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Condensations of Cartesian products

OLEG PAVLOV

Abstract. We consider when one-to-one continuous mappings can improve normality-type and compactness-type properties of topological spaces. In particular, for any Tychonoff non-pseudocompact space $X$ there is a $\mu$ such that $X^\mu$ can be condensed onto a normal ($\sigma$-compact) space if and only if there is no measurable cardinal. For any Tychonoff space $X$ and any cardinal $\nu$ there is a Tychonoff space $M$ which preserves many properties of $X$ and such that any one-to-one continuous image of $M^\mu$, $\mu \leq \nu$, contains a closed copy of $X^\mu$. For any infinite compact space $K$ there is a normal space $X$ such that $X \times K$ cannot be mapped one-to-one onto a normal space.

Keywords: condensation, one-to-one, compact, measurable

Classification: 54C10, 54A10

0. Introduction

We consider only Tychonoff topological spaces and continuous mappings. A condensation is a one-to-one mapping onto. Throughout the paper $\kappa$ denotes the first Ulam-measurable cardinal, if such a cardinal exists.

It is well-known that many key topological properties are not multiplicative. However, for many examples of a given property $\mathcal{P}$ and a space $(X, \tau)$ which has $\mathcal{P}$, but $X^2$ does not, there is a weaker topology $\tau'$ on $X$ such that the square of $(X, \tau')$ does have $\mathcal{P}$. In fact, many examples are produced starting with the space $(X, \tau')$. This observation motivated A.V. Arhangel'skii to raise the following questions. Is it true that for any Lindelöf space $X$ there is a condensation $f : X \to Z$ such that $Z^2$ is Lindelöf (see [1])? Is it true that the second power of any normal (hereditarily normal, paracompact, Lindelöf, pseudocompact, countably compact, etc.) space can be condensed onto a space with the same property? Can any power of a Lindelöf space be condensed onto a Lindelöf space ([1])? Is it true that $\mathbb{Q}^\mu$ can be condensed onto a Lindelöf (compact) space for any infinite $\mu$? These questions are in line with the most general problem concerning condensations: when can a space from class $\mathcal{A}$ be condensed onto a space from $\mathcal{B}$?, for some $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{B}$ is “better” than $\mathcal{A}$ in some sense.

R. Buzyakova answered several of these questions negatively. She constructed a normal countably compact space in [3] and a Lindelöf space in [4], whose squares cannot be condensed onto a normal space (A.N. Yakivchik constructed earlier in [10] a Hausdorff non-regular finally compact space whose square cannot be condensed onto a Hausdorff finally compact space). We generalize these results
in Corollary 1: for any space $X$ and a cardinal $\nu$ there is a larger space $M$ which preserves many properties of $X$ and contains many clopen copies of $X$ in such a way, that for any $\mu \leq \nu$ and for each condensation $f : M^\mu \to Z$, $Z$ contains a closed copy of $X^\mu$. Thus, condensations cannot improve most non-multiplicative properties of arbitrary large (but a priori fixed) powers. If also all powers of $X$ are $\tau$-compact for some $\tau$, then there is an $M$ such that for any $\mu$, $f(M^\mu)$ contains a closed copy of $X^\mu$.

E.G. Pytkeev proved in [9] that any separable metrizable non $\sigma$-compact Borel space can be condensed onto $I^\omega$. Since $Q^\omega$ is Borel (as a one-to-one continuous image of $N^\omega$, see [8]) and not $\sigma$-compact ($N^\omega$ is closed in $Q^\omega$), $Q^\omega$ can be condensed onto $I^\omega$. Therefore $Q^\mu$ can be condensed onto $I^\mu$ for any infinite $\mu$. This solves one of the mentioned questions. It turns out that a somewhat similar result holds for most Lindelöf spaces. We show in Theorem 1 that for any non pseudocompact $X$ with $|X| < \kappa$, $X^\mu$ can be condensed onto a $\sigma$-compact space for many $\mu < \kappa$. On the contrary, if $\kappa$ does exist, then no power of some non-pseudocompact spaces (of cardinality $\geq \kappa$) can be condensed onto a normal space (Corollary 3).

