Alessandro Fedeli; Attilio Le Donne
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An independency result in connectification theory

ALESSANDRO FEDELI, ATILIO LE DONNE

Abstract. A space is called connectifiable if it can be densely embedded in a connected Hausdorff space.

Let $\psi$ be the following statement: “a perfect $T_3$-space $X$ with no more than $2^c$ clopen subsets is connectifiable if and only if no proper nonempty clopen subset of $X$ is feebly compact”.

In this note we show that neither $\psi$ nor $\neg \psi$ is provable in ZFC.

Keywords: connectifiable, perfect, feebly compact

Classification: 54D25, 54C25, 03E35

The problem of finding those spaces which can be densely embedded in a connected Hausdorff space has been extensively studied in the last years and many results have been obtained (see, e.g., [1], [2], [6], [10] and [13]).

Despite all the efforts, a characterization of connectifiable spaces is still unknown.

In this note we present a characterization of connectifiable perfect $T_3$-spaces with no more than $2^c$ clopen subsets, which can be neither proved nor disproved in ZFC.

We recall that a space $X$ is called:
(i) perfect if every closed subset of $X$ is a $G_\delta$-set;
(ii) $H$-closed if every open cover of $X$ has a finite subfamily whose union is dense, or equivalently, $X$ is a closed subspace of every Hausdorff space in which it is contained;
(iii) feebly compact if every countable open cover of $X$ has a finite subfamily whose union is dense.

As usual, $\mathfrak{p}$ will stand for the smallest cardinality of a maximal subfamily of $[\omega]^\omega$ with the strong finite intersection property (see, e.g., [4] and [12]).

Regarding connectifiability observe that
(1) A connectifiable space contains no proper nonempty open $H$-closed subset ([13]).
(2) Let $X$ be a Hausdorff space with no more than $2^c$ clopen subsets. If every proper nonempty clopen subsets of $X$ is not feebly compact, then $X$ is connectifiable ([10]).
(3) There exists, in ZFC, a nonconnectifiable Hausdorff space of cardinality \( \mathfrak{c} \) with no proper nonempty H-closed subspace ([10]).

(4) It is consistent with ZFC that there is a nonconnectifiable normal Hausdorff space of cardinality \( \mathfrak{c} \) which has no proper nonempty H-closed subspace ([10]).

In our result we will make use of the following set-theoretic statements (which are consistent with ZFC):

(a) \( p > \omega_1 \);

(b) (Jensen’s Combinatorial Principle ♦) There are sets \( A_\alpha \subset \alpha \) for \( \alpha < \omega_1 \) such that for every \( A \subset \omega_1 \) the set \( \{ \alpha < \omega_1 : A \cap \alpha = A_\alpha \} \) is stationary.

We refer the reader to [5] for topological terminology. For set-theoretic terminology see [8] and [4].

**Theorem.** The following statement:

> “a perfect T\(_3\)-space \( X \) with no more than \( 2^\mathfrak{c} \) clopen subsets \( X \) is connectifiable if and only if no proper nonempty clopen subset of \( X \) is feebly compact”

is independent of ZFC.

**Proof:** First let us show that, under \( p > \omega_1 \), the above statement is true.

If \( X \) is not connectifiable then, by one of the above mentioned result, there is a proper nonempty feebly compact clopen subset \( A \) of \( X \).

Now let us suppose that there is a proper nonempty feebly compact clopen subset \( A \) of \( X \). Since \( X \) is T\(_3\) and perfect, it follows that \( A \) is a countably compact perfect T\(_3\)-space. So by a theorem of Weiss (here we use \( p > \omega_1 \) \( A \) is compact ([14], see also [12])

Hence \( X \) is not connectifiable. Therefore the statement is consistent with ZFC.

Now let us prove the independency by showing that, under the Jensen’s principle ♦, there exists a connectifiable T\(_6\)-space with exactly two proper nonempty clopen subsets, each of which is feebly compact.

Let \( S \) be the Ostaszewskii’s space, this space is, under ♦, an example of a noncompact countably compact perfectly normal space ([9]).

Let \( Z \) be the cone over \( S \), i.e., let \( Z \) be the quotient of \( S \times I \) obtained by identifying \( S \times \{ 1 \} \) with a point.

Now \( Z \) is noncompact \( (S \times \{ 0 \} \) is a noncompact space homeomorphic to a closed subspace of \( Z \), countably compact and perfectly normal \( (Z \) is the continuous closed image of the countably compact perfectly normal space \( S \times I \) under the natural mapping). \( Z \) is perfectly normal.

Now let \( X = Z \oplus Z \), \( X \) is a perfectly normal space. The only proper nonempty clopen subsets of \( X \) are the two copies of \( Z \), which are countably compact (= feebly compact) but not compact.

Since a Hausdorff space with open components is connectifiable if and only if it has no proper nonempty open H-closed subspace ([7]), it follows that \( X \) is connectifiable.
Example. It is worth noting that there is a ZFC example of a connectifiable perfect Hausdorff space, with no more than $2^\omega$ clopen subsets, which has proper nonempty feebly compact clopen subsets.

In fact let $\mathcal{F}$ be the set of all free ultrafilters on $\omega$ and let $Y$ be $\omega \cup \mathcal{F}$ endowed with the topology generated by the points of $\omega$ and all sets of the form $G \cup \{p\}$ where $G \in p \in \mathcal{F}$. Now fix $p \in \mathcal{F}$ and let $X$ be the subspace $Y \setminus \{p\}$ of $Y$.

$X$ is a Hausdorff space which is not H-closed (it is not closed in $Y$).

Now let us show that $X$ is feebly compact. By a result in [3] it is enough to show that every locally finite system of pairwise disjoint nonempty open subsets of $X$ is finite.

Suppose that $A = \{A_n : n \in \omega\}$ is an infinite locally finite family of pairwise disjoint nonempty open subsets of $X$. Without loss of generality we may assume that, for every $n$, $A_n = \{\kappa_n\}$ for some $\kappa_n \in \omega$.

Let $q$ be a free open ultrafilter on $\omega$ such that $q \neq p$ and $\{\kappa_n : n \in \omega\} \in q$. Since every neighbourhood of $q$ in $X$ meets infinitely many members of $A$, we reach a contradiction. Therefore $X$ is feebly compact.

Moreover $X$ is perfect. In fact, every open subset $A$ of $X$, is the union of the $F_\sigma$-set $\omega \cap A$ and the closed set $A \setminus \omega$ ($A \setminus \omega$ is a subset of the closed discrete subspace $X \setminus \omega$ of $X$).

Now let $C$ be the cone over $X$ and set $Z = C \oplus C$.

$Z$ is a perfect Hausdorff space, and the only two proper nonempty clopen subsets of $Z$ (namely the copies of $X$) are feebly compact.

Nonetheless $Z$ has open components and no proper nonempty H-closed subspaces, therefore $Z$ is connectifiable.

Remarks. (i) If $L$ is the long line, then $X = L \oplus L$ is a ZFC example of a connectifiable hereditarily normal space of cardinality $\omega_1$ which has proper nonempty feebly compact clopen subsets.

(ii) In [10] it is shown that, under $MA + \neg CH$, a disconnected perfectly normal space with no more than $2^\omega$ clopen subsets is connectifiable if and only if no nonempty clopen subset is relatively pseudocompact.

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**Department of Mathematics, University of L’Aquila, 67100 L’Aquila, Italy**

*E-mail*: fedeli@aquila.infn.it

**Department of Mathematics, University of Rome “La Sapienza” 00100 Rome, Italy**

*E-mail*: ledonne@mat.uniroma1.it

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