \), \( GE \subset E \).

In that case, as from Lemma 2.2 \( G \) is continuous on \( E \) and \( GE \) equicontinuous in \( E \), \( G \) has a fixed point \( \phi \), say, in \( E \) as \( E \) is a closed and convex subset of \( C^1 \) by Schauder-Tychonoff fixed point theorem. \( U(t) := w(t)\phi(t) \) is such a required solution.

For the equation \( (E_\lambda^0) \), with \( B = 1, (3.1) \) reads

\[
(3.1a) \quad (2\lambda_0)^\mu (\Psi_1 + \Psi_1) \leq 1 \quad \text{and} \quad N_2 \lambda_0 A^{\mu \gamma_0} \geq A.
\]

So, for \( \lambda_0 = (1/2)(\Psi_1 + \Psi_1)^{-1/\mu} \) and some \( A \in (0, 1) \), we obtain \( U_0 \) as \( U \) obtained above.

For \( \lambda \in (0, \lambda_0) \), \( \gamma \geq \gamma_0 \) and \( q \geq q_0 \) \( U_0 \) is a supersolution of \( (E_\lambda) \) and Theorem 2.5 applies.
3.2 Proof of Theorem 2. From Theorem 2.5, it suffices to find a supersolution of the problem in \( C^1 \). Define

\[
v(r) := (1 + r^s)^{-\beta}; \quad s > 1; \quad \beta > 0,
\]

then for \( a > 1 \) and \( p \in (1, a + 1) \)

\[
D_au = -r^a \frac{(s\beta)^{p-1}r^{(s-1)(p-1)-1}}{(1 + r^s)^{(p-1)+p}} \{(s-1)(p-1) + a + r^s(a + 1 - p - s\beta(p-1))\}.
\]

For \( s = p/(p-1) \) and \( \beta = (a + 1 - p)/\tau p, \quad \tau > 1, \)

\[
D_au + r^a \left\{ \frac{a + 1 - p}{\tau(p-1)} \right\}^{p-1} \left\{ \frac{a + 1 + [(a + 1 - p)(\tau - 1)/\tau]r^s}{(1 + r^s)^{(p-1)+p}} \right\} = 0.
\]

This implies that

\[
D_au + \left\{ \frac{a + 1 - p}{\tau} \right\}^p (p - 1)^{1-p}(\tau - 1) r^a (1 + r^s)^{-(p-1)(\beta+1)} \leq 0
\]

whence \( \forall \theta \geq 0 \)

\[
\begin{cases}
D_au + D \frac{r^a u^\gamma}{(1+r)^\sigma} \leq 0, \quad r \geq 0 \\
\forall \gamma \geq \gamma(\tau, \theta) := (p - 1) \frac{a+1-p+\tau(p-\theta)}{a+1-p} ; \\
D := D(a, p, \tau) = \left( \frac{a+1-p}{\tau} \right)^p (p - 1)^{1-p}(\tau - 1).
\end{cases}
\]

For \( v_0 = \max\{1, \frac{a+1-p}{\tau(p-1)}\} \) and \( V(r) = v(r)/v_0, \quad V(r), |V(r)'| \in [0,1] \quad \forall r \geq 0 \)

hence

\[
\begin{cases}
\forall \gamma \geq \gamma(\tau, \theta), \sigma \in (0, \quad v_0^{1+\gamma-p}D/2] \quad \text{and} \quad q \geq 0 \\
D_au + \sigma \frac{r^a u^\gamma}{(1+r)^\sigma} (1 + |V'|^q) \leq 0, \quad r \geq 0,
\end{cases}
\]

\( V \) is then a supersolution of \( (F_\tau) \). The proof is completed by the fact that \( \forall \gamma > (p - 1)(a + 1 - p + \tau(p - \theta))/(a + 1 - p) \) and \( \theta \leq p, \) there is \( \tau > 1 \) such that \( \gamma = \gamma(\tau, \theta). \) For \( \tau = 1 \) in (3.4) and \( v_0 = (a + 1 - p)/(p - 1) \), (3.6) becomes

\[
\begin{cases}
D_au + \sigma \frac{r^a u^\gamma}{(1+r)^\sigma} (1 + |V'|^q) \leq 0, \quad r \geq 0 \\
\forall q \geq 0, \quad \sigma < \sigma_1 \quad \text{and} \quad \gamma \geq \gamma_1.
\end{cases}
\]

The proof is completed by Theorem 2.5.

3.3 Proof of Theorem 3. (1) Adapting the proof of Theorem 1 to \( (E^1_\lambda) \), we see that \( GE \subset E \) if for any \( \lambda > 0, \) there are \( B > A > 0 \) such that

\[
\lambda^\mu B^{\mu\gamma_0}(\Psi^1 + \Psi_1) \leq B \quad \text{and} \quad \lambda^\mu A^{\mu\gamma_0} N_2 \geq A;
\]

the fact that \( \mu \gamma_0 < 1 \) ensures the existence of such \( A \) and \( B. \)

As \( \mu \gamma_0 < 1, \) this part of (1) follows the same process as for Theorem 1. In the same manner, the part (2) of the theorem is obtained by a simple adaptation of the proof of Theorem 2.
References

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