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Abstract. The aim of this paper is to present an existence theorem for the operator equation of Hammerstein type \( x + KF(x) = 0 \) with the discontinuous semimonotone operator \( F \). Then the result is used to prove the existence of solution of the equations of Urysohn type. Some examples in the theory of nonlinear equations in \( L^p(\Omega) \) are given for illustration.

Keywords: semimonotone operators, uniformly convex Banach spaces

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1. Introduction

Let \( X \) be a real Banach space and \( X^* \) be its dual which are uniformly convex. For the sake of simplicity, the norms of \( X \) and \( X^* \) will be denoted by one symbol \( \| \cdot \| \). We write \( \langle x^*, x \rangle \) instead of \( x^*(x) \) for \( x \in X^* \) and \( x \in X \). Let \( F : X \to X^* \) be a bounded, discontinuous and semimonotone operator and \( K : X^* \to X \) a bounded (i.e. image of any bounded subset is bounded), linear and nonnegative operator.

Consider the nonlinear operator equation of Hammerstein type

\[
(1.1) \quad x + KF(x) = 0.
\]

Integral equations of Hammerstein type with a nonlinear smooth operator \( F \) are studied in [1]–[3], [6], [17]. When \( F \) is discontinuous, they are investigated in [5], [7], [15], [16] by introducing a new concept of solution. But, throughout this paper, the word ‘solution’ is meant in the classical sense. We shall prove an existence theorem for solution for discontinuous \( F \). Using this result, we get a new result regarding the solvability of a class of nonlinear equations of Urysohn type

\[
(1.2) \quad x + \sum_{j=1}^{m} K_jF_j(x) = 0,
\]

where each \( K_j \) and \( F_j \) has the properties as \( K \) and \( F \), respectively. Then, these theoretical results are applied to study the nonlinear integral equations in the spaces of type \( L^p(\Omega) \). It should be mentioned that quasilinear elliptic equations
with nonlinear discontinuous part are usually used to describe the state of the systems with variable structure (see [10]). These equations are studied recently (see [12]–[14]) and can be transformed to equations of Hammerstein type (see [12]).

Below, the symbols $\rightarrow$ and $\rightharpoonup$ denote convergence in norm and weak convergence, respectively.

2. Main result

**Definition 1** (see [13]). A point $x \in X$ is called a point of h-continuity of the operator $G : X \rightarrow X^*$ if

$$ \forall l \in X \lim_{t \to 0^+} \langle G(x + tl), l \rangle = \langle G(x), l \rangle. $$

A point $x \in X$ is called a point of discontinuity if $x$ does not satisfy the condition in Definition 1.

**Definition 2.** A point of discontinuity $x$ of $G$ is called regular if

$$ \exists l \in X : \lim_{t \to 0^+} \langle G(x + tl), l \rangle < 0. $$

**Theorem 2.1.** Assume that all the above conditions hold, all the points of discontinuity of $F$ are regular and that there exists a positive constant $r$ such that

$$ \langle F(x), x \rangle > 0 \text{ if } \|x\| > r. $$

Then equation (1.1) has a solution $x$.

**Proof:** As in [6], consider the regularized equation

\begin{equation}
(2.1) \quad x + B_n F(x) = 0, \quad B_n = B + \alpha_n V,
\end{equation}

where $V$ is the standard dual mapping of $X^*$, i.e. $V : X^* \rightarrow X$,

$$ \langle V(x^*), x^* \rangle = \|V(x^*)\| \|x^*\| = \|x^*\|^2, \quad \forall x^* \in X^*, $$

and $\alpha_n$ is a sequence of positive real numbers such that $\alpha_n \to 0$ as $n \to +\infty$. Then $R(B_n) = X$, $B_n^{-1}(0) = 0$, $B_n^{-1}$ is an one-to-one mapping and $B_n^{-1}$ is continuous (see [4]). Therefore, all the points of discontinuity of $F$ are points of discontinuity of $\tilde{B}_n + F$ and, conversely, all points of discontinuity of $\tilde{B}_n + F$ are points of discontinuity of $F$, where $\tilde{B}_n(x) = -B_n^{-1}(-x)$. Obviously, we can rewrite equation (2.1) in the form

\begin{equation}
(2.2) \quad \tilde{B}_n(x) + F(x) = 0.
\end{equation}
Moreover, by virtue of [17], equation (2.2) has a unique solution, henceforth denoted by \( x_n \). Moreover, \( ||x_n|| \leq r, \forall n \). As \( F \) is bounded, the sequence \( \{F(x_n)\} \) is bounded, too. Without loss of generality, assume that

\[
x_n \to x_0 \quad \text{and} \quad F(x_n) \to y_0^*.
\]

