Minoru Matsuda
A remark on localized weak precompactness in Banach spaces


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A remark on localized weak precompactness in Banach spaces

MINORU MATSUDA

Abstract. We give a characterization of $K$-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions.

Keywords: $K$-weakly precompact set, uniform Gateaux differentiability

Classification: 46B07, 46B22, 49J50

We begin with the requisite definition. Throughout this paper $X$ denotes a real Banach space with topological dual $X^*$. If $g : X \to \mathbb{R}$ is a continuous convex function, for $x, y \in X$, we define $Dg(x, y)$ by

$$
\lim_{t \to 0} \frac{g(x + ty) - g(x)}{t}
$$

provided that this limit exists, and we also define the subdifferential of $g$ at $x \in X$ to be the set $\partial g(x)$ of all elements $x^* \in X^*$ satisfying that $(u, x^*) \leq g(x + u) - g(x)$ for any $u \in X$. Then $\partial g(x)$ is a non-empty weak* compact convex subset of $X^*$ for every $x \in X$. The triple $(I, \Lambda, \lambda)$ refers to the Lebesgue measure space on $I (= [0, 1])$, $\Lambda^+$ to the sets in $\Lambda$ with positive $\lambda$-measure. We always understand that $I$ is endowed with $\Lambda$ and $\lambda$. We denote the set $\{ \chi_E/\lambda(E) : E \in \Lambda^+ \}$ by $\Delta(I)$. A function $f : I \to X^*$ is said to be weak*-measurable if $(x, f(t))$ is $\lambda$-measurable for each $x \in X$. If $f : I \to X^*$ is a bounded weak*-measurable function, we obtain a bounded linear operator $T_f : X \to L_1(I, \Lambda, \lambda)$ given by $T_f(x) = x \circ f$ for every $x \in X$, where $(x \circ f)(t) = (x, f(t))$ for every $t \in I$, and the dual operator of $T_f$ is denoted by $T_f^* : L_\infty(I, \Lambda, \lambda) \to X^*$.

According to Bator and Lewis [1], let us define the notion of localized weak precompactness in Banach spaces as follows.

Definition 1. Let $A$ be a bounded subset of $X$ and $K$ a weak*-compact subset of $X$. Then we say that $A$ is $K$-weakly precompact if every sequence $\{x_n\}_{n \geq 1}$ in $A$ has a pointwise convergent subsequence $\{x_{n(k)}\}_{k \geq 1}$ on $K$.

Then, in [1], they have made a systematic study of $K$-weakly precompact sets $A$ in Banach spaces and obtained various characterizations of such sets.

Succeedingly, in our paper [4], we also have obtained measure theoretic characterizations of $K$-weakly precompact sets $A$ by the effective use of a $K$-valued
weak*-measurable function constructed in the case where $A$ is non-$K$-weakly precompact. In this paper we wish to add a characterization of $K$-weakly precompact sets in terms of uniform Gateaux differentiability of certain continuous convex functions, which is our aim. This can be regarded as a slight generalization and refinement of Corollary 10 in [1]. And it should be noted that even here this $K$-valued function also becomes an effective means to an end. Before giving our characterization theorem, let us define some special continuous convex functions on $X$ as follows.

**Definition 2.** Let $H$ be a non-empty bounded subset of $X^*$. Then the continuous convex function associated with $H$, which is denoted by $g_H$, is defined by $g_H(x) = \sup\{(x, x^*) : x^* \in H\}$ for every $x \in X$.

In what follows, all notations and terminology used and not defined are as in [1].

