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Uniformly \( \mu \)-continuous topologies on Köthe-Bochner spaces and Orlicz-Bochner spaces

Krzysztof Feledziak

Abstract. Some class of locally solid topologies (called uniformly \( \mu \)-continuous) on Köthe-Bochner spaces that are continuous with respect to some natural two-norm convergence are introduced and studied. A characterization of uniformly \( \mu \)-continuous topologies in terms of some family of pseudonorms is given. The finest uniformly \( \mu \)-continuous topology \( T^\mu_i (X) \) on the Orlicz-Bochner space \( L^\varphi (X) \) is a generalized mixed topology in the sense of P. Turpin (see [11, Chapter I]).

Keywords: Orlicz spaces, Orlicz-Bochner spaces, Köthe-Bochner spaces, locally solid topologies, generalized mixed topologies, uniformly \( \mu \)-continuous topologies, inductive limit topologies

Classification: 46E30, 46E40, 46A70

1. Preliminaries.

For notation and terminology concerning locally solid Riesz spaces we refer to [1].

Throughout the paper let \((\Omega, \Sigma, \mu)\) be a complete \(\sigma\)-finite measure space and let \(L^0\) denote the corresponding space of equivalence classes of all \(\Sigma\)-measurable real valued functions. Then \(L^0\) is a super Dedekind complete Riesz space under the ordering \(u_1 \leq u_2\) whenever \(u_1(\omega) \leq u_2(\omega)\) \(\mu\)-a.e. on \(\Omega\).

For \(u \in L^0\) let us put
\[
\|u\|_\mu = \inf\{\lambda > 0 : \mu(\{\omega \in \Omega : |u(\omega)| > \lambda\}) \leq \lambda\}.
\]

It is easy to see that a sequence \((u_n)\) in \(L^0\) is convergent to \(u \in L^0\) in measure on \(\Omega\) (in symbols \(u_n \to u (\mu - \Omega)\)) iff \(\|u_n - u\|_\mu \to 0\). We will denote by \(T_\mu\) the topology on \(L^0\) of \(\| \cdot \|_\mu\).

For a subset \(A\) of \(\Omega\) let \(\chi_A\) stand for its characteristic function.

Let \([x]\) denote the greatest integer which is less or equal to a real number \(x\).

Let \((E, \| \cdot \|_E)\) be an \(F\)-normed function space, that is \(E\) is an ideal of \(L^0\) with \(\text{supp } E = \Omega\) and \(\| \cdot \|_E\) is a complete Riesz \(F\)-norm. The Köthe dual \(E'\) of \(E\) is defined by
\[
E' = \{v \in L^0 : \int_\Omega |u(\omega)v(\omega)| \, d\mu < \infty \text{ for all } u \in E\}.
\]

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In case \((E, \| \cdot \|_{E})\) is a Banach function space the associated norm \(\| \cdot \|_{E'}\) on \(E'\) can be defined for \(v \in E'\) by
\[
\|v\|_{E'} = \sup \left\{ \left| \int_{\Omega} u(\omega)v(\omega) \, d\mu \right| : u \in E, \|u\|_{E} \leq 1 \right\}.
\]

We will write \(A_{n} \searrow \emptyset\) when \((A_{n})\) is a decreasing sequence in \(\Sigma\) such that \(\mu(A_{n} \cap A) \to 0\) for every \(A \in \Sigma\) with \(\mu(A) < \infty\).

We denote by \(E_{a}\) the ideal of elements of absolutely continuous norm in \(E\), i.e. \(E_{a} = \{ u \in E : \| \chi_{A_{n}}u\|_{E} \to 0 \text{ as } A_{n} \searrow \emptyset \}\).

Let \((X, \| \cdot \|_{X})\) be a real Banach space, and let \(S_{X}\) and \(B_{X}\) denote the unit sphere and the closed unit ball in \(X\), respectively.

By \(L^{0}(X)\) we will denote the linear space of equivalence classes of all strongly \(\Sigma\)-measurable functions \(f : \Omega \to X\).

For \(f \in L^{0}(X)\) let us put
\[
\|f\|_{\mu}^{X} = \inf \{ \lambda > 0 : \mu(\{ \omega \in \Omega : \|f(\omega)\|_{X} > \lambda \}) \leq \lambda \}.
\]

We say that a sequence \((f_{n})\) in \(L^{0}(X)\) is convergent to \(f \in L^{0}(X)\) in measure on \(\Omega\) (in symbols \(f_{n} \to f (\mu - \Omega)\)) whenever \(\mu(\{ \omega \in \Omega : \|f_{n}(\omega) - f(\omega)\|_{X} > \varepsilon \}) \to 0\) for every \(\varepsilon > 0\). It can be seen that a sequence \((f_{n})\) in \(L^{0}(X)\) is convergent to \(f \in L^{0}(X)\) in measure on \(\Omega\) iff \(\|f_{n} - f\|_{\mu}^{X} \to 0\). The topology on \(L^{0}(X)\) of \(\| \cdot \|_{\mu}^{X}\) will be denoted by \(T_{\mu}(X)\).

For \(f \in L^{0}(X)\) let
\[
\tilde{f}(\omega) := \|f(\omega)\|_{X} \quad \text{for } \omega \in \Omega.
\]

The linear space \(E(X) = \{ f \in L^{0}(X) : \tilde{f} \in E \}\) provided with the norm \(\|f\|_{E(X)} := \|\tilde{f}\|_{E}\) is called a Köthe-Bochner space (see [2], [3]).

Now we recall some concepts and terminology concerning locally solid topologies on vector-valued function spaces as set out in [3].

A subset \(H\) of \(E(X)\) is said to be solid whenever \(\|f_{1}(\omega)\|_{X} \leq \|f_{2}(\omega)\|_{X}\) \(\mu\)-a.e. and \(f_{1} \in E(X)\), \(f_{2} \in H\) imply \(f_{1} \in H\).

A pseudonorm \(\rho\) on \(E(X)\) is said to be solid whenever for \(f_{1}, f_{2} \in E(X)\), \(\|f_{1}(\omega)\|_{X} \leq \|f_{2}(\omega)\|_{X}\) \(\mu\)-a.e. imply \(\rho(f_{1}) \leq \rho(f_{2})\).

A linear topology \(\tau\) on \(E(X)\) is said to be locally solid if it has a basis for neighbourhoods of zero consisting of solid sets.

A linear topology \(\tau\) on \(E(X)\) that is at the same time locally solid and locally convex will be called a locally convex-solid topology on \(E(X)\).

**Theorem 1.1** (see [3, Theorem 2.2, Theorem 2.3]). For a linear topology \(\tau\) on \(E(X)\) the following statements are equivalent:

(i) \(\tau\) is a locally solid topology (respectively \(\tau\) is a locally convex-solid topology);

(ii) \(\tau\) is generated by some family of solid pseudonorms (respectively seminorms).
Now we are going to explain the relationship between locally solid topologies on $E$ and $E(X)$ (see [3]).

