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On some fan-tightness type properties

J. Valuyeva

Abstract. Properties similar to countable fan-tightness are introduced and compared to countable tightness and countable fan-tightness. These properties are also investigated with respect to function spaces and certain classes of continuous mappings.

Keywords: \(\omega\)-cover, \(\text{vet}^*(X)\), tightness, fan-tightness, biquotient mapping, \(C_p(X)\)

Classification: 54A25, 54C35

In this paper, all spaces are assumed to be Hausdorff. We denote by \(\mathbb{R}\) the set of real numbers; \(\beta X\) denotes the Stone-Čech compactification of a Tychonoff space \(X\) and \(C_p(X)\) stands for the space of all real-valued continuous functions on \(X\) with the topology of pointwise convergence. A basic open neighborhood of a function \(f \in C_p(X)\) is of the form \(W(x_1, \ldots, x_k; f; \varepsilon) = \{g \in C_p(X) : |f(x_i) - g(x_i)| < \varepsilon, i = 1, 2, \ldots, k\}\), where \(k \in \omega\), \(x_i \in X\) and \(\varepsilon > 0\). We denote by \(C_0^p(X)\) the set of all bounded continuous functions on \(X\) equipped with the topology of pointwise convergence. A cover \(\gamma\) of \(X\) is said to be an \(\omega\)-cover if for any finite subset \(F\) of \(X\) there is a \(G \in \gamma\) such that \(F \subseteq G\). The notion of countable fan-tightness was introduced in [1]: a space \(X\) is said to have countable fan-tightness (denoted \(\text{vet}(X) \leq \omega\)) if for each point \(x\) in \(X\) and any countable family \(\{A_n\}_{n \in \omega}\) of subsets of \(X\) satisfying \(x \in \bigcap\{A_n : n \in \omega\}\), there exist finite sets \(H_n \subseteq A_n\) such that \(x \in \bigcup\{H_n : n \in \omega\}\). A space \(X\) is said to have countable strong fan-tightness if for each \(x \in X\) and for each countable family \(\{A_n : n \in \omega\}\) of subsets of \(X\) such that \(x \in \bigcap\{A_n : n \in \omega\}\), there exist \(a_i \in A_i\) such that \(x \in \{a_i : i \in \omega\}\).

A space \(X\) is said to have property \(\text{vet}^*(X) \leq \omega\) if for each point \(x\) in \(X\) and any countable family \(\{A_n\}_{n \in \omega}\) of subsets of \(X\) satisfying \(x \in \bigcap\{\overline{A_n} : n \in \omega\}\), there exist sets \(H_n \subseteq A_n\) with \(|H_n| \leq n\) such that \(x \in \bigcup\{H_n : n \in \omega\}\). Clearly, every space \(X\) of countable strong fan-tightness has \(\text{vet}^*(X) \leq \omega\), and \(\text{vet}^*(X) \leq \omega\) in turn implies that the fan-tightness of \(X\) is countable.

The following theorems were proved in [1] and [5], respectively:

**Theorem 1** (Arhangelskii). For a Tychonoff space \(X\), the following are equivalent:

(a) \(\text{vet} C_p(X) \leq \omega\);
(b) for each \(n \in \omega\), \(X^n\) is a Hurewicz space.
Theorem 2 (Sakai). For a Tychonoff space $X$, the following are equivalent:

(a) $C_p(X)$ has countable strong fan-tightness;
(b) for each sequence $\{\gamma_n : n \in \omega\}$ of open $\omega$-covers of $X$ there exist $U_n \in \gamma_n$ such that $\bigcup U_n \in \gamma_n$ is an $\omega$-cover of $X$.