Let $A$ be a bounded subset of $X$, $K$ a weak*-compact subset of $X^*$, $\{x_n\}_{n \geq 1}$ a sequence in $A$ and $Y$ the closed linear span of $\{x_n : n \geq 1\}$ in $X$. In the following, we always understand that $Y$ is a such space. Let $j : Y \rightarrow X$ be the inclusion mapping and $j^*$ its dual mapping. For any non-empty subset $H$ of $K$, the continuous convex function $g_H : Y \rightarrow \mathbb{R}$ satisfies that $\partial g_H(y) \subset \overline{co}^*(j^*(K))$ for each $y \in Y$. Further let us note two preliminary facts for the proof of Theorem. One concerns separably related sets in the case where $A$ is $K$-weakly precompact. Let $\{x_n\}_{n \geq 1}$ be a sequence in $A$ and suppose that there exists a subsequence $\{x_{n(k)}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$ such that $\lim_{k \rightarrow \infty}(x_{n(k)}, x^*)$ exists for every $x^* \in K$. Then this implies that $\lim_{k \rightarrow \infty}(x_{n(k)}, y^*)$ exists for every $y^* \in \overline{co}^*(j^*(K))$. Hence, by considering the mapping $L : \overline{co}^*(j^*(K)) \rightarrow c$ (the Banach space of all convergent sequences of real numbers equipped with the supremum norm $\| \cdot \|_\infty$) defined by $L(y^*) = \{(x_{n(k)}, y^*)\}_{k \geq 1}$, we easily know that $\overline{co}^*(j^*(K))$ is separably related to $\{x_{n(k)} : k \geq 1\}$, since $c$ is separable. The other concerns the construction of a $K$-valued weak*-measurable function $h$ and a sequence $\{x_n\}_{n \geq 1}$ in $A$ in the case where $A$ is non-$K$-weakly precompact. Then, although the construction of this function $h$ and the sequence $\{x_n\}_{n \geq 1}$ in $A$ is exactly the same as in §3 of [4], for the sake of completeness, we state its outline briefly in the following. Since $A$ is not $K$-weakly precompact, by the celebrated argument of Rosenthal [5], we have a sequence $\{x_n\}_{n \geq 1}$ in $A$ and real numbers $r$ and $\delta$ with $\delta > 0$ such that putting $A_n = \{x^* \in K : (x_n, x^*) \leq r\}$ and $B_n = \{x^* \in K : (x_n, x^*) \geq r + \delta\}$, $A_n, B_n \geq 1$ is an independent sequence of pairs of weak*-closed subsets of $K$ (that is, for every $\{\varepsilon_j\}_{1 \leq j \leq k}$ with $\varepsilon_j = 1$ or $-1$, $\bigcap \{\varepsilon_j A_j : 1 \leq j \leq k\}$ is a non-empty set, where $\varepsilon_j A_j = A_j$ if $\varepsilon_j = 1$ and $\varepsilon_j A_j = B_j$ if $\varepsilon_j = -1$). Putting $\Gamma = \bigcap_{n \geq 1}(A_n \cup B_n)$, $\Gamma$ is a non-empty weak*-compact subset of $K$, since $(A_n, B_n)_{n \geq 1}$ is independent. Define $\varphi : \Gamma \rightarrow \mathcal{P}(N)$ (Cantor space, with its usual compact metric topology) by $\varphi(x^*) = \{p : (x_p, x^*) \leq r\} (= \{p : A_p \ni x^*\}) \in \mathcal{P}(N)$. Then $\varphi$ is a continuous surjection from $\Gamma$ to $\mathcal{P}(N)$ (here, $\Gamma$ is endowed with the weak*-topology $\sigma(X^*, X)$) and so we have a Radon probability measure $\gamma$ on $\Gamma$ such that $\varphi(\gamma) = \nu$ (the normalized Haar measure if we identify $\mathcal{P}(N)$ with $\{0, 1\}^N$).
and \( \{ f \circ x : f \in L_1(\mathcal{P}(N), \Sigma, \nu) \} = L_1(\Gamma, \Sigma, \gamma) \) where \( \Sigma, \nu \) (resp. \( \gamma, \Gamma \)) is the family of all \( \nu \) (resp. \( \gamma \))-measurable subsets of \( \mathcal{P}(N) \) (resp. \( \Gamma \)). Further, consider a function \( \tau : \mathcal{P}(N) \to I \) defined by \( \tau(D) = \Sigma\{1/2^m : m \in D\} \) for every \( D \in \mathcal{P}(N) \). Then \( \tau \) is a continuous surjection such that \( \tau(\nu) = \lambda \) and \( \{ u \circ \tau : u \in L_1(I, \Lambda, \lambda) \} = L_1(\mathcal{P}(N), \Sigma, \nu) \). Then, making use of the lifting theory, we have a weak*-measurable function \( h : I \to \Gamma (\subset K) \) such that