Let $p$ be a Riesz pseudonorm (respectively seminorm) on $E$, and let

$$
\overline{p}(f) := p(\tilde{f}) \text{ for } f \in E(X).
$$

Then $\overline{p}$ is a solid pseudonorm (respectively seminorm) on $E(X)$.

Next, fix $x \in S_X$. Given $u \in E$ let us put $u(\omega) := u(\omega) \cdot x$ for $\omega \in \Omega$. Then $\overline{u} \in L^0(X)$ and $\|\overline{u}(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $\overline{u} \in E(X)$.

Let $\rho$ be a solid pseudonorm (respectively seminorm) on $E(X)$, and let

$$
\tilde{\rho}(u) := \rho(\overline{u}) \text{ for } u \in E.
$$

Then $\tilde{\rho}$ is a Riesz pseudonorm (respectively seminorm) on $E$.

**Theorem 1.2** (see [3, Lemma 3.1]).

(i) If $\rho$ is a solid pseudonorm on $E(X)$, then $\tilde{\rho}(f) = \rho(f)$ for $f \in E(X)$.

(ii) If $p$ is a Riesz pseudonorm on $E$, then

$$
\overline{p}(u) = p(u) \text{ for } u \in E.
$$

Let $\tau$ be a locally solid topology on $E(X)$ generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms defined on $E(X)$. By $\tilde{\tau}$ we will denote the locally solid topology on $E$ generated by the family $\{\tilde{\rho}_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on $E$. If $\tau$ is a Hausdorff topology, then so is $\tilde{\tau}$.

In turn, let $\xi$ be a locally solid topology on $E$ generated by some family $\{p_\alpha : \alpha \in \{\alpha\}\}$ of Riesz pseudonorms on $E$. By $\tilde{\xi}$ we will denote the locally solid topology on $E(X)$ generated by the family $\{\overline{p}_\alpha : \alpha \in \{\alpha\}\}$ of solid pseudonorms on $E(X)$. Then $\tilde{\xi}$ is a Hausdorff topology, whenever $\xi$ is Hausdorff.

**Theorem 1.3** (see [3, Theorem 3.2]).

(i) For a locally solid topology $\tau$ on $E(X)$ we have: $\tilde{\tau} = \tau$.

(ii) For a locally solid topology $\xi$ on $E$ we have: $\tilde{\xi} = \xi$.

Now we recall some notation and terminology concerning Orlicz spaces (see [5], [6], [11] for more details).

By an Orlicz function we mean a function $\varphi : [0, \infty) \to [0, \infty]$ which is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$ and not identically equal to 0.

A convex Orlicz function is usually called a Young function. For a Young function $\varphi$ we denote by $\varphi^*$ the function complementary to $\varphi$ in the sense of Young, i.e.

$$
\varphi^*(s) = \sup\{ts - \varphi(t) : t \geq 0\} \text{ for } s \geq 0.
$$

Let $\varphi$ and $\psi$ be a pair of Orlicz functions vanishing only at zero (respectively taking only finite values). We say that $\varphi$ increases essentially more rapidly than $\psi$
for small $t$ (respectively for large $t$) denoted $\psi \preceq \varphi$ (respectively $\psi \prec \varphi$), whenever for any $c > 0$, $\psi(ct)/\varphi(t) \to 0$ as $t \to 0$ (respectively $t \to \infty$). We will write $\psi \ll \varphi$ when $\psi \preceq \varphi$ and $\psi \prec \varphi$ hold. For $\varphi$ and $\psi$ being Young functions the condition $\psi \ll \varphi$ (respectively $\psi \prec \varphi$) implies $\varphi^* \ll \psi^*$ (respectively $\varphi^* \prec \psi^*$) (see [5, Lemma 13.1]).

An Orlicz function $\varphi$ determines a functional $m_\varphi : L^0 \to [0, \infty]$ by

$$m_\varphi(u) = \int_\Omega \varphi(|u(\omega)|) \, d\mu.$$ 

The Orlicz space generated by $\varphi$ is the ideal of $L^0$ defined by

$$L^\varphi = \{ u \in L^0 : m_\varphi(\lambda u) < \infty \text{ for some } \lambda > 0 \}.$$ 

$L^\varphi$ can be equipped with the complete metrizable topology $T_\varphi$ of the $F$-norm

$$\|u\|_\varphi = \inf \left\{ \lambda > 0 : m_\varphi \left( \frac{u}{\lambda} \right) \leq \lambda \right\}.$$ 

Let

$$\varphi_0(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1 \\ 1 & \text{for } t > 1. \end{cases}$$

It is known that $L^{\varphi_0}$ is the largest Orlicz space and consists of all those $u \in L^0$ that are bounded outside of some set of finite measure and $\|u\|_{\varphi_0} = \|u\|_\mu$ for all $u \in L^{\varphi_0}$. (see [11, 0.3.4]).

Moreover one can check that $L^{\varphi_0}$ is the largest linear subspace of $L^0$ such that the functional $\| \cdot \|_\mu$ restricted to $L^{\varphi_0}$ is an $F$-norm.

We will write $\| \cdot \|_\mu$ and $T_\mu$ instead of $\| \cdot \|_{\varphi_0}$ and $T_{\varphi_0}$, respectively.

Moreover, if $\varphi$ is a Young function, then the topology $T_\varphi$ can be generated by the Luxemburg norm:

$$|||u|||_\varphi = \inf \left\{ \lambda > 0 : m_\varphi \left( \frac{u}{\lambda} \right) \leq 1 \right\}.$$ 

For an Orlicz function $\varphi$ let

$$E^\varphi = \{ u \in L^0 : m_\varphi(\lambda u) < \infty \text{ for all } \lambda > 0 \}$$

and

$$L_{\hat{a}}^\varphi = \{ u \in L^\varphi : \|u_{A_n}\|_\varphi \to 0 \text{ as } A_n \searrow \emptyset \}.$$ 

It is well known that $E^\varphi = L_{\hat{a}}^\varphi$ whenever $\varphi$ takes only finite values. Moreover, for every Young function $\varphi$ the identity $(L^\varphi)' = L^{\varphi^*}$ holds.
Let $M_\phi : L^0(X) \to [0, \infty]$ be defined by
\[ M_\phi(f) = \int_\Omega \phi(\|f(\omega)\|_X) \, d\mu. \]
Thus $M_\phi(f) = m_\phi(\tilde{f})$. The Köthe-Bochner space $L_\phi(X) = \{ f \in L^0(X) : \tilde{f} \in L_\phi \}$ is usually called an Orlicz-Bochner space and is equipped with the $F$-norm $\|f\|_{L_\phi(X)} = \|\tilde{f}\|_\phi$ for $f \in L_\phi(X)$.