1. Condensation onto a $\sigma$-compact space

**Theorem 1.** Let $X$ be a non-pseudocompact Tychonoff space and let $|X|$ be non Ulam-measurable. Let $|X| \leq \mu_0 < \kappa$ and for each $k \in \omega$, $\mu_{k+1} = exp(\mu_k)$ and $\mu = sup\{\mu_k : k \in \omega\}$. Then $X^\mu$ can be condensed onto a regular $\sigma$-compact space.

**Proof:** Let $\alpha_0 = |\beta X|$ and for any $k \in \omega$, $\alpha_{k+1} = exp(\alpha_k)$. Then for $\alpha = sup\{\alpha_n : n \in \omega\}$, $\alpha = \mu$. Let $f \in C(X, [0, \infty))$ be such that for each $i \in \omega$ there is $b_i \in f^{-1}(i + 0.5)$. Let $K = \beta X$, $K = \{x \in K : f$ can be extended on $X \cup \{x\}\}$ and let $\tilde{f}$ be an extension of $f$ on $\tilde{K}$. We denote $K = \tilde{K} \times \prod\{K_\gamma : 1 \leq \gamma < \alpha\}$ and $\mathcal{X} = \prod\{X_\gamma : \gamma < \alpha\}$, where $K_\gamma$ and $X_\gamma$ are copies of $K$ and $X$ respectively. Then $K$ is a $T_1$ regular $\sigma$-compact space.

For any $i \in \omega$, let $A_i = \{a_{ij} \in \omega : a_{i0} = i\}$ be an increasing sequence such that for $i \neq j$, $A_i^+ \cap A_j^+ = 0$ where $A_i^+ = A_i \setminus \{a_{i0}\}$. By induction, a mapping $\phi : \omega \to \omega$ can be defined such that

1. if $i \notin \cup\{A_j^+ : i \in \omega\}$, then $\phi(i) = 0$, and
2. if $j \geq 1$, then $\phi(a_{ij}) = \phi(i) + j + 1$.

Let $C_0 = \tilde{f}^{-1}([0; 1])$ and for $i \in \omega$, $C_{i+1} = \tilde{f}^{-1}((i + 1/2; i + 2)) \setminus C_i$; $C_i = C_i \times \prod\{K_\gamma : 1 \leq \gamma < \alpha\}$. 


For \( i, j \in \omega, j \geq 1 \), let \( F_{ij,0} = b_{aij} \times \prod \{K_\gamma : 1 \leq \gamma \leq \alpha_i(a_{ij})\} \), and for \( 1 \leq \Delta < \alpha \), 
\[
F_{ij,\Delta} = \prod \{K_\gamma : \alpha_i(a_{ij}) \cdot \Delta < \gamma \leq \alpha_i(a_{ij}) \cdot (\Delta + 1)\} \tag{4.24}
\]
(here we use a product of ordinals, see [7]). Then \( b_{aij} \times \prod \{K_\gamma : 1 \leq \gamma < \alpha\} = \prod \{F_{ij,\Delta} : \Delta < \alpha\}. \)

For any \( i, j \in \omega, j \geq 1 \) and \( \Delta \geq 1 \) we denote \( M_{ij,0} = b_{aij} \times \prod \{X_\gamma : 1 \leq \gamma \leq \alpha_i(a_{ij})\} \) and \( M_{ij,\Delta} = \prod \{X_\gamma : \alpha_i(a_{ij}) \cdot \Delta < \gamma \leq \alpha_i(a_{ij}) \cdot (\Delta + 1)\} \). Then \( M_{ij,0} \subset F_{ij,0} \) and \( M_{ij,\Delta} \subset F_{ij,\Delta}. \) Each \( M_{ij,\Delta}, \Delta \geq 0 \), contains a closed discrete subset \( H_{ij,\Delta} \) of cardinality \( \alpha_i(a_{ij})-1 \) which is also \( C^* \)-embedded in \( F_{ij,\Delta}. \) Indeed, \( M_{ij,0} \approx M_{ij,0} \times M_{ij,0}. \)

The first factor contains a closed discrete subset of cardinality \( \alpha_i(a_{ij})-1 \) by a theorem from [6] (since \( M_{ij,0} \) is a \( \alpha_i(a_{ij}) \)-power of a non countably compact space \( X \)). The second factor contains a \( C^* \)-embedded subset of the same cardinality. The diagonal product of these subsets is a required set \( H_{ij,\Delta}. \) Let us denote \( \tilde{H}_{ij,\Delta} = \overline{F_{ij,\Delta}} \). For each \( \tau, C_{i|\leq \tau} \) denotes projection of \( C \) onto ordinals not greater than \( \tau \).

