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On the selector of twin functions

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Abstract. A theorem is proved which could be considered as a bridge between the combinatorics which have a beginning in the dyadic spaces theory and the partition calculus.

Keywords: twin functions, selector of twin functions, transfixed selector

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The aim of this paper is to prove a pure set-theoretical theorem which could be considered as a bridge between the combinatorics which have a beginning in the dyadic spaces theory (see [1]) and partition calculus (see [3]). As an application of this theorem, the proofs of Erdős-Rado Theorem [2], Strong Sequences Theorem [5] and the Bolzano-Weierstrass Method [4] will be given.

The Erdős-Rado Theorem has been used several times for proving important theorems (for more information see [3]). The same we can say about the role of the strong sequences theorem in dyadic spaces theory (see [1], [6]).

Main theorem

Let $X$ be a set and $\phi$ be an ordinal.

A pair $(F, G)$ of two multifunctions

\[ G : \phi \rightarrow 2^X \]
\[ F : X \rightarrow 2^\phi \]

such that

\[ (*) \text{ for any conditions } \beta < \alpha < \phi \text{ there exists } b \in G(\beta) \text{ such that } \alpha \in F(b) \]

is said to be twin functions.

A map $g : K \rightarrow X$, $K \subset \phi$, is said to be a selector of twin functions if

1. for any $\beta \in K$ there is $g(\beta) \in G(\beta)$,
2. for any $\alpha, \beta \in K; \beta < \alpha < \phi$ implies $\alpha \in F(g(\beta))$.

Fix twin functions $(F, G)$. The main result of this note is the following:
**Theorem.** If $\phi = (\kappa^+)^+$ and $\text{card}(G(\alpha)) \leq \kappa$ for each $\alpha < \phi$ then there is a selector $g : K \to X$ of the twin functions such that $\text{card}(K) \geq \lambda^+$.

A selector $g : K \to X$ is said to be **transfixed** if there is an $\alpha > \sup K$ with

$$\alpha \in \bigcap \{F(g(\beta)) : \beta \in K\}.$$ 

Denote by $\alpha(g)$ the least ordinal of this property.

For a given $\lambda$ let us denote by $\lambda^*$, $\lambda < \lambda^* \leq \phi$, the ordinal having the following property:

(I) if $g : K \to X$ is a transfixed selector such that $K \subset \lambda^*$, $\text{card}(K) \leq \lambda$, then $\alpha(g) < \lambda^*$.

**Lemma 1.** If $\lambda^* < \phi$, then there is a selector $g : K \to X$ of the twin functions such that $\text{card}(K) \geq \lambda^+$.

**Proof:** Consider the family $\mathcal{K}$ of all transfixed selectors $g : K \to X$ such that

$$\lambda^* \in \bigcap \{F(g(\beta)) : \beta \in K, K \subset \lambda^*, \text{card}(K) \leq \lambda\}$$

with the partial ordering

$$g_1 \preceq g_2 \text{ if } \text{dom } g_1 \subset \text{dom } g_2 \text{ and } g_2 \mid \text{dom } g_1 = g_1.$$ 

The set $\mathcal{K}$ is non-empty because for each $\alpha < \lambda^*$ in view of $(*)$ we have $\lambda^* \in F(G(\alpha))$. Hence there exists $a \in G(\alpha)$ such that $\lambda^* \in F(a)$. It is clear that the map $g : \{\alpha\} \to a$, $g(\alpha) = a$, belongs to $\mathcal{K}$.

Let us observe that there are no maximal elements in $\mathcal{K}$. To see this, fix $g \in \mathcal{K}$, $g : \mathcal{K} \to X$ and define $g_1 : K_1 \to X$ with $K_1 = K \cup \{\alpha(g)\}$, $g_1(\beta) = g(\beta)$ for $\beta \in K$ and $g_1(\alpha(g)) = x$, where according to the condition $(*)$ one can find $x \in G(\alpha(g))$ such that $\lambda^* \in F(x)$. Since $\alpha(g) < \lambda^*$, the map $g_1 : K \to X$ is well defined, $g_1 \in \mathcal{K}$, $g_1 \neq g$ and $g \preceq g_1$.

