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## A short proof on lifting of projection properties in Riesz spaces

MAREK WÓJTOWICZ

*Abstract.* Let  $L$  be an Archimedean Riesz space with a weak order unit  $u$ . A sufficient condition under which Dedekind  $[\sigma]$ -completeness of the principal ideal  $A_u$  can be lifted to  $L$  is given (Lemma). This yields a concise proof of two theorems of Luxemburg and Zaanen concerning projection properties of  $C(X)$ -spaces. Similar results are obtained for the Riesz spaces  $B_n(T)$ ,  $n = 1, 2, \dots$ , of all functions of the  $n$ th Baire class on a metric space  $T$ .

*Keywords:* Dedekind completeness, spaces of continuous functions, spaces of Baire functions

*Classification:* 46A40, 26A99, 46B30

The purpose of this note is to give a short and concise proof of the following result established by Luxemburg and Zaanen ([3, Theorems 43.2 and 43.3]).

**Theorem.** *Let  $C(X)$  and  $C_b(X)$ , respectively, denote the Riesz spaces of all real continuous and continuous and bounded, respectively, functions on a topological space  $X$ . Then the following conditions are equivalent.*

- (i)  $C(X)$  has the [principal] projection property.
- (ii)  $C(X)$  is Dedekind  $[\sigma]$ -complete.
- (iii)  $C_b(X)$  has the [principal] projection property.
- (iv)  $C_b(X)$  is Dedekind  $[\sigma]$ -complete.

As remarked in ([3, p. 283]), the only nontrivial implication is (iv)  $\Rightarrow$  (ii). Our proof replaces a large part of the direct argument in [3] by an appeal to a lemma (see below), inspired by the classical proof of the Tietze extension theorem ([1, p. 158], the unbounded case).

Let  $S$  be a nonempty set. In the rest of the paper  $L$  denotes a Riesz subspace of the Riesz space  $\mathbb{R}^S$  (pointwise ordering) containing the constant-one on  $S$  function  $e$ , and  $B_e$  denotes the set  $\{f \in L : |f(s)| < 1, s \in S\}$ . It is obvious that  $B_e$  is a (nonlinear) sublattice of  $A_e$ . The symbol  $\circ$  denotes composition of functions.

**Lemma.** *If there exists a strictly increasing and continuous function  $\phi$  from  $\mathbb{R}$  onto  $(-1, 1)$  such that both*

- (a)  $\phi \circ f \in B_e$  for every  $f \in L$ , and
- (b)  $\phi^{-1} \circ g \in L$  for every  $g \in B_e$ ,

then  $L$  and  $B_e$  are order isomorphic as partially ordered sets. In particular, Dedekind  $[\sigma]$ -completeness of  $A_e$  implies Dedekind  $[\sigma]$ -completeness of  $L$ .

**Examples.** 1. If  $L = C(X)$  then every strictly increasing, continuous and onto function  $\phi : \mathbb{R} \rightarrow (-1, 1)$  fulfills both (a) and (b), and the same holds for the Riesz spaces  $B_n(T)$ ,  $n = 1, 2, \dots$ , of all functions  $T \rightarrow \mathbb{R}$  of the  $n$ th class on a metric space  $T$ .

2. If  $L$  consists of all continuous and piecewise functions on  $[0, 1]$ , then  $\phi$  must be piecewise linear to fulfil the condition (a).

PROOF OF LEMMA: By (a) and (b),  $L$  and  $B_e$  are order isomorphic as partially ordered sets (in the sense of the definition given in [3, p.186]) via the mapping  $\hat{\phi}(f) = \phi \circ f$ ,  $f \in L$ . Since, by ([3, Definitions 1.1 and 23.1]), Dedekind  $[\sigma]$ -completeness both is invariant under such isomorphisms and is hereditated from  $A_e$  by  $B_e$ , the result follows.  $\square$

PROOF OF THEOREM (the nontrivial implication (iv)  $\Rightarrow$  (ii)): It follows by Lemma and Example 1.  $\square$

**Remark.** Since bounded functions of the  $n$ th Baire class  $B_n^b(T)$ ,  $n = 1, 2, \dots$ , endowed with the sup-norm form AM-spaces with units ([2, Theorem 12.3.7]), the notions of the [principal] projection property and Dedekind  $[\sigma]$ -completeness coincide (by Theorem). Moreover, Lemma and Example 1 prove that  $B_n^b(T)$  and  $B_n(T)$  are Dedekind  $[\sigma]$ -complete simultaneously. These observations yield the result similar to that of Theorem when  $C(X)$  is replaced by  $B_n(T)$  and  $C_b(X)$  by  $B_n^b(T)$ .

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