If \( i \in \omega, k \geq 1 \) and \( \phi(i) = 0 \), let 
\[
C_{i0} = C_{i|\leq \alpha_0} \setminus \prod \{X_\gamma : \gamma \leq \alpha_0\},
\]
and 
\[
C_{ik} = \{x \in (C_{i|\leq \alpha_k} \setminus \prod \{X_\gamma : \gamma \leq \alpha_k\}) : x|_{\alpha_{k-1}} = \prod \{X_\gamma : \gamma \leq \alpha_{k-1}\}\}.
\]

If \( n, k \geq 1 \) and \( i = a_{jn} \), let 
\[
C_{i0} = C_{i|\leq \alpha_0(i)} \setminus (\prod \{X_\gamma : \gamma \leq \alpha_0(i)\}) \cup \tilde{H}_{jn,0},
\]
and 
\[
C_{ik} = \{x \in (C_{i|\leq \alpha_0(i)+k} \setminus \prod \{\tilde{H}_{jn,\Delta} : \Delta < \alpha\}) : x|_{\alpha_{(\delta(i)+k)}}, \text{ and } x|_{\alpha_{(\delta(i)+k-1)}} \in \prod \{X_\gamma : \gamma \leq \alpha_0(i)+k-1\}\}.
\]

Then for every \( i, j \in \omega, |C_{ij}| = \exp(\alpha_0(i)+j) = \alpha_0(i)+j+1. \) Let also \( C_{ik} = C_{ik} \times \prod \{K_\gamma : \alpha_0(i)+k < \gamma < \alpha\}. \) Therefore, if \( \phi(i) = 0 \), then \( \{C_{ik} : k \in \omega\} \) is a partition of \( C_{i|\leq \alpha_0(i)+k} \). If \( \phi(i) \neq 0 \) and \( i = a_{jn} \), then \( \{C_{ik} : k \in \omega\} \) is a partition of \( C_{i|\leq \alpha_0(i)+k} \). For \( i, j \in \omega, j \geq 1 \), let \( \psi_{ij,0} \) be a one-to-one mapping of \( H_{ij,0} \) onto \( C_{i(j-1)} \). Such a mapping exists since \( |H_{ij,0}| = \alpha_i(a_{ij})-1 = \alpha_i(\phi(i)+j+1)-1 = \alpha_0(i)+j = |C_{i(j-1)}| \). This mapping can be extended to a continuous mapping \( \tilde{\psi}_{ij,0} : H_{ij,0} \rightarrow \overline{C_{i(j-1)}} \)
\[
\prod \{K_\gamma : 1 \leq \gamma \leq \alpha_i(\phi(i)+j-1)\}.
\]
In the same way for \( i, j \in \omega, j \geq 1 \) and \( 1 \leq \Delta < \alpha \) there is a one-to-one continuous mapping \( \tilde{\psi}_{ij,\Delta} \) of \( H_{ij,\Delta} \) onto \( F_{ij(j-1),\Delta}. \) This mapping can be extended to a continuous mapping \( \tilde{\psi}_{ij,\Delta} : H_{ij,\Delta} \rightarrow F_{ij(j-1),\Delta}. \) For any \( i, j \in \omega, j \geq 1 \), let \( \tilde{\psi}_{ij} = \prod \{\tilde{\psi}_{ij,\Delta} : \Delta < \alpha\} \rightarrow \overline{C_i} \) and \( \psi_{ij} = \tilde{\psi}_{ij}|\chi. \) It then follows that \( \tilde{\psi}_{ij} \) is a mapping “onto” and that \( \psi_{ij} \) is a condensation of \( \prod \{H_{ij,\Delta} : \Delta < \alpha\} \) onto \( C_{i(j-1)} \).
For $i, j \in \omega$, $j \geq 1$, let $D_{ij} = \text{Dom}(\tilde{\psi}_{ij})$, then $\tilde{\psi}_{ij}$ induces an upper semicontinuous decomposition $E_{ij}$ of $D_{ij}$ since $D_{ij}$ is compact. We define a decomposition $E$ of $\mathcal{K}$ as follows:

1. if $x \notin \bigcup\{D_{ij} : i, j \in \omega, j \geq 1\}$, then $xEy \leftrightarrow x = y$;
2. if $j_0 \geq 1$ and $x \in D_{i_0j_0}$, then $xEy$ if and only if $y \in D_{i_0j_0}$ and $xE_{i_0j_0}y$.

This decomposition is well defined and it is upper semicontinuous since $\{D_{ij} \subset \mathcal{K} : i, j \in \omega, j \geq 1\}$ is a locally finite family of disjoint closed subsets of $\mathcal{K}$. Then the quotient mapping $q : \mathcal{K} \to \mathcal{K}' = \mathcal{K}/E$ is closed, therefore $\mathcal{K}'$ is a $T_1$ regular $\sigma$-compact space. For $i \in \omega$, let $D_{i0} = \overline{C}_i$, $D_i = \bigcup\{D_{ij} : j \in \omega\}$, $\mathcal{K}_i = \bigcup\{D_j : j \leq i\}$ and $G_i = \bigcup\{\overline{C}_j : j \leq i\}$. By a theorem from [2] the space $\mathcal{K}$ is an inductive limit of its closed subsets $\mathcal{K}_i$ and also of the compacta $G_i$. The same is true for the space $\mathcal{K}'$ and sets $\mathcal{K}'_i = q(\mathcal{K}_i)$ and $G'_i = q(G_i)$ since $q$ is a quotient mapping. Let $D'_i = q(D_i)$, $D'_{ij} = q(D_{ij})$ and $\mathcal{X}' = q(\mathcal{X})$.

We claim that $q_{|\mathcal{X}'}$ is a condensation. To see this, note that from the definition of the decomposition $E$ it is sufficient to prove that $q_{|D_{ij} \cap \mathcal{X}'}$ is a condensation. But this is obvious since $E_{ij}$ is generated by a mapping $\tilde{\psi}_{ij}$ whose restriction $\psi_{ij}$ is a condensation. In general, $\mathcal{X}'$ is not a $\sigma$-compact space. The desired condensation of $\mathcal{X}'$ onto a $\sigma$-compact space will be a restriction $q_{|\mathcal{X}'}$ of a quotient map $g : \mathcal{K}' \to g(\mathcal{K}')$ which we define at the end of the proof. $g$ will be the limit of maps $g_i$, $i \in \omega$, which are defined below, in the sense of Lemma 1. It will be constructed in such a way that $g(\mathcal{X}') = g(\mathcal{K}')$ which ensures that $g(\mathcal{X}')$ is $\sigma$-compact. In the next paragraph we introduce an auxiliary notation which will be used in the definition of maps $g_i$.

Let $H$ be a closed subset of some topological space $M$, and let $h$ be a quotient mapping of $H$. Then $h$ induces a decomposition $E_H$ of $H$ and an associate decomposition $E_M$ of $M$ by the rules: if $x \notin H$, then $xEMy \leftrightarrow x = y$; if $x \in H$, then $xEMy \leftrightarrow y \in H$ and $xEHy$. The decomposition $E_M$ defines a quotient mapping of $M$, which we will denote by $h_{H,M}$. It is clear that if $h$ is closed then so is $h_{H,M}$, that $h_{H,M|M\setminus H}$ is a homeomorphism, and that $h_{H,M}(M \setminus H) \cap h_{H,M}(H) = \emptyset$.