We will denote by $T_\phi(X)$ the topology on $L_\phi(X)$ generated by the $F$-norm $\| \cdot \|_{L_\phi(X)}$. Moreover, if $\phi$ is a Young function, then $T_\phi(X)$ is generated by the Luxemburg norm: $\|f\|_{L_\phi(X)} = \|\tilde{f}\|_\phi$ for $f \in L_\phi(X)$. We will write $\| \cdot \|_\mu$ and $T_\mu(X)$ instead of $\| \cdot \|_{L_\phi(X)}$ and $T_{\phi_0}(X)$, respectively.

2. Uniformly $\mu$-continuous topologies on Köthe-Bochner spaces

**Definition 2.1.** (i) A solid pseudonorm $\rho$ on $E(X)$ is said to be uniformly $\mu$-continuous, whenever $f_n \in E(X), f_n \to 0 (\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $\rho(f_n) \to 0$.

(ii) A locally solid topology $\tau$ on $E(X)$ is said to be uniformly $\mu$-continuous whenever $f_n \in E(X), f_n \to 0 (\mu - \Omega)$ with $\sup_n \|f_n\|_{E(X)} < \infty$ imply $f_n \xrightarrow{\tau} 0$.

In view of [3, Theorem 2.3] a locally solid topology $\tau$ on $E(X)$ is uniformly $\mu$-continuous iff it is generated by some family $\{\rho_\alpha : \alpha \in \{\alpha\}\}$ of uniformly $\mu$-continuous pseudonorms defined on $E(X)$.

It is easy to prove the following lemma.

**Lemma 2.1.** (i) If $\rho$ is a uniformly $\mu$-continuous pseudonorm on $E(X)$, then $\tilde{\rho}$ is a uniformly $\mu$-continuous pseudonorm on $E$ (i.e. $u_n \in E, u_n \to 0 (\mu - \Omega)$ with $\sup_n \|u_n\|_E < \infty$ imply $\tilde{\rho}(u_n) \to 0$).

(ii) If $p$ is a uniformly $\mu$-continuous pseudonorm on $E$, then $\overline{p}$ is a uniformly $\mu$-continuous pseudonorm on $E(X)$.

From Lemma 2.1 we easily get the following theorem that explains the relationship between uniformly $\mu$-continuous topologies on $E$ and $E(X)$.

**Theorem 2.2.** (i) If $\tau$ is a uniformly $\mu$-continuous topology on $E(X)$, then $\tilde{\tau}$ is a uniformly $\mu$-continuous topology on $E$.

(ii) If $\xi$ is a uniformly $\mu$-continuous topology on $E$, then $\overline{\xi}$ is a uniformly $\mu$-continuous topology on $E(X)$.

We shall need the following result.
Theorem 2.3. (i) If \( \tau \) is the finest uniformly \( \mu \)-continuous topology on \( E(X) \), then \( \tilde{\tau} \) is the finest uniformly \( \mu \)-continuous topology on \( E \).

(ii) If \( \xi \) is the finest uniformly \( \mu \)-continuous topology on \( E \), then \( \tilde{\xi} \) is the finest uniformly \( \mu \)-continuous topology on \( E(X) \).

Proof: (i) Let \( \xi \) be a uniformly \( \mu \)-continuous topology on \( E \). By Theorem 2.2 \( \tilde{\xi} \) is a uniformly \( \mu \)-continuous topology on \( E(X) \), so \( \tilde{\xi} \subset \tau \). By [3, Theorem 3.3] and Theorem 1.3 \( \xi = \tilde{\xi} \subset \tilde{\tau} \), as desired.

(ii) Let \( \tau \) be a uniformly \( \mu \)-continuous topology on \( E(X) \). By Theorem 2.2 \( \tilde{\tau} \) is a uniformly \( \mu \)-continuous topology on \( E \), so \( \tilde{\tau} \subset \xi \). By [3, Theorem 3.3] and Theorem 1.3 \( \tau = \tilde{\tau} \subset \tilde{\xi} \), as desired.

Now we are going to give a description of uniformly \( \mu \)-continuous topologies on Orlicz-Bochner spaces. We start with the following definition.

Definition 2.2. A solid pseudonorm \( \rho \) on \( E(X) \) is said to be uniformly summable whenever the following conditions hold:

For every \( r > 0 \)

\[ (*) \quad \sup \{ \rho(\chi_{A(f,\lambda)}f) : f \in E(X), \| f \|_{E(X)} \leq r \} \to 0 \quad \text{as} \quad \lambda \to 0_+ , \]

where \( A(f, \lambda) = \{ \omega \in \Omega : \| f(\omega) \|_X \leq \lambda \text{ or } \| f(\omega) \|_X > \frac{1}{\lambda} \} \) for \( 0 < \lambda < 1 \) and

\[ (**) \quad \rho(\chi_A) \to 0 \quad \text{as} \quad \mu(A) \to 0. \]

Theorem 2.4. Let \( \varphi \) be an arbitrary Orlicz function and \( \psi \) be a finite valued Orlicz function such that \( \psi \prec \varphi \). Then the \( F \)-norm \( \| \cdot \|_{L^\varphi(X)} \) (restricted to \( L^\varphi(X) \)) is uniformly summable on \( L^\varphi(X) \).

Proof: Since \( \varphi \prec \varphi \), so \( L^\varphi \subset L^\psi \) (see [11, 0.2.5, 0.3.5]). Hence \( L^\varphi(X) \subset L^\psi(X) \). Let \( r > 0 \), \( \varepsilon > 0 \) be given. Choose \( \eta > 0 \) such that \( \eta(r + 1) < \varepsilon \) and let \( c = \frac{\varepsilon}{\varepsilon + 1} \). Then there exist \( 0 < t_1 < t_2 \) such that \( \psi(t) \leq \eta\varphi(ct) \) for \( 0 \leq t < t_1 \) or \( t > t_2 \), and choose \( \lambda_0 \in (0,1) \) such that \( \lambda_0 \leq \varepsilon t_1 \) and \( \frac{1}{\lambda_0} > \varepsilon t_2 \). Hence for \( f \in L^\varphi(X) \) and \( \| f \|_{L^\varphi(X)} \leq r \) we have:

\[ M_{\psi}(\frac{\chi_{A(f,\lambda)}f}{\varepsilon}) = \int_{A(f,\lambda)} \psi(\frac{\| f(\omega) \|_X}{\varepsilon}) \, d\mu \leq \int_{A(f,\lambda)} \eta\varphi(c\frac{\| f(\omega) \|_X}{\varepsilon}) \, d\mu \]
\[ \leq \int_{\Omega} \eta\varphi(\frac{\| f(\omega) \|_X}{r + 1}) \, d\mu \leq \eta(r + 1) < \varepsilon \]

for every \( 0 < \lambda \leq \lambda_0 \). It follows that \( \| \chi_{A(f,\lambda)}f \|_{L^\psi(X)} \leq \varepsilon \) for every \( f \in L^\varphi(X) \), \( \| f \|_{L^\varphi(X)} \leq r \) and \( 0 < \lambda \leq \lambda_0 \). This means that for \( r > 0 \)

\[ \sup \{ \| \chi_{A(f,\lambda)}f \|_{L^\psi(X)} : f \in L^\varphi(X), \| f \|_{L^\varphi(X)} \leq r \} \to 0 \quad \text{as} \quad \lambda \to 0_+. \]
Remark 2.1. Let ϕ be an Orlicz function such that ϕ(u) → ∞ as u → ∞. Then ϕ_0 ≪ ϕ and it follows that the F-norm ∥ · ∥_μ^X is uniformly summable on L^ϕ(X).