\[
(\alpha) \quad \rho(x \circ h)(t) = (x, h(t)) \quad \text{for every } x \in X \text{ and every } t \in I,
\]

\[
(\beta) \quad \int_E (x, h(t)) d\lambda(t) = \int_{\varphi^{-1}(\tau^{-1}(E))} (x, x^*) d\gamma(x^*)
\]

for every \( E \in \Lambda \) and every \( x \in X \). Here \( \rho \) denotes a lifting on \( L_\infty(I, \Lambda, \lambda) \).

Now we are ready to state our characterization theorem (a localized version of Theorem 8 in [1]). Its main part is that (3) implies (1), whose proof is significant in the point that the characters of the \( K \)-valued function \( h \) and the sequence \( \{ x_n \}_{n \geq 1} \) in \( A \) obtained above are used concretely and effectively. And there, we can get a result that for every \( y \in Y \) and every subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \), \( Dg_H(y, x_{n(k)}) \) does not exist uniformly in \( k \), where \( H = h(I) (\subset K) \).

**Theorem.** Let \( A \) be a bounded subset of \( X \) and \( K \) a weak*-compact (not necessarily convex) subset of \( X^* \). Then the following statements about \( A \) and \( K \) are equivalent.

1. The set \( A \) is \( K \)-weakly precompact.

2. If \( \{ x_n \}_{n \geq 1} \) is a sequence in \( A \) and \( g : Y \to \mathbb{R} \) is a continuous convex function such that \( \partial g(y) \subset \overline{O}^*(j^*(K)) \) for every \( y \in Y \), then there exists a dense \( G_\delta \)-subset \( G \) of \( Y \) and a subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \) such that \( Dg(y, x_{n(k)}) \) exists uniformly in \( k \) for each \( y \in G \).

3. If \( \{ x_n \}_{n \geq 1} \) is a sequence in \( A \) and \( H \) is a non-empty subset of \( K \), then there exists an element \( y \) of \( Y \) and a subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \) such that \( Dg_H(y, x_{n(k)}) \) exists uniformly in \( k \).

**Proof:** (1) \( \Rightarrow \) (2). The proof is analogous to that of the corresponding part of Theorem 8 in [1]. Suppose that (1) holds. Take any sequence \( \{ x_n \}_{n \geq 1} \) in \( A \) and any continuous convex function \( g : Y \to \mathbb{R} \) such that \( \partial g(y) \subset \overline{O}^*(j^*(K)) \) for every \( y \in Y \). As \( A \) is \( K \)-weakly precompact, we have a subsequence \( \{ x_{n(k)} \}_{k \geq 1} \) of \( \{ x_n \}_{n \geq 1} \) such that \( \lim_{k \to \infty}(x_{n(k)}, x^*) \) exists for every \( x^* \in K \). Therefore, by the first preliminary fact preceding Theorem, \( \overline{O}^*(j^*(K)) \) is separably related to \( B (= \{ x_{n(k)} : k \geq 1 \}) \). So it is separably related to \( \text{aco}(B) \) (i.e., the absolutely convex hull of \( B \)). Since \( \partial g(y) \subset \overline{O}^*(j^*(K)) \) for every \( y \in Y \), by the same argument as in Theorem 3.14 and Proposition 3.15 of [2], we have a dense \( G_\delta \)-subset \( G \) of \( Y \) such that \( g \) is \( \text{aco}(B) \)-differentiable (cf. [2]) at every \( y \in G \), whence (2) holds.