Let us define quotient mappings $g_{-1}$, $g_{-1,0}$ and $g_i$, $g_{i,i+1}$ as follows:

1. $g_{-1} \equiv id_{\mathcal{X}'}$;
2. if $g_{i-1}$ is already defined, then $g_{i-1,i} = g_{i-1,i}g_{i-1}(D'_i),g_{i-1}(\mathcal{K}')$ and $g_i = g_{i-1,i} \circ g_{i-1}$;
3. let $g_{i-1,i}|g_{i-1}(D')$ be a quotient mapping corresponding to decomposition $E'_i$ of the space $g_{i-1}(D')$, where for $y \in \overline{C}_i$, $E'_i(g_{i-1}q(y)) = \{g_{i-1}(q(y))\} \cup \{g_{i-1}(q(X)) : there is j \geq 1, x \in D_{i,j} and \tilde{\psi}_{i,j}(x) = y\}.$

The following are the properties of the mappings $g_{i-1}, g_{i-1,i}$ for $i \in \omega$:

a) $g_{i-1}(\mathcal{K})$ is a $T_1$ normal space;

b) every compact $g_{i-1}(D'_{\omega,n})$ ($n \in \omega$) has a neighborhood $U_{i,n}$ in $g_{i-1}(\mathcal{K}')$ such that $\{U_{i,n} : n \in \omega\}$ is a discrete family in $g_{i-1}(\mathcal{K})$;
(c) \( g_{i-1}(D') \) is closed in \( g_{i-1}(\mathcal{K}') \);
(d) for any \( i, j \in \omega \), \( g_{i-1}|_{D'_i} \) is a homeomorphism;
(e) \( g_{i-1}|_{D'_i} \) is a homeomorphism in a closed subset of \( g_{i-1}(\mathcal{K}') \);
(f) \( B_{i-1} = g_{i-1}(\mathcal{K}') \) is compact for \( i > 0 \);
(g) \( g_{i-1,i}|_{B_{i-1}} \) is a homeomorphism for \( i > 0 \).

First, let us check properties (a)–(g) for \( i = 0 \). (a) holds trivially. The family
\( \{U_0 \subset \mathcal{K} : n \in \omega \} \), where \( U_{00} = q(\tilde{f}^{-1}[0; \frac{4}{3}]) \) and \( U_{0i} = q(\tilde{f}^{-1}(b_{a0j} - \frac{1}{3}; b_{a0j} + \frac{1}{3})) \) for \( i \geq 1 \) satisfies (b). (c) follows from (b) and the fact that \( D'_0 = \bigoplus \{D'_{0,n} : n \in \omega \} \) and each \( D'_{0,n} \) is compact. (d) holds trivially, (e) follows directly from (b)–(d).

Now let mappings \( g_k, g_{k-1,k} \) be constructed for all \( k \leq i-1 \) and satisfy properties (a)–(e).

**Lemma 1.** Let a \( T_1 \) normal space \( M \) be an inductive limit of an increasing sequence of its closed subsets \( M_n \), where \( n \in \omega \). Let \( \{h_{n,n+1} : n \in \omega \} \) be a family of quotient mappings such that \( \text{Dom}(h_{0,1}) = M \), \( \text{Dom}(h_{n,n+1,2}) = \text{Ran}(h_{n,n+1}) \) and \( h_{n+1} = h_{n,n+1} \circ \ldots \circ h_{0,1} \). Let \( \mathcal{M} \) be an equivalence relation on \( M \) such that \( x\mathcal{M}y \Leftrightarrow h_k(x) = h_k(y) \) for some \( n \in \omega \). Let also for \( n \in \omega \) sets \( B_n = h_n(M_n) \) be normal and closed subsets of \( h_n(M) \) and \( h_{n,n+1}|_{B_n} \) be a homeomorphism onto a closed subset of \( B_{n+1} \). Then the image \( H/\mathcal{M} \) of a natural quotient mapping \( h \) of \( M \) is a \( T_1 \) normal space.

**Proof of Lemma 1:** For any \( x \in M \), \( h^{-1}(h(x)) = \bigcup \{h_n^{-1}(h_n(x)) : n \in \omega \} \). For each \( i \in \omega \), \( h_{n+i}^{-1}(h_{n+i}(x)) \cap M_n = h_n^{-1}(h_n(x)) \cap M_n \), therefore \( h^{-1}(h(x)) \cap M_n = h_n^{-1}(h_n(x)) \cap M_n \). The latter set is closed in \( M_n \), hence \( h^{-1}(h(x)) \) is closed in \( M \) and \( M/\mathcal{M} \) is a \( T_1 \) space.