Now let $\mathcal{L} \subset \mathcal{K}$ be a chain. Denote by $g_{\mathcal{L}} : K_{\mathcal{L}} \to X$ a selector such that $K_{\mathcal{L}} = \bigcup \{\text{dom } g : g \in \mathcal{L}\}$ and $g_{\mathcal{L}} \mid \text{dom } g = g$. Observe that if $\text{card}(\mathcal{L}) \leq \lambda$, then $g_{\mathcal{L}} \in \mathcal{K}$. Since there are no maximal elements in $\mathcal{K}$, by the Zorn Lemma there is a chain $\mathcal{L} \subset \mathcal{K}$ such that $\lambda^+ \leq \text{card}(\mathcal{L})$. It is clear that $g_{\mathcal{L}}$ is a selector with $\lambda^+ \leq \text{card}(\text{dom } g_{\mathcal{L}})$. \hfill $\square$

**Lemma 2.** If $\text{card}(G(\alpha)) \leq \kappa$, then for each $\lambda$ such that $(\kappa^+)^+ \leq \phi$ we have $\lambda^* < \phi$.

**Proof:** Let us observe that if $(G, F)$ are twin functions, then for each $\kappa$ and $\lambda$ such that $(\kappa^+)^+ \leq \phi$, the system $(G, F)$, for which we take $\phi = (\kappa^+)^+$, is a system of twin functions.

Consider the set $\mathcal{M}$ of all transfixed selectors. By induction we shall define an increasing sequence of ordinals $\{\lambda_\alpha : \alpha < \lambda^+\}$ satisfying the following conditions:

1° $\lambda_0 = \lambda$,

2° if $\alpha$ is a limit ordinal then $\lambda_\alpha = \sup \{\lambda_\beta : \beta < \alpha\}$,

3° if $\alpha = \beta + 1$ then $\lambda_\alpha = \sup \{\alpha(g) : g \in \mathcal{M}, \text{dom } g \subset \lambda_\beta, \text{card}(\text{dom } g) \leq \lambda\} + 1$. 

Let us put \( \lambda^* = \sup \{ \lambda_\alpha : \alpha < \lambda \} \). To see that \( \lambda^* < (\kappa^\lambda)^+ \), let us observe that if \( \lambda_\beta < (\kappa^\lambda)^+ \) then the set

\[
M_\beta = \{ g \in M : \text{dom} \, g \subset \lambda_\beta, \, \text{card}(g) \leq \lambda \}
\]

has cardinality less or equal to \( \kappa^\lambda \). Therefore \( \lambda_{\beta+1} < (\kappa^\lambda)^+ \).

Now let us verify that if \( g \in M, \, \text{card}(\text{dom} \, g) \leq \lambda \) and \( \text{dom} \, g \subset \lambda^* \), then \( \alpha(g) < \lambda^* \). Indeed, if \( \text{card}(\text{dom} \, g) \leq \lambda \), \( \text{dom} \, g \subset \lambda^* \), then there is \( \beta < \lambda^+ \) such that \( \text{dom} \, g \subset \lambda_\beta \). By our construction we have \( \alpha < \lambda_{\beta+1} < \lambda^* \). \( \square \)

The Theorem is an easy corollary of Lemmas 1 and 2.

**Proof of the Theorem:** From Lemma 2 it follows that \( \lambda^* < (\kappa^\lambda)^+ \). Hence, by Lemma 1, there exists a selector \( g : K \rightarrow X \) of the twin functions such that \( \text{card}(K) \geq \lambda^+ \). \( \square \)

**Applications**

We shall prove the following theorem of P. Erdős and R. Rado [2]. By \( [X]^2 \) denote the family of all exactly two points subsets of \( X \).

**Theorem (Erdős-Rado [2]).** Suppose \( \lambda \) is an infinite cardinal number and \( F \) is a partition of \( [X]^2 \) of cardinality not greater than \( \lambda \). If the cardinality of the set \( X \) is greater than \( 2^\lambda \), then there exists a subset \( \Gamma \subset X \) of cardinality greater than \( \lambda \) such that the family \( [\Gamma]^2 \) is contained in some element of \( F \).