Theorem 2.5. Let ϕ be an Orlicz function such that ϕ(u) → ∞ as u → ∞. For a solid pseudonorm ρ on L^ϕ(X) the following statements are equivalent:

(i) ρ is uniformly summable;
(ii) ρ is uniformly µ-continuous.

Proof: (i) ⇒ (ii) Take a sequence (f_n) in L^ϕ(X) such that f_n → 0 (μ − Ω) and sup_n ∥f_n∥_{L^ϕ(X)} ≤ r for some r > 0. Fix ε > 0. There exists λ_0 ∈ (0, 1) such that sup_n ρ(χ_A(f_n, λ_0), f_n) < ε. Moreover, there exists δ > 0 such that

\[ ρ(χ_A) < \frac{ε}{2(\left\lfloor \frac{1}{λ_0} \right\rfloor + 1)} \]

whenever A ∈ Σ with μ(A) ≤ δ.

Since f_n → 0 (μ − Ω), we can find a natural number k such that for all n ≥ k

μ(Ω \ A(f_n, λ_0)) ≤ μ(\{ω ∈ Ω : ∥f_n(ω)∥_X > λ_0\}) ≤ δ.

Hence for n ≥ k

\[ ρ(f_n) = ρ(χ_A(f_n, λ_0), f_n + χ_Ω \ A(f_n, λ_0), f_n) ≤ ρ(χ_A(f_n, λ_0), f_n) \]

\[ + ρ(χ_Ω \ A(f_n, λ_0), f_n) \]

\[ ≤ \frac{ε}{2} + ρ(\left\lfloor \frac{1}{λ_0} \right\rfloor + 1) ρ(χ_Ω \ A(f_n, λ_0)) \]

\[ ≤ \frac{ε}{2} + (\left\lfloor \frac{1}{λ_0} \right\rfloor + 1) \frac{ε}{2(\left\lfloor \frac{1}{λ_0} \right\rfloor + 1)} ≤ ε. \]

Thus ρ(f_n) → 0.

(ii) ⇒ (i) For r > 0 let B^ϕ_ρ(r) = \{f ∈ L^ϕ(X) : ∥f∥_{L^ϕ(X)} ≤ r\}, B^ϕ_0(r) = \{f ∈ L^ϕ(X) : ρ(f) ≤ r\}, B^ϕ_μ(r) = \{f ∈ L^ϕ_0(X) : ∥f∥^X_μ ≤ r\}. By (ii) the identity map

\[ id : (B^ϕ_ρ(r), T_μ(X)|_{B^ϕ_ρ(r)}) \rightarrow (B^ϕ_ρ(r), T(ρ)|_{B^ϕ_ρ(r)}) \]

is continuous at zero for any r > 0, where T(ρ) denotes the topology on L^ϕ(X) generated by ρ. Let ε > 0, r > 0 be given. There exists η > 0 such that B^ϕ_X(η) ∩ B^ϕ_ρ(r) ⊂ B^ϕ_X(ε). Since ∥ · ∥^X_μ is uniformly summable on L^ϕ(X) (see Remark 2.1) there exists λ_0 ∈ (0, 1) such that

\[ sup \{∥χ_A(f, λ)f∥^X_μ : f ∈ L^ϕ(X), ∥f∥_{L^ϕ(X)} ≤ r\} ≤ η \]

whenever 0 < λ ≤ λ_0.
Then \( \sup \{ \rho(\chi_A(f,\lambda) f) : f \in L^\varphi(X), \|f\|_{L^\varphi(X)} \leq r \} \leq \varepsilon \) whenever \( 0 < \lambda \leq \lambda_0 \).

Hence \( \sup \{ \rho(\chi_A(f,\lambda) f) : f \in L^\varphi(X), \|f\|_{L^\varphi(X)} \leq r \} \to 0 \) as \( \lambda \to 0_+ \).

Moreover, there exists \( \delta > 0 \) such that \( \|\chi_A\|_{X^\mu} \leq \eta \) for \( A \in \Sigma \) with \( \mu(A) \leq \delta \).

Then \( \rho(\chi_A) \leq \varepsilon \) whenever \( A \in \Sigma \) with \( \mu(A) \leq \delta \). It follows that \( \rho(\chi_A) \to 0 \) as \( \mu(A) \to 0 \).

Thus \( \rho \) is a uniformly summable pseudonorm on \( L^\varphi(X) \).

**Theorem 2.6.** Let \( \varphi \) be an Orlicz function such that \( \varphi(u) \to \infty \) as \( u \to \infty \). For a locally solid topology \( \tau \) on \( L^\varphi(X) \) the following statements are equivalent:

(i) \( \tau \) is uniformly \( \mu \)-continuous;
(ii) \( \tau|_{B_X^\varphi(r)} \subset T_\mu(X)|_{B_X^\varphi(r)} \) for every \( r > 0 \);
(iii) \( \tau \) is generated by some family of uniformly summable pseudonorms.

**Proof:** (i) \( \Rightarrow \) (ii) Since \( T_\mu(X) \) is a linear metrizable topology, it follows from Definition 2.1(ii).

(ii) \( \Rightarrow \) (i) Obvious.

(i) \( \Rightarrow \) (iii) Let \( \tau \) be defined by the family \( \{ \rho_\alpha : \alpha \in \{ \alpha \} \} \) of solid pseudonorms. Then by Definition 2.1 and Theorem 2.5 \( \tau \) is generated by the family \( \{ \rho_\alpha : \alpha \in \{ \alpha \} \} \) of uniformly summable pseudonorms.

(iii) \( \Rightarrow \) (i) It follows from Theorem 2.5.

**3. Generalized mixed topologies on Orlicz-Bochner spaces**

In this section we consider some kind of inductive limit topology on Orlicz-Bochner space \( L^\varphi(X) \).