Let \( F, G \) be disjoint closed subsets of \( M \) such that \( h^{-1}(h(F)) = F, h^{-1}(h(G)) = G \). Let \( O_0 \) and \( U_0 \) be functionally disjoint in \( B_0 \) neighborhoods of \( h_0(F) \) and \( h_0(G) \) respectively. The sets \( V_0 = h_0^{-1}(O_0) \cap M_0 \) and \( W_0 = h^{-1}(U_0) \cap M_0 \) satisfy the following conditions for \( n = 0 \):

1. \( h_0^{-1}(h_0(V_0)) \cap M_n = V_n, h_0^{-1}(h_0(W_0)) \cap M_n = W_n \);
2. \( F_n \subset V_n \) and \( G_n \subset W_n \) where \( F_n = F \cap M_n \) and \( G_n = G \cap M_n \);
3. \( \overline{h_n(V_n)B_n} \cap \overline{h_n(W_n)B_n} = \emptyset \);
4. \( V_n \supset V_{n-1} \) and \( W_n \supset W_{n-1} \) for all \( n \geq 1 \).

Let \( V_n, W_n \) be constructed for all \( n < k, k \geq 1 \), and satisfy (1)–(4). By (3) \( h_{k-1,k}(\overline{h_{k-1}(V_{k-1}B_{k-1})}) \cap h_{k-1,k}(\overline{h_{k-1}(W_{k-1}B_{k-1})}) = \emptyset \). From the definition of \( F \) and \( G \) and by (1), (2) \( h_{k-1,k}(\overline{h_{k-1}(V_{k-1}B_{k-1})}) \cap h_k(G) = \emptyset \) and \( h_k(F) \cap h_{k-1,k}(\overline{h_{k-1}(W_{k-1}B_{k-1})}) = \emptyset \), then \( h_k(V_{k-1}U_F)^B \cap h_k(W_{k-1}G_{k-1}B_k) = \emptyset \), and these sets have functionally disjoint in \( B_k \) neighborhoods \( O_k \) and \( U_k \) respectively. Let \( V_k = h_k^{-1}(O_k) \cap M_k, W_k = h_k^{-1}(U_k) \cap M_k \). \( V_k \) and \( W_k \) satisfy (1)–(4) for \( n = k \), therefore the construction of \( V_n, W_n \) can be carried out for all \( n \in \omega \).
Now let $V = \cup \{ V_k : k \in \omega \}$ and $W = \cup \{ W_k : k \in \omega \}$. $V$ and $W$ are open in $M$ since $M$ is an inductive limit of $M_n$. By (1) $h^{-1}(h(V)) = V$ and $h^{-1}(h(W)) = W$.

by (2) $F \subset V$ and $G \subset W$. Lemma 1 is proved. \[ \Box \]

Let $M = g_{i-1}(K')$ and $M_n = g_{i-1}(G_n)$. Let $h_n$ be a natural quotient mapping for the decomposition $\mathcal{M}_n$ of the space $g_{i-1}(K')$, where for $x \in M_n$, $x\mathcal{M}_n y \Leftrightarrow xE_i' y$ and for $x \notin M_n$, $x\mathcal{M}_n y \Rightarrow x = y$. Since any element of $\mathcal{M}_n$ is a subset of some element of $\mathcal{M}_{n+1}$, the composition mapping $h_{n-1,n} = h_n \circ h_{n-1}^{-1}$ also is a quotient mapping. $M = g_{i-1}(K')$ is an inductive limit of compacta $M_n$ since $K'$ is an inductive limit of compacta $G'_n$ and $g_{i-1}$ is a quotient mapping.