Lemma 3. For a topological space $X$, the following are equivalent:

(a) $\operatorname{vet}^*(X) \leq \omega$;
(b) for each mapping $\phi : \omega \to \omega$ such that $\phi(n) \geq n$ for each $n \in \omega$, for each point $x \in X$ and for each (decreasing) family $\{A_n\}_{n \in \omega}$ of subsets of $X$ satisfying $x \in \bigcap \{A_n : n \in \omega\}$, there exist $H_i \subseteq A_i$ such that $x \in \bigcap H_i$ and $|H_i| \leq \phi(n)$;
(c) for each point $x \in X$ and for each decreasing family $\{A_n\}_{n \in \omega}$ of subsets of $X$ satisfying $x \in \bigcap \{A_n : n \in \omega\}$, there exist $a_i \in A_i$ such that $x \in \bigcap \{a_i : n \in \omega\}$.

Proof: (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Assume (b) and fix $x \in X$ and a decreasing family $\{A_n\}_{n \in \omega}$ of subsets of $X$ such that $x \in \bigcap \{A_n : n \in \omega\}$. Consider the subset $\{n_k : k \in \omega\}$ of $\omega$ defined as follows: $n_1 = 1$ and $n_k = n_{k-1} + \phi(k)$. Since $x \in A_{n_k}$ for each $k$, select $H_k \subseteq A_{n_k}$ with $|H_k| \leq \phi(k)$ and $x \in \bigcup H_k$. Without loss of generality it may be assumed that $H_k = \{x_1^k, x_2^k, \ldots, x_{\phi(k)}^k\}$. For $i \in \omega$ such that $n_{k-1} < i \leq n_k$, put $a_i = x_{i-n_{k-1}}^k$. Clearly, $a_i \in A_i$ and $x \in \bigcap a_i = x_{\omega}$.

(c) $\Rightarrow$ (a). Assume (c) and fix $\{B_n\}_{n \in \omega}$ with $x \in \bigcap \{B_n : n \in \omega\}$. Put $A_i = \bigcup \{B_k : k \geq i\}$. The family $\{A_i : i \in \omega\}$ satisfies (c); select $a_i \in A_i$ with $x \in \{a_i : n \in \omega\}$ and put $H_i = B_i \cap \{a_n : 1 \leq n \leq i\}$. Clearly, $|H_i| \leq i$ and $x \in \bigcup H_i$.

Proposition 4. Let $X$ be a Fréchet space of countable fan-tightness. Then $\operatorname{vet}^*(X) \leq \omega$.

Proof: Fix a decreasing sequence $\{A_n : n \in \omega\}$ of subsets of $X$ and a point $x \in X$ such that $x \in \bigcap \{A_n \setminus A \subseteq A_n : n \in \omega\}$. There exist finite sets $H_n \subseteq A_n$ with $x \in \bigcup H_n : n \in \omega\}$. Choose a sequence $\{a_n : n \in \omega\} \subseteq \bigcup H_n : n \in \omega\}$ converging to $x$ and define a countable subset of $A_1$ as follows: for each $i \in \omega$, put $k_i = \max \{a_n : a_n \in H_i\}$ if $\{a_n : n \in \omega \cap H_i \neq \emptyset\}$ and $b_i = a_{k_i}$, if $\{a_n : n \in \omega \cap H_i \neq \emptyset\}$ and $b_i = b_{i+1}$ otherwise. Since each $H_i$ is finite, the sequence $\{b_i\}_{i \in \omega}$ is well-defined and $b_i \in A_i$ for each $i$. Since $\{b_i\}_{i \in \omega}$ contains a subsequence of $\{a_n\}_{n \in \omega}$, we have $x \in \bigcup b_i : i \in \omega\}$ and therefore $\operatorname{vet}^*(X) \leq \omega$.

Theorem 5. Let $X$ be a Tychonoff space. Then the following are equivalent:

(a) $\operatorname{vet}^*C_p(X) \leq \omega$;
(b) for every sequence $\{\gamma_n : n \in \omega\}$ of open $\omega$-covers of $X$ there exist $\lambda_n \subseteq \gamma_n$ such that $|\lambda_n| \leq n$ and $\bigcup \{\lambda_n : n \in \omega\}$ is an $\omega$-cover of $X$. 