Let \( \varphi \) be an arbitrary Orlicz function, and let

\[ F_n^X = B_X^\varphi(2^n) \quad \text{and} \quad T_n(X) = T_\mu(X)|_{F_n^X} \quad \text{for} \quad n \geq 0. \]

It can be seen that the metric bounded sets \( F_n^X \) \( (n \geq 0) \) are balanced subsets of \( L^\varphi(X) \). Moreover, the sequence \( (F_n^X, T_n(X)) \) \( (n \geq 0) \) of balanced topological spaces satisfies the following conditions:

(i) \( L^\varphi(X) = \bigcup_{n \geq 0} F_n^X \);
(ii) \( F_n^X + F_n^X \subset F_{n+1}^X \), and the function

\[ F_n^X \times F_n^X \ni (f, g) \mapsto f + g \in F_{n+1}^X \]

is continuous \( (n \geq 0) \);
(iii) the function \([-1,1] \times F_n^X \ni (\lambda, f) \mapsto \lambda \cdot f \in F_n^X \) is continuous \( (n \geq 0) \);
(iv) \( T_{n+1}(X)|_{F_n^X} = T_n(X) \) for \( n \geq 0 \).

Thus the space \( L^\varphi(X) \) with the system \( \{(F_n^X, T_n(X)) : n \geq 0\} \) comes under the definition of the strict inductive limit of balanced topological spaces (in the sense of Turpin; see [11, Definition 1.1.1]).
Definition 3.1. Let \( \varphi \) be an Orlicz function and let \((\varepsilon_n)\) be a sequence of positive numbers. The family of all sets of the form:
\[
(*) \quad \bigcup_{N=0}^{\infty} \left( \sum_{n=0}^{N} B^\varphi_X(2^n) \cap B^\mu_X(\varepsilon_n) \right)
\]
forms a base of neighbourhoods of zero for a linear topology \( T_1^\varphi(X) \) on \( L^\varphi(X) \) that will be called **generalized mixed topology.** \( T_1^\varphi(X) \) is exactly the strict inductive limit topology of balanced topological spaces \( \{(B^\varphi_X(2^n), T_\mu(X)|B^\varphi_X(2^n)) : n \geq 0\} \) in the sense of Turpin [11, Chapter I].

Using the solid decomposition property (see [3, Lemma 1.1]) it is easy to verify that the sets of the form \((*)\) are solid, so \( T_1^\varphi(X) \) is locally solid.

According to [11, Theorem 1.1.6] \( T_1^\varphi(X) \) is the finest of all linear topologies \( \tau \) on \( L^\varphi(X) \), which satisfy the condition
\[
(1) \quad \tau|B^\varphi_X(2^n) \subset T_\mu(X)|B^\varphi_X(2^n) \quad \text{for} \quad n \geq 0.
\]
Moreover, in view of [11, Theorem 1.1.8] we have
\[
(2) \quad T_1^\varphi(X)|B^\varphi_X(2^n) = T_\mu(X)|B^\varphi_X(2^n) \quad \text{for} \quad n \geq 0.
\]
Since \( T_\mu(X)|L^\varphi(X) \subset T_\varphi(X) \) we have \( T_1^\varphi(X) \subset T_\varphi(X) \); hence \( T_\mu(X)|L^\varphi(X) \subset T_1^\varphi(X) \subset T_\varphi(X) \).

Henceforth, we assume in this section that \( \varphi(u) \to \infty \) as \( u \to \infty \).

Theorem 3.1. The topology \( T_1^\varphi(X) \) is the finest uniformly \( \mu \)-continuous topology on \( L^\varphi(X) \).

Proof: It follows from (1) and Theorem 2.6.

The generalized mixed topology \( T_1^\varphi \) on Orlicz spaces \( L^\varphi \) has been studied in [11], [8], [9], [10]. Now we will extend the study of the generalized mixed topology to the Orlicz-Bochner spaces.

Theorem 3.2. The space \( (L^\varphi(X), T_1^\varphi(X)) \) is complete.

Proof: First we show that the balls \( B^\varphi_X(2^n) \) are closed subsets of \( (L^{\varphi_0}(X), T_\mu(X)) \). Indeed, let \((f_k)\) be a sequence in \( B^\varphi_X(2^n) \) and let \( f \in L^{\varphi_0}(X) \) be such that \( f_k \to f \) for \( T_\mu(X) \). This means that \( \mu(\{\omega \in \Omega : \|f_k(\omega) - f(\omega)\|_X > \varepsilon\}) \to 0 \) for any \( \varepsilon > 0 \). Hence \( \mu(\{\omega \in \Omega : \|f_k(\omega)\|_X - \|f(\omega)\|_X > \varepsilon\}) \to 0 \) for every \( \varepsilon > 0 \). Thus \( f_k \to \tilde{f} \) for \( T_\mu \) in \( L^{\varphi_0} \). It is known that the balls \( B^\varphi(2^n) \) are closed subsets of \( (L^{\varphi_0}, T_\mu) \) (see [11, 0.3.6]). But \( \tilde{f}_k \in B^\varphi(2^n) \quad (k = 1, 2, \ldots) \), \( \tilde{f} \in L^{\varphi_0} \), so we get \( \tilde{f} \in B^\varphi(2^n) \). It follows that \( f \in B^\varphi_X(2^n) \).

Since the spaces \( (B^\varphi_X(2^n), T_\mu(X)|B^\varphi_X(2^n)) \quad (n \geq 0) \) are complete, by [11, Theorem 1.1.10] the space \( (L^\varphi(X), T_1^\varphi(X)) \) is complete.
Theorem 3.3. For a subset $Z \subset L^\varphi(X)$ the following statements are equivalent:

(i) $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$;
(ii) $Z$ is bounded for $T^\varphi(X)$.

Proof: Observe that the balls $B^\varphi_X(2^n)$ are bounded subsets of $(L^\varphi(X), T_\mu(X)|_{L^\varphi(X)})$. In fact, fix an $r > 0$, let $f_n \in B^\varphi_X(r) \ (n = 1, 2, \ldots)$ and let $\lambda_n \to 0$. For $\varepsilon > 0$ let $\Omega_n(\varepsilon) = \{ \omega \in \Omega : \|\lambda_n f_n(\omega)\|_X > \varepsilon \}$. Then we have

$$\mu(\Omega_n(\varepsilon)) \cdot \varphi\left(\varepsilon \frac{r}{\|\lambda_n\|}\right) \leq \int_{\Omega_n(\varepsilon)} \varphi\left(\frac{\|f_n(\omega)\|_X}{r}\right) d\mu \leq M\varphi\left(\frac{f_n}{r}\right) \leq r.$$ 

Since $\varphi(u) \to \infty$ as $u \to \infty$ we get $\mu(\Omega_n(\varepsilon)) \to 0$ and this means that $\lambda_n f_n \to 0$ for $T_\mu(X)$.