Since $\mathcal{M}_n\big|_{M_n} \equiv \mathcal{M}_{n+1}\big|_{M_n}$, $h_{n,n+1} : h_n(M_n)$ is a homeomorphism for any $n \in \omega$. All conditions of the lemma are satisfied, therefore $h$ maps $M$ onto a normal space $\mathcal{M}/M_n \equiv E_i'/M_n$, $n \in \omega$. $\cup \{ M_n : n \in \omega \} = M = g_{i-1}(K')$ and $M = Dom(\mathcal{M})$, $g_{i-1}(K') = Dom(E_i')$, thus $\mathcal{M} \equiv E_i'$ and the quotient mappings $H$ and $g_i$ (which are generated by $\mathcal{M}$ and $E_i'$) coincide. Therefore $g_i(K')$ is a $T_1$ normal space. Let us prove properties (b)–(e). For $U_{i_0} = g_i(q(\tilde{f}^{-1}[0; i + \frac{1}{4}]))$ and $U_{i_j} = g_i(q(\tilde{f}^{-1}(b_{a_{i,j}} - \frac{1}{3}; b_{a_{i,j}} + \frac{1}{3})))$ for $j \geq 1$, the family $\{ U_{i_n} : n \in \omega \}$ satisfies (b). Equality $D_{i+1} = \cup \{ D_{i+1,n} : n \in \omega \}$ and (c) follow from (b) and the fact that each subset $D_{i+1,n}$ is compact, and therefore $g_i(D_{i+1,n})$ is closed in $g_i(K')$. Each $D_{j,n}$ is compact and $E_i|D_{j,n}$ is a trivial decomposition into singletons, therefore (d) is true. (e) follows from (b)–(d).

Therefore, $g_{i-1,i}$ and $g_i$ can be constructed for all $i \in \omega$ and satisfy (a)–(e). Let us prove (f) and (g) for $i \geq 1$. $B_i = g_i(K) = g_i(G_i')$, hence $B_i$ is compact. Map $g_{i,i+1}$ is defined by the decomposition $E_i'|_{i+1}$, $E_i'|_{i+1}|B_i$, which is a decomposition into singletons, therefore $g_{i,i+1}|B_i$ is a homeomorphism.

Now let $M = K'$, $h_n = g_{n,n+1} = g_{n,n+1}$ and $M_n = D_n$ for $n \in \omega$. Conditions of the lemma follows from (f), (g). The resulting mapping $g$ is defined by the decomposition $E_i'$ of $K$: $xE_i'y \Leftrightarrow g_i(x) = g_i(y)$ for some $i \in \omega$, and $g$ maps $K'$ onto a $T_1$ regular $\sigma$-compact space.

The conclusion of Theorem 1 follows from the following properties:

(h) $B_i \subset g_i(\mathcal{X}')$;

(k) $g_i|\mathcal{X}'$ is a condensation.

Assume the contrary to (h). Then there is the minimal $i_0 \in \omega$ such that for some $x \in C_{i_0} \big| \mathcal{X}$, $g_{i_0}(q(x)) \notin g_i(x')$. If $i_0 = a_{i_0,k_0}$ and $x \in H_{j_0,k_0}$, then $\tilde{\psi}_{i_0,k_0}(x) \in C_{i_0}$, $j_0 < i_0$ and by the assumption $g_{i_0}(q(x)) \notin g_{i_0}(q(C_{j_0} \big| \mathcal{X}' \big)) \subset g_{i_0}(x')$. That contradicts the minimality of $i_0$. If $x \notin \cup \{ H_{j,k} : j < j_0, k \in \omega \}$, then $x \in C_{i_0,j_0}$ for some $j_0 \in \omega$. Since $\psi_{i_0,j_0}$ maps $H_{i_0,j_0}$ onto $C_{i_0,j_0}$ and from the definition of $E_{i_0}$, $g_{i_0}(q(x)) \subset g_{i_0}(q(H_{i_0,j_0})) \subset g_{i_0}(x')$ and (h) is proved.