□
PROOF: Assume (a). Fix a sequence \( \{\gamma_n : n \in \omega\} \) of open \( \omega \)-covers of \( X \) and for each natural number \( n \) put \( A_n = \{ f \in C_p(X) : \exists U \in \gamma_n \text{ such that } f(X \setminus U) = \{0\}\} \). Put \( f^*(x) = 1 \) for each \( x \in X \). Clearly, \( f^* \in A_n \) for each \( n \). Choose \( H_n \subseteq A_n \) such that \( f^* \in \bigcup\{H_n : n \in \omega\} \) and \( |H_n| \leq n \). For each \( n \) and for each \( f \in H_n \) fix \( U_f \in \gamma_n \) such that \( f(X \setminus U_f) = \{0\} \) and put \( \lambda_n = \{U_f : f \in H_n\} \).

To show that \( \bigcup\{\lambda_n : n \in \omega\} \) is an \( \omega \)-cover of \( X \), fix \( x_1, \ldots, x_k \in X \). There exist \( n \in \omega \) and \( f \in H_n \) such that \( f \in W(x_1, \ldots, x_k); f^*; 1/2 \). Thus for each \( i = 1, \ldots, k \) we have \( f(x_i) > \frac{1}{2} \) and \( x_i \in U_f \).

Assume (b) and fix \( f \in C_p(X) \) and a sequence \( \{A_n\}_{n \in \omega} \) of subsets of \( X \) such that \( f \in \bigcap\{A_n : n \in \omega\} \). Put \( \gamma_n = \{(g - f)^{-1}(-\frac{1}{n}, \frac{1}{n}) : g \in A_n\} \). To show that \( \gamma_n \) is an \( \omega \)-cover of \( X \), fix \( x_1, \ldots, x_k \in X \). Since \( W(x_1, \ldots, x_k; f; \frac{1}{n}) \cap A_n = \emptyset \), there exists \( g \in A_n \) such that \( x_i \in (g - f)^{-1}(-\frac{1}{n}, \frac{1}{n}) \) for each \( i = 1, \ldots, k \).

Case 1. There exists a subsequence \( \{n_k\}_{k \in \omega} \) such that \( X \in \gamma_{n_k} \) for each \( k \). Fix \( g_{n_k} \in A_{n_k} \) such that \( X = (g_{n_k} - f)^{-1}(-\frac{1}{n_k}, \frac{1}{n_k}) \). It is easy to see that \( f \in \{g_{n_k} : k \in \omega\} \).

Case 2. \( X \) is an element of finitely many members of \( \{\gamma_n\} \). Without loss of generality we may assume that \( X \notin \gamma_n \) for each \( n \). Choose \( \lambda_n \subseteq \gamma_n \) with \( |\lambda_n| \leq n \) and for each \( U \in \lambda_n \), fix \( g_U \in A_n \) with \( U = (g_U - f)^{-1}(-\frac{1}{n}, \frac{1}{n}) \). Put \( H_n = \{g_U : U \in \lambda_n\} \). Fix a basic open neighborhood \( W(x_1, \ldots, x_k; f; \frac{1}{n}) \) of \( f \). Since \( X \notin \bigcup\{\lambda_n : n \in \omega\} \), \( \{U \in \bigcup\{\lambda_n : n \in \omega\} : x_i \in U \}, \text{ for each } i = 1, \ldots, k \} = \omega \) and there exists \( N \geq n \) such that for some \( U \in \lambda_N \), \( x_i \in (g_U - f)^{-1}(-\frac{1}{N}, \frac{1}{N}) \) for all \( i \). Hence \( g_U \in W(x_1, \ldots, x_k; f; \frac{1}{n}) \cap H_N \) and \( f \in \bigcup\{H_n : n \in \omega\} \). This completes the proof. \( \square \)

Question 1. Does \( vet^*(X) \leq \omega \) imply that \( X \) has countable strong fan-tightness? In particular, are these two properties equivalent for function spaces (equivalently, are condition (b) of Theorem 2 and condition (b) of Theorem 5 equivalent)?