Moreover the balls $B^\varphi_X(2^n)$ are also closed in $(L^\varphi(X), T_\mu(X)|_{L^\varphi(X)})$. In view of (1) and (2) $T^\varphi_t(X)$ is the finest of all linear topologies $\tau$ on $L^\varphi(X)$ such that $\tau|_{B^\varphi_X(2^n)} = T_\mu(X)|_{B^\varphi_X(2^n)} \ (n = 0, 1, 2, \ldots)$. Hence by [11, Corollary 1.1.12] the equivalence (i) $\iff$ (ii) holds. □

Theorem 3.4. For a subset $Z \subset L^\varphi(X)$ the following statements are equivalent:

(i) $Z$ is relatively compact for $T^\varphi_t(X)$;
(ii) $Z$ is relatively compact for $T_\mu(X)|_{L^\varphi(X)}$ and $\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$.

Proof: follows from Theorem 3.3 and (2). □

Definition 3.2. A sequence $(f_n)$ in $L^\varphi(X)$ is said to be $\gamma^X_\varphi$-convergent to $f \in L^\varphi(X)$, in symbols $f_n \xrightarrow{\gamma^X_\varphi} f$, whenever

$$f_n \to f \ (\mu - \Omega) \text{ and } \sup_n \|f_n\|_{L^\varphi(X)} < \infty.$$ 

Theorem 3.5. For a sequence $(f_n)$ in $L^\varphi(X)$ the following statements are equivalent:

(i) $f_n \to 0$ for $T^\varphi_t(X)$;
(ii) $f_n \xrightarrow{\gamma^X_\varphi} 0$.

Moreover, $T^\varphi_t(X)$ is the finest of all linear topologies $\tau$ on $L^\varphi(X)$ which satisfy the condition:

$$(+) \quad f_n \xrightarrow{\gamma^X_\varphi} 0 \text{ implies } f_n \to 0 \text{ for } \tau.$$ 

Proof: The equivalence (i) $\iff$ (ii) follows immediately from Theorem 3.3 and (2). Now let $\tau$ be a linear topology on $L^\varphi(X)$ for which the condition $(+)$ holds. Then $\tau|_{B^\varphi_X(r)} \subset T_\mu(X)|_{B^\varphi_X(r)}$ for $r > 0$, because $T_\mu(X)$ is a linear metrizable topology. Hence by (1) we get $\tau \subset T^\varphi_t(X)$. □
Definition 3.3. Let \((Y, \eta)\) be a linear topological space. A linear mapping \(T : L^\varphi(X) \to Y\) is said to be \(\gamma\varphi\)-linear, whenever
\[ f_n \xrightarrow{\gamma\varphi} 0 \text{ implies } T(f_n) \to 0 \text{ for } \eta. \]

Then following theorem gives a characterization of \(\gamma\varphi\)-linear operators on \(L^\varphi(X)\).

Theorem 3.6. For a linear topological space \((Y, \eta)\) and a linear mapping \(T : L^\varphi(X) \to Y\) the following statements are equivalent:

(i) \(T\) is \((T^\varphi_I(X), \eta)\)-continuous;
(ii) \(T\) is \(\gamma\varphi\)-linear;
(iii) for every \(r > 0\), the restriction \(T|_{B^\varphi_X(r)}\) is \((T^\mu_X|_{B^\varphi_X(r)}, \eta)\)-continuous.

Proof: (i) \(\Rightarrow\) (ii) It follows from Theorem 3.5.

(ii) \(\Rightarrow\) (iii) Obvious.

(iii) \(\Rightarrow\) (i) Let \(W\) be a neighbourhood of zero in \(Y\) for \(\eta\). Since \(\eta\) is a linear topology, there exists a sequence \((W_n : n \geq 0)\) of neighbourhoods of zero for \(\eta\) such that \(\sum_{n=0}^{N} W_n \subset W\) for every \(N \geq 0\). By (iii) we can find a sequence \((\varepsilon_n : n \geq 0)\) of positive numbers such that \(T(B^\varphi_X(2^n) \cap B^\mu_X(\varepsilon_n)) \subset W_n\) for \(n \geq 0\).

Thus for \(N \geq 0\) we have
\[ T\left( \sum_{n=0}^{N} (B^\varphi_X(2^n) \cap B^\mu_X(\varepsilon_n)) \right) \subset \sum_{n=0}^{N} W_n \subset W, \]
so
\[ T\left( \bigcup_{N=0}^{\infty} \left( \sum_{n=0}^{N} (B^\varphi_X(2^n) \cap B^\mu_X(\varepsilon_n)) \right) \right) \subset \bigcup_{N=0}^{\infty} T\left( \sum_{n=0}^{N} (B^\varphi_X(2^n) \cap B^\mu_X(\varepsilon_n)) \right) \subset W. \]

It follows that \(T\) is \((T^I(X), \eta)\)-continuous. \(\square\)

Theorem 3.7. Assume that \((\Omega, \Sigma, \mu)\) is an atomless measure space or that \(\mu\) is the counting measure on \(N\). If \((L^\varphi(X), T^\varphi(X))\) is a locally bounded space then for a subset \(Z\) of \(L^\varphi(X)\) the following statements are equivalent:

(i) \(Z\) is bounded for \(T^\varphi_I(X)\);
(ii) \(\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty\);
(iii) \(Z\) is bounded for \(T^\varphi(X)\).

Proof: (i) \(\iff\) (ii) See Theorem 3.3.

(ii) \(\Rightarrow\) (iii) In view of [11, 0.3.10.2] \(\sup\{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty\) iff \(Z\) is additively bounded (see [11, 0.3.10.1]), so arguing as in the proof of [9, Lemma 2.5] we obtain that \(Z\) is bounded for \(T^\varphi(X)\).

(iii) \(\Rightarrow\) (i) Obvious. \(\square\)

The next theorem compares the topology \(T^\varphi_I(X)\) with the mixed topology \(\gamma[T^\varphi(X), T^\mu(X)|_{L^\varphi(X)}]\) in the sense of Wiweger (see [12]).
**Theorem 3.8.** Assume that \((\Omega, \Sigma, \mu)\) is an atomless measure space or that \(\mu\) is the counting measure on \(\mathbb{N}\). If \((L^\varphi(X), T^\varphi(X))\) is a locally bounded space, then the generalized mixed topology \(T^\varphi_I(X)\) coincides with the mixed topology \(\gamma[T^\varphi(X), T^\mu(X)|_{L^\varphi(X)}]\).

**Proof:** In view of Theorem 3.7 it follows from [12, 2.2.1, 2.2.2]. □

An Orlicz function \(\varphi\) continuous for all \(u \geq 0\), taking only finite values, vanishing only at zero and not bounded is usually called a \(\varphi\)-function. By \(\Phi\) we will denote the collection of all \(\varphi\)-functions.