(2) \( \Rightarrow \) (3). This follows immediately from the fact that \( \partial g_H(y) \subset \overline{O}^*(j^*(K)) \) for every non-empty subset \( H \) of \( K \) and every \( y \in Y \).
(3) ⇒ (1). The proof of this part is crucial. Suppose that (1) fails. By the second preliminary fact preceding Theorem, we have a function \( h : I \to K \) and a sequence \( \{x_n\}_{n \geq 1} \) in \( A \) as stated above. Take \( H = h(I) \), and let \( \{U(n,k) : n = 0, 1, \ldots; k = 0, \ldots, 2^n - 1\} \) be a system of open intervals in \( I \) given by 
\[
U(n,k) = (k/2^n, (k+1)/2^n) \text{ if } n \geq 0, \quad 0 \leq k \leq 2^n - 1.
\]
Then we get that 
\[
\varphi^{-1}(\tau^{-1}(U(n,2k))) \subset B_{n} \quad \text{and} \quad \varphi^{-1}(\tau^{-1}(U(n,2k+1))) \subset A_{n},
\]
for \( n = 1, 2, \ldots \) and \( k = 0, \ldots, 2^n - 1 \). Further we note a following elementary fact: Let \( E \in \Lambda^+ \) and \( \{n(i)\}_{i \geq 1} \) be a strictly increasing sequence of natural numbers. Then there exists a natural number \( i \) and a non-negative number \( q \) with \( 0 \leq 2q < 2^n(i) - 1 \) such that both \( E \cap U(n(i),2q) \) and \( E \cap U(n(i),2q+1) \) are in \( \Lambda^+ \), which can be easily shown by an argument used in Lemma 2 of [3].

Now, let us show that for every subsequence \( \{x_{n(k)}\}_{k \geq 1} \) of \( \{x_n\}_{n \geq 1} \) and every \( y \in Y \), \( D_{g_H}(y,x_{n(k)}) \) does not exist uniformly in \( k \). To this end, take any point \( y \) in \( Y \) and any subsequence \( \{x_{n(k)}\}_{k \geq 1} \) of \( \{x_n\}_{n \geq 1} \), and set \( y_k = x_{n(k)} \) for every \( k \). Consider a family of weak*-open slices of \( \overline{co}^*(j^*(T_h^*(\Delta(I)))) \) and \( \{S(y,\delta/3i,M) : i \geq 1\} \). Then we have that for every \( i \)
\[
S(y,\delta/3i,M) = \left\{ y^* \in M : (y,y^*) > \sup_{z^* \in M} (y,z^*) - \delta/3i \right\}
\]
\[
= \left\{ y^* \in M : (y,y^*) > \text{ess-sup}_{t \in I} (j(y),h(t)) - \delta/3i \right\}
\]
\[
= \left\{ y^* \in M : (y,y^*) > g_H(y) - \delta/3i \right\},
\]
since \( g_H(y) = \sup_{t \in I} (j(y),h(t)) = \text{ess-sup}_{t \in I} (j(y),h(t)) \) by virtue of (α) above. So, letting \( E_i = \{t \in I : (j(y),h(t)) > g_H(y) - \delta/3i\} \), we easily get that \( E_i \in \Lambda^+ \) and \( j^*(h(E_i)) \subset S(y,\delta/3i,M) \) for every \( i \). Hence, by the elementary fact stated above, there exists a natural number \( k(i) \) and a non-negative number \( q(i) \) with \( 0 \leq 2q(i) < 2^{n(k(i))} - 1 \) such that both \( E_i \cap U(n(k(i)),2q(i)) \) and \( E_i \cap U(n(k(i)),2q(i)+1) \) are in \( \Lambda^+ \). For every \( i \), let \( F_i = E_i \cap U(n(k(i)),2q(i)) \) and \( G_i = E_i \cap U(n(k(i)),2q(i)+1) \), and let \( u_i^* = j^*(T_h^*(\chi_{F_i}/\lambda(F_i))) \) and \( v_i^* = j^*(T_h^*(\chi_{G_i}/\lambda(G_i))) \). Then we have that for every \( i \)
\begin{itemize}
  \item[(a)] \( (y,u_i^*) > g_H(y) - \delta/3i \) and \( (y,v_i^*) > g_H(y) - \delta/3i \),
  \item[(b)] \( (y_k(i),u_i^* - v_i^*) \geq \delta \),
  \item[(c)] \( g_H(y + y_k(i)/i, u_i^*) \) and \( g_H(y - y_k(i)/i, v_i^*) \).
\end{itemize}
Indeed, we have that
\[
(y,u_i^*) = (j(y),T_h^*(\chi_{F_i}/\lambda(F_i)))
\]
\[
= \left\{ \int_{F_i} (j(y),h(t)) \, d\lambda(t) \right\} / \lambda(F_i) > g_H(y) - \delta/3i,
\]
since \( j^*(h(F_i)) \subset S(y,\delta/3i,M) \). Similarly, \( (y,v_i^*) > g_H(y) - \delta/3i \). Thus we have (a). And we can prove (b) as follows. In virtue of (β), we have that for
every $i$