Corollary 6. Condition (b) of Theorem 5 is preserved by \( t \)-equivalence.

Remark. It can be shown that a space \( X \) satisfies condition (b) of Theorem 5 if and only if for each finite power \( X^k \) of \( X \) and for each sequence \( \{\gamma_n : n \in \omega\} \) of open covers of \( X^k \) there exist \( \lambda_n \subseteq \gamma_n \) such that \( |\lambda_n| \leq n \) and \( \bigcup\{\lambda_n : n \in \omega\} \) is a cover of \( X^k \). It can also be shown that every Tychonoff space \( X \) satisfying condition (b) of Theorem 5 is zero-dimensional.

Example 7. Countable fan-tightness does not imply that \( vet^*(X) \leq \omega \): Consider \( X = C_p(0, 1) \), where \( (0, 1) \) is the open unit interval. By Arhangel’skii’s theorem, \( vet C_p(0, 1) \leq \omega \). It is easy to see, however, that the sequence \( \{\gamma_n\} \) of open covers of \( (0, 1) \), where \( \gamma_n = \{\bigcup\{(a_i, b_i) : 1 \leq i \leq k\} : k \in \omega, a_i, b_i \in (0, 1), \text{ and } \sum_{i=1}^{k}(b_i - a_i) < \frac{1}{3^n}\} \), does not admit the choice of \( \lambda_n \subseteq \gamma_n \) satisfying condition (b) of Theorem 5 and, therefore, \( vet^*C_p(0, 1) \not\leq \omega \).

Denote by \( S_c \) the space obtained by identifying the limit points of continuum many convergent sequences.
Theorem 8. Let $X$ be a topological space such that $t(S_c \times X) \leq \omega$. Then $vet^*(X) \leq \omega$.

Proof: Enumerate the convergent sequences of $S_c$ by the elements of $\mathbb{R}$: $S_c = \{C_\alpha : \alpha \in \mathbb{R}\} \cup \{O\}$, where $C_\alpha = \{\frac{n}{n} : n \in \omega\}$ and $O$ is the only non-isolated point of $S_c$.

Fix $x^* \in X$ and a countable family $\{A_n : n \in \omega\}$ of subsets of $X$ such that $x^* \in \bigcap \{A_n : n \in \omega\}$. Since $t(X) \leq t(S_c \times X) = \omega$, we may assume without loss of generality that $|A_n| = \omega$.

Consider $K = \{(a_i)_{i \in \omega} : a_i \in A_i \forall i \in \omega\}$. Since $|K| = 2^\omega$, $K = \{\xi_\alpha : \alpha \in \mathbb{R}\}$, where each $\xi_\alpha = (a_\alpha^i)_{i \in \omega}$ and $\xi_\alpha \neq \xi_\alpha'$ whenever $\alpha \neq \alpha'$.

For each $\alpha \in \mathbb{R}$, put $\zeta_\alpha = \{(\frac{n}{n}, a_n^\alpha) : n \in \omega\}$. Let $B = \bigcup \{\zeta_\alpha : \alpha \in \mathbb{R}\} \subseteq S_c \times X$.

Claim 1: $\overline{B} \ni (O, x^*)$. Fix a neighborhood $O_{x^*}$ of $x^*$ in $X$ and a neighborhood $V$ of $O$ in $S_c$. For each $n \in \omega$ there exists an $a_n^* \in O_{x^*} \cap A_n$. Also, there is a real number $\alpha^*$ such that $\xi_{\alpha^*} = (a_i^*)_{i \in \omega}$. Since $V$ contains all but finitely many points of $C_{\alpha^*}$, $\xi_{\alpha^*} \cap (V \times O_{x^*}) \neq \emptyset$.