A Young function \(\varphi\) vanishing only at zero and taking only finite values is called an \(N\)-function whenever \(\frac{\varphi(t)}{t} \to 0\) as \(t \to 0\) and \(\frac{\varphi(t)}{t} \to \infty\) as \(t \to \infty\). By \(\Phi_N\) we will denote the collection of all \(N\)-functions.

Let \(\Phi^1\) be the set of all Orlicz functions \(\varphi\) vanishing only at zero and such that \(\frac{\varphi(t)}{t} \to \infty\) as \(t \to \infty\). Denote by

\[
\Phi^1_{11} = \{ \varphi \in \Phi^1 : \varphi(t) < \infty \text{ for } t \geq 0 \}, \\
\Phi^1_{12} = \{ \varphi \in \Phi^1 : \varphi \text{ jumps to } \infty \}.
\]

Then \(\Phi^1 = \Phi^1_{11} \cup \Phi^1_{12} \).

**Theorem 3.9.** Let \(\varphi \in \Phi^1_i (i = 1, 2)\). Then the topology \(T^\varphi_I(X)\) is generated by the family of solid \(F\)-norms:

\[
\{ \| \cdot \|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i} \},
\]

where \(\Psi^\varphi_{1i} = \{ \psi \in \Phi : \psi \prec \varphi \} \), \(\Psi^\varphi_{12} = \{ \psi \in \Phi : \psi \nsim \varphi \} \).

Moreover, the following identities hold:

\[
L^\varphi(X) = \bigcap \{ L^\psi(X) : \psi \in \Psi^\varphi_{1i} \} = \bigcap \{ E^\psi(X) : \psi \in \Psi^\varphi_{1i} \}.
\]

**Proof:** Let \(\varphi \in \Phi^1_i (i = 1, 2)\). Then \(T^\varphi_I\) is the finest uniformly \(\mu\)-continuous topology on \(L^\varphi\) (see [10, Theorem 2.4]) and is generated by the family \(\{ \| \cdot \|_\psi : \psi \in \Psi^\varphi_{1i} \} \) (see [10, Theorem 4.5, Theorem 3.8]). Then the topology \(\overline{T^\varphi_I}\) on \(L^\varphi(X)\) is generated by the family \(\{ \| \cdot \|_{L^\psi(X)} : \psi \in \Psi^\varphi_{1i} \}\) of solid \(F\)-norms and by Theorem 2.3 \(\overline{T^\varphi_I}\) is the finest uniformly \(\mu\)-continuous topology on \(L^\varphi(X)\). By Theorem 3.1 \(\overline{T^\varphi_I} = T^\varphi_I(X)\), and we are done.

The identities (3) follow from [10, Theorem 3.1]. □

Let \(\Phi^c_i\) be the set of all Young functions \(\varphi\) vanishing only at zero and such that \(\frac{\varphi(t)}{t} \to \infty\) as \(t \to \infty\). Denote by

\[
\Phi^c_{11} = \{ \varphi \in \Phi^c_i : \varphi(t) < \infty \text{ for } t \geq 0 \text{ and } \frac{\varphi(t)}{t} \to 0 \text{ as } t \to 0 \}, \\
\Phi^c_{12} = \{ \varphi \in \Phi^c_i : \varphi \text{ jumps to } \infty \text{ and } \frac{\varphi(t)}{t} \to 0 \text{ as } t \to 0 \},
\]
\( \Phi_{13}^c = \{ \varphi \in \Phi_1^c : \varphi(t) < \infty \text{ for } t \geq 0 \text{ and } \frac{\varphi(t)}{t} \to a \text{ as } t \to 0 \} \),
\( \Phi_{14}^c = \{ \varphi \in \Phi_1^c : \varphi \text{ jumps to } \infty \text{ and } \frac{\varphi(t)}{t} \to a \text{ as } t \to 0 \} \).

Then \( \Phi_1^c = \bigcup_{i=1}^4 \Phi_{1i}^c \) and the sets \( \Phi_{1i}^c \) \( (i = 1, 2, 3, 4) \) are pairwise disjoint. It can be seen that \( \Phi_{11}^c = \Phi_N \).

**Theorem 3.10.** Let \( \varphi \in \Phi_{1i}^c \) \( (i = 1, 2, 3, 4) \). Then the topology \( T_1^\varphi(X) \) is generated by the family of solid norms

\[
\{ \| \cdot \|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi(N) \},
\]

where \( \Psi_{11}^\varphi(N) = \{ \psi \in \Phi_N : \psi \prec \varphi \} \), \( \Psi_{12}^\varphi(N) = \{ \psi \in \Phi_N : \psi \lesssim \varphi \} \), \( \Psi_{13}^\varphi(N) = \{ \psi \in \Phi_N : \psi \ll \varphi \} \), \( \Psi_{14}^\varphi(N) = \Phi_N \).

Moreover, the following identities hold:

\[
(4) \quad L^\varphi(X) = \bigcap \{ L^\psi(X) : \psi \in \Psi_{1i}^\varphi(N) \} = \bigcap \{ E_{\psi}(X) : \psi \in \Psi_{1i}^\varphi(N) \}.
\]

**Proof:** Let \( \varphi \in \Phi_{1i}^c \) \( (i = 1, 2, 3, 4) \). Then \( T_1^\varphi \) is the finest uniformly \( \mu \)-continuous topology on \( L^\varphi \) (see [10, Theorem 2.4]) and is generated by the family \( \{ \| \cdot \|_{\psi} : \psi \in \Psi_{1i}^\varphi(N) \} \) (see [10, Theorem 3.12 and Theorem 4.5]). Then the topology \( T_1^\varphi \) on \( L^\varphi(X) \) is generated by the family \( \{ \| \cdot \|_{L^\psi(X)} : \psi \in \Psi_{1i}^\varphi(N) \} \) of solid norms, and by Theorem 2.3 \( T_1^\varphi \) is the finest uniformly \( \mu \)-continuous topology on \( L^\varphi(X) \).

By Theorem 3.1 \( T_1^\varphi = T_1^\psi(X) \), as desired.

The identities (4) follow from [10, Theorem 3.2]. \( \square \)

As an application of Theorem 3.10 we get a characterization of uniformly \( \mu \)-continuous seminorms on \( L^\varphi(X) \).