\[
\begin{align*}
(y_{k(i)}, u_i^* - v_i^*) \\
= (j(y_{k(i)}), T_{h*}(x_{F_i}/\lambda(F_i))) - (j(y_{k(i)}), T_{h*}(x_{G_i}/\lambda(G_i))) \\
= (j(x_n(k(i))), T_{h*}(x_{F_i}/\lambda(F_i))) - (j(x_n(k(i))), T_{h*}(x_{G_i}/\lambda(G_i))) \\
= \left\{ \int_{F_i} (j(x_n(k(i))), h(t)) \, d\lambda(t) \right\} / \lambda(F_i) \\
- \left\{ \int_{G_i} (j(x_n(k(i))), h(t)) \, d\lambda(t) \right\} / \lambda(G_i) \\
= \left\{ \int_{\varphi^{-1}(\tau^{-1}(F_i))} (j(x_n(k(i))), x^*) \, d\gamma(x^*) \right\} / \lambda(F_i) \\
- \left\{ \int_{\varphi^{-1}(\tau^{-1}(G_i))} (j(x_n(k(i))), x^*) \, d\gamma(x^*) \right\} / \lambda(G_i) \\
\geq (r + \delta) - r = \delta,
\end{align*}
\]

since $\varphi^{-1}(\tau^{-1}(F_i)) \subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i)))) \subset B_{n(k(i))}$, $\varphi^{-1}(\tau^{-1}(G_i)) \subset \varphi^{-1}(\tau^{-1}(U(n(k(i)), 2q(i) + 1))) \subset A_{n(k(i))}$ and $\tau(\varphi(\gamma)) = \lambda$. As to (c), we have that for every $i$

\[
g_H(y + y_{k(i)} / i) = \sup_{t \in I} (j(y + y_{k(i)} / i), h(t)) \\
\geq \left\{ \int_{F_i} (j(y + y_{k(i)} / i), h(t)) \, d\lambda(t) \right\} / \lambda(F_i) = (y + y_{k(i)} / i, u_i^*).
\]

Similarly, $g_H(y - y_{k(i)} / i) \geq (y - y_{k(i)} / i, v_i^*)$. Then, making use of (a), (b) and (c), we have that for every $i$

\[
\begin{align*}
g_H(y + y_{k(i)} / i) + g_H(y - y_{k(i)} / i) - 2 \cdot g_H(y) \\
\geq (y + y_{k(i)} / i, u_i^*) + (y - y_{k(i)} / i, v_i^*) - \{(y, u_i^* + v_i^*) + 2\delta / 3i\} \\
= (y_{k(i)}, u_i^* - v_i^*) / i - 2\delta / 3i \geq \delta / 3i.
\end{align*}
\]

Consequently, we have that for every $i$

\[
\frac{g_H(y + y_{k(i)} / i) + g_H(y - y_{k(i)} / i) - 2 \cdot g_H(y)}{1/i} > \delta / 3,
\]

which implies that $D_{g_H}(y, x_{n(k)})$ does not exist uniformly in $k$. Thus the proof is complete.

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References


Department of Mathematics, Faculty of Science, Shizuoka University, Ohya, Shizuoka 422–8529, Japan

E-mail: smmmmatsu@ipcs.shizuoka.ac.jp

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