**Proof:** Order well the elements of \( F \) into the size \( \lambda \), i.e. \( F = \{ F_\beta : \beta < \lambda \} \).

Order well the set \( X \) into the size \( (2^\lambda)^+ \), i.e. \( X = \{ \alpha : \alpha < (2^\lambda)^+ \} \). For each \( \alpha < (2^\lambda)^+ \) let \( F_\gamma(\alpha) = \{ \beta : \{ \alpha, \beta \} \in F_\gamma \} \). Let \( Z = \{ F_\gamma(\alpha) : \alpha < (2^\lambda)^+ \text{ and } \gamma < \lambda \} \).

Let us define the functions

\[
G : (2^\lambda)^+ \rightarrow 2^Z; \, \alpha \longmapsto \{ F_\gamma(\alpha) \} : \gamma < \lambda
\]

and

\[
F : Z \rightarrow 2^{(2^\lambda)^+} : \{ F_\gamma(\alpha) \} \longmapsto F_\gamma(\alpha).
\]

We shall show that \( (F, G) \) are twin functions. For this purpose, take \( \beta < \alpha \),

\[
G(\beta) = \{ F_\gamma \} : \gamma < \lambda \text{ and } \bigcup \{ F_\gamma(\beta) : F_\gamma(\beta) \in G(\beta) \} = (2^\lambda)^+ \setminus \{ \beta \}.
\]

Hence we have \( \alpha \in F_\gamma(\beta) = F(\{ F_\gamma(\beta) \}) \) for some \( \gamma < \lambda \). Hence, by the Theorem there exists a selector \( g : K \rightarrow Z \) such that \( \lambda^+ \leq \text{card}(K) \). From this it follows that there exist \( \gamma < \lambda \) and \( \Gamma \subset K \), \( \text{card}(\Gamma) = \lambda^+ \) such that \( g(\beta) = \{ F_\gamma(\beta) \} \) for each \( \beta \in \Gamma \). Hence for each \( \alpha \) and \( \beta \) from \( \Gamma \), the condition \( \beta < \alpha \) implies that \( \alpha \in F_\gamma(\beta) \). This means that for each \( \alpha, \beta \) from \( \Gamma \) we have \( \{ \alpha, \beta \} \in F_\gamma \). Hence \( [\Gamma]^2 \subset F_\gamma \). \( \square \)

Let \( X \) be a set. Let \( r \subset [X]^\omega \times [X]^\omega \). Let \( S_\phi \) be a finite subset of \( X \) and \( H_\phi \subset X \) for \( \phi < \alpha \).
Definition. A sequence \((S_\phi, H_\phi); \phi < \alpha\) is called a strong sequence if

1° for each \(T, S \in [S_\phi \cup H_\phi]^\omega\) there is \(\text{Tr}S\),
2° for each \(\beta > \phi\) there exist \(T, S \in [S_\beta \cup H_\phi]^\omega\) such that \(\sim (\text{Tr}S)\).

Theorem (On strong sequences [1], [5], [6]). Let \(X\) be a set and \(r\) be a relation on \([X]^\omega\). Let \((S_\phi, H_\phi); \phi < (\kappa \lambda)^+\) be a strong sequence such that \(\text{card}(H_\phi) \leq \kappa\) for each \(\phi < (\kappa \lambda)^+\). Then there exists a strong sequence \((S_\phi, T_\phi); \phi < \lambda^+\), where \(\text{card}(T_\phi) < \omega\) for each \(\phi < \lambda^+\).

Proof: For each \(H_\phi\) let

\[G(\phi) = \{T : T \subset H_\phi, \text{card}(T) < \omega \text{ and there exists } \beta > \phi \text{ such that } \sim (\text{Tr}S_\beta)\}\.

Let \(X = \{T : T \in G(\phi) \text{ for some } \phi\}\). Let us define the functions:

\[G : (\kappa \lambda)^+ \rightarrow 2^X : \phi \mapsto G(\phi)\]

and

\[F : X \rightarrow (\kappa \lambda)^+ : T \mapsto \{\beta : \sim (\text{Tr}S_\beta)\}.