From (2.1) it follows that

\[
(2.3) \quad x_0 + By_0^* = 0.
\]

Now, we have to prove that \( y_0^* = F(x_0) \). Since \( F \) is semimonotone, we have \( F = T + C \), with a monotone operator \( T \) and a compact operator \( C \). Therefore,

\[
\langle F(x) - C(x) - (F(x_n) - C(x_n)) , x - x_n \rangle > 0, \quad \forall x \in X.
\]

Hence,

\[
\langle F(x) - C(x) , x - x_0 \rangle - \langle F(x_n) - C(x_n) , x \rangle \geq \langle F(x_n) , BF(x_n) \rangle
\]

\[
- \langle C(x_n) , x_n \rangle = \alpha_n \langle F(x_n) , VF(x_n) \rangle.
\]

By passing \( n \to +\infty \) in the last equality, because of

\[
\lim_{n \to +\infty} \inf \langle F(x_n) , BF(x_n) \rangle \geq \langle y_0^* , By_0^* \rangle,
\]

\[
\lim_{n \to +\infty} \alpha_n \langle F(x_n) , VF(x_n) \rangle = 0,
\]

\[
\lim_{n \to +\infty} \langle C(x_n) , x_n \rangle = \langle C(x_0) , x_0 \rangle,
\]

and (2.3) we obtain

\[
\langle F(x) - C(x) , x - x_0 \rangle - \langle y_0^* - C(x_0) , x \rangle \geq \langle y_0^* , By_0^* \rangle - \langle C(x_0) , x_0 \rangle.
\]

Thus,

\[
(2.4) \quad \langle T(x) - (y_0^* - C(x_0)) , x - x_0 \rangle \geq 0.
\]

Replacing \( x \) by \( x_0 + tl \) for any \( l \in X \) and \( t > 0 \) in (2.4) we see that

\[
\langle F(x_0 + tl) - (y_0^* + C(x_0)) , l \rangle \geq 0, \quad \forall l \in X.
\]

Hence, \( x_0 \) is a point of h-continuity of \( T \). Consequently, from (2.4) and Minty’s lemma (see [17]) \( T(x_0) = y_0^* - C(x_0) \), i.e. \( y_0^* = F(x_0) \).

Now, consider equation (2.1). Let the following conditions hold:

\begin{itemize}
  \item \( K_j : X^* \to X \) are linear and bounded operators satisfying the condition:
    \[
    \sum_{j=1}^{m} \langle K_j x_j^* , x^* \rangle \geq 0, \quad x^* = \sum_{i=1}^{m} x_i^* , \quad x_i^* \in X^*;
    \]
  \item \( F_j : X \to X^* \) are bounded, discontinuous and semimonotone, and
  \item \( \langle F_j(x) , x \rangle \geq a_j \|x\|^2 - b_j \|x\| - c_j, \quad a_j, b_j, c_j > 0 \) (see [8]).
\end{itemize}

Operator equation (1.2) is investigated in [8]–[9], [11] with some smoothness property of \( F_j \). Here, applying Theorem 2.1, we can prove the following result.
Theorem 2.2. Under the above conditions on $K_j$ and $F_j$, equation (1.2) has a solution in $X$.

Proof: Denote $Z = X \times \cdots \times X$ ($m$ times). For $z = (x_1, \ldots, x_m) \in Z$, let

$$
\|z\| = \left( \sum_{j=1}^{m} \|x_j\|^2 \right)^{1/2}.
$$

Then, $Z$ is uniformly convex Banach space with respect to this norm with dual $Z^* = X^* \times \cdots \times X^*$. $(x_1, \ldots, x_m)$ means the column vector $(x_1, \ldots, x_m)^T$. Let $K : Z^* \to Z$ and $F : Z \to Z^*$ be defined as follows

$$
(2.5) \quad K = \begin{bmatrix}
K_1 & K_2 & \cdots & K_m \\
K_1 & K_2 & \cdots & K_m \\
\vdots & \vdots & \ddots & \vdots \\
K_1 & K_2 & \cdots & K_m
\end{bmatrix}, \quad F = \begin{bmatrix}
F_1 & 0 & \cdots & 0 \\
0 & F_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & F_m
\end{bmatrix}.
$$