Choose a countable subset $M$ of $B$ such that $\overline{M} \ni (O, x^*)$. Without loss of generality, it may be assumed that $M = \bigcup \{\zeta_k : k \in \omega\}$. Put $H_i = \{a_i^\alpha : 1 \leq k \leq i\}$. Clearly, each $H_i$ is a subset of $A_i$ and $|H_i| \leq i$.

Claim 2: $x^* \in \bigcup \{H_i : i \in \omega\}$. Fix a neighborhood $O_{x^*}$ of $x^*$ in $X$. Put $V = S_c \setminus \{\frac{a_k^\alpha}{n} : k \in \omega, n \in \omega, n < k\}$. Clearly, $V$ is an open neighborhood of $O$, and, consequently, $M \cap (V \times O_{x^*}) \neq \emptyset$. Fix natural numbers $k$ and $n$ such that $(\frac{a_k^\alpha}{n}, a_n^\alpha) \in (V \times O_{x^*}) \cap M$. Since $\frac{a_k^\alpha}{n} \in V$, we have $n \geq k$ and, therefore, $a_n^\alpha k \in H_n \cap O_{x^*}$. This completes the proof.

The following two corollaries provide answers to questions posed in [3]:

Corollary 9. Let $X$ be a Hausdorff space such that $t(S_c \times X) \leq \omega$. Then $vet(X) \leq \omega$.

Remark. Example 7 shows that the last corollary cannot be reversed: It follows from theorem (on product) that $t(C_p(0, 1) \times S_c) > \omega$. In fact, it follows from Example 2 in [6] that $t(C_p(0, 1) \times S_c) \geq 2^\omega$.

Question 2. Can Theorem 8 be reversed? Also, is it true that for a space $X$ of countable strong fan-tightness we have $t(S_c \times X) \leq \omega$?

Corollary 10. Let $X$ be a Tychonoff space such that $t(C_p(X) \times S_c) \leq \omega$. Then $X$ is a Hurewicz space.

It was shown in [3] that for a regular countably compact space $X$ of countable tightness, the tightness of the product space $S_c \times X$ is countable. It was also proved that for a regular countably compact space, countable fan-tightness and countable tightness are equivalent. The following corollary improves the last result:
Corollary 11. Let $X$ be a regular countably compact space of countable tightness. Then $\text{vet}^*(X) \leq \omega$.

Question 3. Let $X$ be a regular pseudocompact space of countable tightness. Is it true that $t(S_c \times X) \leq \omega$? In particular, is it true that $\text{vet}^*(X) \leq \omega$?

Definition. A mapping $f : X \to Y$ is said to be countably biquotient, if for each point $y \in Y$ and for each increasing open cover $\{U_n : n \in \omega\}$ of $f^{-1}(y)$ there is a number $n$ such that $y \in \text{Int}(f(U_n))$.

Proposition 12. Let $X$ be a space such that $\text{vet}^*(X) \leq \omega$ and let $f : X \to Y$ be a continuous countably biquotient mapping onto $Y$. Then $\text{vet}^*(Y) \leq \omega$.

Proof: Use condition (c) of Lemma 3. Fix $y \in Y$ and a countable decreasing family $\{A_n\}_{n \in \omega}$ of subsets of $Y$ such that $y \in \bigcap A_n$. Put $B_n = f^{-1}(A_n)$. There is a point $x \in f^{-1}(y)$ such that $x \in \bigcap \{B_n : n \in \omega\}$. Indeed, otherwise $X \setminus B_n : n \in \omega$ be an increasing cover of $f^{-1}(y)$ and for some $n$, we would have $y = f(x) \in \text{Int}(f(X \setminus f^{-1}(A_n))) \subseteq Y \setminus A_n$, a contradiction to $y \in A_n$.