We shall show that \((F, G)\) are twin functions. Let \(\beta < \alpha < (\kappa \lambda)^+\), then there exists \(T \in G(\beta)\) such that \(\sim (\text{Tr}S_\alpha)\). Hence \(\alpha \in F(T)\). By the theorem there exists a selector \(g : K \rightarrow X, \lambda^+ \leq \text{card}(K)\) such that

1° for each \(\beta \in K\) we have \(g(\beta) \in G(\beta)\),
2° for each \(\alpha, \beta \in K; \beta < \alpha\) implies \(\alpha \in F(g(\beta))\).

By 1° we have that \(g(\beta) \in [H_\beta]^\omega\). By 2° we have that for \(\alpha > \beta, \sim (S_\alpha, Tr g(\beta))\).

Hence \((S_\alpha, g(\alpha)); \alpha \in K\) is a strong sequence. □

In [4] the following theorem has been proved.

Theorem (The Bolzano-Weierstrass Method). Suppose \(\lambda\) and \(\kappa\) are cardinal numbers such that \(\kappa > 1\) and \(\lambda\) is infinite. Assume that \(Y = \{y_\alpha : \alpha < (\kappa \lambda)^+\}\) is a set of different indexed points. If for any \(\alpha < (\kappa \lambda)^+\) the family

\[F_{y_\alpha} = \{F_{y_\alpha}(\beta) : \beta < \kappa\}\]

consists of pairwise disjoint subsets of \(X\) such that

\[(*) \quad \bigcup F_{y_\alpha} \cup \{y_\alpha\} \subset \bigcap \{\bigcup F_{y_\gamma} : \gamma < \alpha\},\]

then there exist a function \(f : \lambda^+ \rightarrow \kappa\) and an indexed subset \(\{p_\gamma : \gamma < \lambda^+\} \subset Y\) such that any condition \(\beta < \tau < \lambda^+\) implies \(p_\tau \in F_{p_\beta}(f(\beta))\).

Proof: Let us define the set

\[X = \{F_{y_\alpha}(\beta) : \alpha < (\kappa \lambda)^+ \text{ and } \beta < \kappa\}.\]
Let $G : (\kappa^\lambda)^+ \rightarrow 2^X : \alpha \mapsto \{F_\gamma(\alpha) : \gamma < \kappa\}$ and let $F : X \rightarrow 2^{(\kappa^\lambda)^+} : F_{y_\alpha}(\beta) \mapsto \{\gamma : y_\gamma \in F_{y_\gamma}(\beta)\}$.

We shall show that $(F, G)$ are twin functions. By $(*)$ we have that for each $\beta < \alpha$, $y_\alpha \in \bigcup F_{y_\beta}$. Hence $y_\alpha \in F_{y_\beta}(\gamma)$ for some $\gamma < \kappa$. Then $\alpha \in F(F_{y_\beta}(\gamma))$. We have $\text{card}(G(\alpha)) \leq \kappa$ for each $\alpha < (\kappa^\lambda)^+$. Then, by the theorem, there exists a selector $g : K \rightarrow X$, $\lambda^+ \leq \text{card}(K)$ such that

1° for each $\beta \in K$ there is $g(\beta) \in G(\beta)$

and

2° for each $\alpha, \beta \in K$ the condition $\beta < \alpha$ implies $\alpha \in F(g(\beta))$.

From this it follows that

for each $\alpha \in K$, $\alpha \in \bigcap F(g(\beta))$, where $\beta \in K$, $\beta < \alpha$.

The selector $g : K \rightarrow X$ and any increasing map $h$ from $\lambda^+$ into $K$ define a map $f : \lambda^+ \rightarrow \kappa$ in the following way: $f(\beta) = \gamma$ if $g(h(\beta)) = F_{y_{h(\beta)}}(\gamma)$ and a set

\[\{p_\gamma : \gamma < \lambda^+, \text{ where } p_\gamma = y_{h(\gamma)}\}\].

\[\square\]

**References**


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