Consider the Hammerstein equation

$$
(2.6) \quad z + KF(z) = 0, \quad z \in Z
$$

with $K$ and $F$ from (2.5). It is easy to see that $K$ is a linear, bounded and nonnegative operator on $Z^*$ and $F$ is a semicontinuous operator on $Z$. Moreover,

$$
\langle F(z), z \rangle = \sum_{j=1}^{m} \langle F_j(x_j) \rangle \geq \sum_{j=1}^{m} \left( a_j \|x_j\|^2 - b_j \|x_j\| - c_j \right)
$$

$$
\geq a \|z\|^2 - b \|z\| - c,
$$

where $a = \min a_j$, $b = \sqrt{m} \max b_j$ and $c = \max c_j$. Therefore, there exists a positive constant $R$ such that $\langle F(z), z \rangle > 0$, if $\|z\| > R$. By virtue of Theorem 2.1, equation (2.6) has a solution $z_* = (x_{1*}, \ldots, x_{m*})$. Consequently, equation (1.2) has a solution $x = x_{1*} = \cdots = x_{m*}$. \hfill \qed

3. Application

a. Consider the nonlinear integral equation of second kind

$$
(3.1) \quad x(s) + \int_{\Omega} k(s, t)F(x(t)) \, dt = 0,
$$

where the kernel function $k(s, t)$ is such that the operator $K$ defined by

$$
(Kx)(s) = \int_{\Omega} k(s, t)x(t) \, dt
$$
is bounded, nonnegative and $K$ acts from $L_q(\Omega)$ into $L_p(\Omega)$ with $\Omega \subset \mathbb{R}^n$ measurable and $p^{-1} + q^{-1} = 1$. The nonlinear function $f(t)$ satisfies the following conditions:

(a) $f(t)t \geq a_0|t|^p + b_0|t|^\gamma + c_0$, $a_0 > 0$, $b_0 < 0$, $c_0 < 0$, $\gamma < p$ (see [14]),
(b) $f(t)$ is nondecreasing, rightcontinuous and at any point of discontinuity $t_0$ $f(t_0 - 0) < 0$, $f(t_0) < 0$,
(c) $|F(t)| \leq a_1 + b_1|t|^{p-1}$, $\forall t \in R^1$, $a_1 + b_1 > 0$, $a_1 \geq 0$, $b_1 \geq 0$.

By virtue of (c) we can define the operator $F : X = L_p(\Omega) \to X^* = L_q(\Omega)$ as

$$F(x)(t) = F(x(t)), \quad \forall x(t) \in L_p(\Omega).$$

Then equation (3.1) can be rewritten in the form (1.1), where the defined operator $F$ possesses all the properties from Section 1. Indeed, condition (a) guarantees the existence of $r$ in Theorem 2.1, the monotone property and the regularity of all points of discontinuity of $F$ follows from (b) (see [13]) and the remaining properties are verified on the base of (c). Therefore, equation (3.1) has a solution, and this solution is unique if one of the operators $K$, $F$ is strictly monotone.

b. Consider the nonlinear integral equation

$$(3.3) \quad x(t) + \sum_{j=1}^{m} \int_{\Omega} k_j(t, s)f_j(x(s)) \, ds = 0.$$ 

If the operators $K_j$ and $F_j$ defined by

$$\begin{align*}
(K_jx)(t) &= \int_{\Omega} k_j(t, s)x(s) \, ds, \\
(F_jx)(t) &= f_j(x(t)),
\end{align*}$$

have the same properties as $K$ and $F$ in a., where only instead of the nonnegativeness of $K$ we assume that

$$\sum_{i=1}^{m} \int_{\Omega} x_i(t) \int_{\Omega} \sum_{j=1}^{m} k_j(t, s)x_j(s) \, ds \, dt \geq 0,$$

then (3.3) can be rewritten in the form (1.2). Therefore, equation (3.3) is solvable by Theorem 2.2.

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