Fix $x \in f^{-1}(y)$ such that $x \in B_n$ for each $n$ and choose $b_i \in B_i$ with the property $x \in \{b_i : i \in \omega\}$. It is easy to see that $f(b_i) \in A_i$ and $y \in \{f(b_i) : i \in \omega\}$. By Lemma 3, $\text{vet}^*(Y) \leq \omega$. 

Corollary 13. If $X$ is a topological space such that $\text{vet}^*(X) \leq \omega$ and $Y$ is an image of $X$ under a continuous open mapping, then $\text{vet}^*(Y) \leq \omega$.

We shall say that a space $X$ has countable omega-fan-tightness ($\text{vet}_\omega(X) \leq \omega$) if for each point $x \in X$ and any countable family $\{A_n\}_{n \in \omega}$ of countable subsets of $X$ satisfying $x \in \bigcap \{A_n : n \in \omega\}$, there exist finite sets $H_n \subseteq A_n$ such that $x \in \bigcup \{H_n : n \in \omega\}$.

Theorem 14. Let $X$ be a Tychonoff space such that for every finite $k$ and for every sequence $\{\gamma_n\}_{n \in \omega}$ of countable open covers of $X^k$ there exist finite subfamilies $\lambda_n \subseteq \gamma_n$ such that $\bigcup \{\lambda_n : n \in \omega\}$ is a cover of $X^k$. Then $\text{vet}_\omega C_p(X) \leq \omega$.

Proof: Fix a family $\{A_n : n \in \omega\}$ of countable subsets of $C_p(X)$ and a function $f \in C_p(X)$ such that $f \in \bigcap A_n$. Fix natural numbers $n$ and $k$ and for each $g \in A_k$, put $V_n(g) = (g - f)^{-1}\{(-\frac{1}{n}, \frac{1}{n})\}$. Put $\gamma_k^n = \{V_n(g)^n : g \in A_k\}$. The family $\gamma_k^n$ is an open cover of $X^n$. Indeed, for $(x_1, x_2, \ldots, x_n) \in X^n$ there is $h \in A_k$ such that $h \in W(x_1, x_2, \ldots, x_n; f; \frac{1}{n})$ and hence $(x_1, x_2, \ldots, x_n) \in V_n(h)^n$.

Consider a sequence $\{\gamma_k^n : k \geq n\}$ of open countable covers of $X^n$ and select finite families $\lambda_k^n = \{V_n(g) : g \in H_k^n\} \subseteq \gamma_k^n$, where $H_k^n$ is a finite subset of $A_n$ and $\bigcup \{\lambda_k^n : k \geq n\} \supseteq X$. Put $H_i = \bigcup \{H_i^n : n \leq i\}$. Clearly, $H_i$ is a finite subset of $A_i$.

Claim: $f \in \bigcup \{H_i : i \in \omega\}$. Fix $x_1, x_2, \ldots, x_n \in X$ and $\epsilon > 0$. It may be assumed without loss of generality that $\frac{1}{n} < \epsilon$. For some natural number $k \geq n$, we have $\bigcup \lambda_k^n \supseteq (x_1, x_2, \ldots, x_n)$ and there is an $h \in H_k^n \subseteq H_k$ such that
(x_1, x_2, \ldots, x_n) \in V_n(h)^n, \text{ i.e. } h \in W(x_1, x_2, \ldots, x_n; f; \epsilon) \cap H_k. \text{ The proof is complete.} \qed

The following two theorems were proved by Professor Arhangelskii, who kindly permitted me to include them in this paper.