**Theorem 3.11.** Let \( \varphi \in \Phi_{1i}^c \) \( (i = 1, 2, 3, 4) \). Then for a solid seminorm \( \rho \) on \( L^\varphi(X) \) the following statements are equivalent:

(i) \( \rho \) is uniformly \( \mu \)-continuous;
(ii) there exist \( \psi \in \Psi_{1i}^\varphi(N) \) and a number \( a > 0 \) such that

\[
\rho(f) \leq a \| f \|_{L^\psi(X)} \quad \text{for all } f \in L^\varphi(X).
\]

**Proof:** (i) \( \Rightarrow \) (ii) Since \( T_1^\varphi(X) \) is the finest uniformly \( \mu \)-continuous topology on \( L^\varphi(X) \) (see Theorem 3.1), in view of Theorem 3.10 and [4, Chapter 4, §18(4)] there exist \( \psi_1, \ldots, \psi_n \in \Psi_{1i}^\varphi(N) \) and a number \( a > 0 \) such that

\[
\rho(f) \leq a \max(\| f \|_{L^\psi_1(X)}, \ldots, \| f \|_{L^\psi_n(X)}) \quad \text{for all } f \in L^\varphi(X).
\]
Let $\psi(u) = \max(\psi_1(u), \ldots, \psi_n(u))$ for $u \geq 0$. Then $\psi \in \Psi_1^\varphi(N)$ and 

$$
\|f\|_{L^\psi_i(X)} \leq \|f\|_{L^\psi(X)} \quad \text{for } i = 1, \ldots, n \text{ and all } f \in L^\varphi(X),
$$

so

$$
\rho(f) \leq a\|f\|_{L^\psi(X)} \quad \text{for all } f \in L^\varphi(X).
$$

(ii) ⇒ (i) It is obvious, because for each $\psi \in \Psi_1^\varphi(X)$, $\|\cdot\|_{L^\psi(X)}$ is a uniformly

$\mu$-continuous norm on $L^\varphi(X)$. □

To present the general form of $T_I^\varphi(X)$-continuous linear functionals on $L^\varphi(X)$ we recall the terminology concerning some spaces of $X$-weak measurable functions (see [2]).

Given a function $g : \Omega \to X^*$ and $x \in X$ we denote by $g_x$ the real function on $\Omega$ defined by $g_x(\omega) = g(\omega)(x)$. A function $g$ is said to be $X$-weak measurable if the functions $g_x$ are measurable for each $x \in X$. We say that two $X$-weak measurable functions $g_1$, $g_2$ are equivalent whenever $g_1(\omega)(x) = g_2(\omega)(x) \mu$-a.e. for all $x \in X$.

By $L^0(X^*, X)$ we denote the linear space of equivalence classes of all $X$-weak measurable functions $g : \Omega \to X^*$. It is known that the set $\{|g_x| : x \in B_X\}$ is order bounded in $L^0$ for every $g \in L^0(X^*, X)$.

The function $\vartheta : L^0(X^*, X) \to L^0$ defined by

$$
\vartheta(g) = \sup\{|g_x| : x \in B_X\} \quad \text{for } g \in L^0(X^*, X)
$$

is called an abstract norm.

It is known that for $f \in L^0(X), g \in L^0(X^*, X)$ the function $\langle f, g \rangle : \Omega \to R$ defined by $\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle = g(\omega)(f(\omega))$ is measurable and

$$
|\langle f, g \rangle(\omega)| \leq \|f(\omega)\|_X \cdot \vartheta(g)(\omega) \quad \mu\text{-a.e.}
$$

For an ideal $I$ of $L^0$ let

$$
I(X^*, X) = \{g \in L^0(X^*, X) : \vartheta(g) \in I\}.
$$

**Theorem 3.12.** Let $\varphi \in \Phi_1^\varphi (i = 1, 2, 3, 4)$. Then for a linear functional $F$ on $L^\varphi(X)$ the following statements are equivalent:

(i) $F$ is continuous for $T_I^\varphi(X)$;

(ii) $F$ is $\gamma_\varphi$-linear;

(iii) there exists a unique $g \in E^\varphi^*(X^*, X)$ such that

$$
F(f) = F_g(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for } f \in L^\varphi(X).
$$
Proof: (i) $\Leftrightarrow$ (ii) The equivalence follows from Theorem 3.6.

(i) $\Rightarrow$ (iii) Let $\varphi \in \Phi_{1i}^c$ ($i = 1, 2, 3, 4$). In view of Theorem 3.10 (see also the proof of Theorem 3.11) there exist $\psi \in \Psi_{1i}^\varphi(N)$ and $r > 0$ such that $F$ is bounded on $B_X^{(\psi)}(r) \cap L^\varphi(X)$, where $B_X^{(\psi)}(r) = \{ f \in L^\psi(X) : \| f \|_{L^\psi(X)} \leq r \}$. This means that $F$ is continuous on the linear subspace $(L^\varphi(X), T_\psi(X)|_{L^\varphi(X)})$ of the normed space $(E^\psi(X), T_\psi(X)|_{E^\psi(X)})$. Hence by the Hahn-Banach extension theorem there exists a $T_\psi(X)|_{E^\psi(X)}$-continuous linear functional $\overline{F}$ on $E^\psi(X)$ such that $\overline{F}(f) = F(f)$ for $f \in L^\varphi(X)$. Since $E^\psi = L^\psi_\alpha$, we get $E^\psi(X) = L^\psi_\alpha(X)$. By [2, Corollary 4.1] there exists a unique $g \in (L^\psi_\alpha)'(X^*, X)$ such that

$$\overline{F}(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle d\mu \quad \text{for} \quad f \in L^\psi_\alpha(X).$$

But $(L^\psi_\alpha)' = L^{\psi^*}$ (see [6, p.56]), so by [10, Corollary 3.5] we get $L^{\psi^*} \subset E^{\psi^*}$. Finally, there exists a unique $g \in E^{\psi^*}(X^*, X)$ such that

$$\overline{F}(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle d\mu \quad \text{for} \quad f \in L^\psi_\alpha(X).$$

Hence

$$F(f) = F_g(f) = \int_\Omega \langle f(\omega), g(\omega) \rangle d\mu \quad \text{for} \quad f \in L^\varphi(X).$$

(iii) $\Rightarrow$ (i) Let $\varphi \in \Phi_{1i}^c$ ($i = 1, 2, 3, 4$). According to [10, Corollary 3.5] there exists $\psi \in \Psi_{1i}^\varphi(N)$ such that $g \in L^{\psi^*}(X^*, X)$. Then $L^\varphi(X) \subset E^\psi(X) \subset L^\psi(X)$. Moreover, by [2, Theorem 1.1] using the Hölder inequality we get for $f \in L^\varphi(X)$

$$|F_g(f)| \leq \int_\Omega |\langle f(\omega), g(\omega) \rangle| d\mu \leq \int_\Omega \| f(\omega) \|_{X} \cdot \| g(\omega) \|_{\psi^*} d\mu \leq 2\| f \|_{L^\varphi(X)} \cdot \| g \|_{\psi^*}.$$ 

This means that $T_\psi(X)|_{L^\varphi(X)}$-continuous, so $F_g$ is $T_I^\varphi(X)$-continuous, because $T_\psi(X)|_{L^\varphi(X)} \subset T_I^\varphi(X)$ by Theorem 3.10.

Thus the proof is complete. \qed

References


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