**Theorem 15.** Let \( X \) be a Tychonoff pseudocompact space. Then \( \text{vet}_\omega C_p(X) \leq \omega. \)

**Proof:** The restriction mapping \( r : C_p(\beta X) \to C_p(X) \) is a continuous bijection. Fix a countable subset \( A \subseteq C_p(X) \). Then the restriction of the inverse mapping \( r^{-1}|_A : A \to C_p(\beta X) \) is continuous. Indeed, for each \( z \in \beta X \setminus X \) the set \( F = \bigcap \{ \tilde{g}^{-1}(g(z)) : g \in A \} \), where \( \tilde{g} \) is an extension of function \( g \) to a continuous function on \( \beta X \), is a \( G_\delta \)-set in \( \beta X \) and, therefore, there exists a \( y_z \in X \cap F \), i.e. \( g(z) = g(y_z) \) for each \( g \in A \). From here, \( r(W(z, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(y_z, f, \epsilon) \cap A \) for any \( f \in A \). Clearly, for each \( x \in X \) and each \( f \in A \) we have \( r(W(x, r^{-1}(f), \epsilon) \cap r^{-1}(A)) = W(x, f, \epsilon) \cap A \).

Fix \( f \in C_p(X) \) and a sequence \( \{ A_n \}_{n \in \omega} \) of subsets of \( C_p(X) \) such that \( f \in \bigcup A_n \). Then \( r^{-1}(f) \in r^{-1}(A_n) \) and by Arhangelskii’s Theorem, \( \text{vet}(C_p(\beta X)) \leq \omega. \) Fix finite \( H_n \subseteq r^{-1}(A_n) \) such that \( r^{-1}(f) \in \bigcup H_n : n \in \omega \). It follows that \( f \in \bigcup \{ r(H_n) : n \in \omega \} \) and each \( r(H_n) \) is a finite subset of \( A_n \). \( \Box \)

**Remark.** The last theorem shows that Theorem 14 cannot be reversed: a pseudocompact Tychonoff space not satisfying the assumptions of Theorem 14 would be a counterexample.

It is known that countable fan-tightness is preserved by continuous open surjective mappings. The next theorem shows that it is not true for countable omega-fan-tightness.

**Theorem 16.** Let \( Y \) be a Tychonoff space. Then there exist a Tychonoff space \( X \) with \( \text{vet}_\omega(X) \leq \omega \) and a continuous open surjection \( f : X \to Y. \)

**Proof:** Consider the space \( Z = (\{\omega_1 + 1\} \times \beta(C_p(Y))) \setminus (\{\omega_1\} \times (\beta(C_p(Y)) \setminus C_p(Y))). \) Since \( Z \) contains a dense countably compact space \( \omega_1 \times \beta(C_p(Y)), \) the space \( Z \) is pseudocompact, and therefore \( \text{vet}_\omega(C_p(Z)) \leq \omega. \) It is easy to see that \( \{\omega_1\} \times C_p(Y) \) is closed in \( Z \) and every bounded continuous function on \( \{\omega_1\} \times C_p(Y) \) can be extended to a continuous function on \( Z. \) Thus the restriction mapping \( r : C_p(Z) \to C_p(\{\omega_1\} \times C_p(Y)) = C_p(C_p(Y)) \) is an open mapping; a topological copy of \( Y \) is contained in \( C_p^0(Z) \subseteq r(C_p(Z)). \) Put \( X = r^{-1}(Y) \) and put \( f = r|_X. \) It is easy to see that \( f \) is a continuous open mapping onto \( Y \) and \( \text{vet}_\omega(X) \leq \omega. \) \( \Box \)

**Remark.** After this paper had been submitted, the author proved independently from S. Garcia-Ferreira and A. Tamariz-Mascarua that for a Tychonoff space \( X, \) \( \text{vet}^* C_p(X) \leq \omega \) implies that \( C_p(X) \) has countable strong fan-tightness. In the
article “Some generalizations of rapid ultrafilters in topology and id-fan tight-
ness”, Tsukuba J. Math, 19 (1) (1995), 173–185, the two authors also showed that vet*(X) ≤ ω does not imply in general that X has countable strong fan-tightness. This provides a complete answer to Question 1. It is not clear, however, whether the two properties coincide for topological groups.

Also, A. Bella noticed that the answer to Question 2 is negative.

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References

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