Suppose it is proved that $g_i|\mathcal{X}'$ is a condensation for all $i < k$, $k \in \omega$. Since $g_k = g_{k-1,k} \circ g_{k-1}$, it is sufficient to prove that $g_{k-1,k}|g_{k-1}(\mathcal{X}')$ is a condensation.
By (d) $g_k|D'_{kj}$ is a homeomorphism for any $i \in \omega$. It is sufficient to prove that for any $j_0, j_1 \in \omega$, $0 < j_0 < j_1$, and $x_0 \in D_{k,j_0} \cap \mathcal{X}$, $x_1 \in D_{k,j_1} \cap \mathcal{X}$ and $y \in D_0 \cap \mathcal{X}$ the following inequalities hold: $g_k(q(x_0)) \neq g_k(q(x_1)) \neq g_k(q(y)) \neq g_k(q(x_0))$. $\psi_{k,j_0}(x_0) \in \mathcal{C}_{k,j_0-1}$, $\psi_{k,j_1}(x_1) \in \mathcal{C}_{k,j_1-1}$, therefore $g_k(q(x_0)) \neq g_k(q(x_1))$ since $\mathcal{C}_{k,j_0-1} \cap \mathcal{C}_{k,j_1-1} = \emptyset$. From the definition of $\psi_{ij}$, $\tilde{\psi}_{kj_0}$ maps $D'_{k,j_0} \cap \mathcal{X}$ in $\mathcal{C}_{k,j_0-1} \in D'_{k_0} \setminus \mathcal{X}$ and $\tilde{\psi}_{kj_1}$ maps $D'_{k,j_1} \cap \mathcal{X}$ in $\mathcal{C}_{k,j_1-1} \in D'_{k_1} \setminus \mathcal{X}$. Hence other inequalities also hold. 

A cardinal $\mu$ is called $\tau$-measurable, if there is a $\tau$-centered ultrafilter on $\mu$, so the Ulam-measurable cardinals are exactly those which are $\omega$-centered. The same method allows us to prove the following

**Theorem 2.** Let $\mu_0$ be a non $\tau$-measurable cardinal and for every $k \in \omega$, $\mu_{k+1} = \exp(\mu_k)$ and $\mu = \sup\{\mu_k : k \in \omega\}$. Let $X_0$ be a Tychonoff non-pseudocompact space and $\{X_\alpha : 1 \leq \alpha \leq \mu\}$ be a family of spaces such that $\text{ext}(X_\alpha) \geq \tau$ for $1 \leq \alpha < \tau$ and $|X_\alpha| < \mu$ for $0 \leq \alpha < \mu$. Then $\prod\{X_\alpha : \alpha < \mu\}$ can be condensed onto a regular $\sigma$-compact space.

2. A case of $\tau$-compact spaces

For any cardinal $\tau$, let $\tilde{\tau}$ be the set of all isolated ordinals less than $\tau$. A space $X$ is called $\tau$-compact if each of its subsets of cardinality $\tau$ has a complete accumulation point in $X$. For any space $X$, a compactification $cX$, and cardinals $\tau_1$, $\tau_2$ let $M(X, cX, \tau_1, \tau_2) = ((\tau_1 + 1) \times (\tau_2 + 1) \times cX) \setminus ((\tau_1 \times \tau_2 \times (cX \setminus X)))$. This construction is related to the space $((\tau + 1) \times \beta X) \setminus (\tau \times (\beta X \setminus X))$ for certain $X$ and $\tau$ which was described by R. Buzyakova in [4].

We have shown in Section 1 that for many spaces $X$ there are certain powers $\mu$, which depend on $X$, such that $X^\mu$ can be condensed onto a $\sigma$-compact space. The original space can be as bad as we wish and fail all the properties of $\sigma$-compact spaces. Thus, in that situation condensations can improve topological properties of powers. In this section we prove somewhat reverse result by producing examples of good spaces $M$ whose (small) powers are so bad that they cannot even be improved by condensations. Let $\mu$ be an ordinal, and let $\tau_i$, $i = 1, 2, 3, 4$, be cardinals which depend on $\tau$ and on the size of $X$ as it is stated in Theorem 3. We denote $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ and $M_\nu \approx M$ for $\nu < \mu$. $M$ consists of a compact “skeleton” $K = \{(((\tau_1 + 1) \times (\tau_2 + 1)) \setminus (\tau_1 \times \tau_2)) \bigoplus (((\tau_3 + 1) \times (\tau_4 + 1)) \setminus (\tau_3 \times \tau_4))\} \times cX$ and of many clopen copies of $X$. If $f : M^\mu \to Z$ is a condensation, then $f|_{K^\mu}$ is a homeomorphism since $K^\mu$ is compact. $K^\mu$ is only a part of $M^\mu$, but the copies of $X$ are inserted in $M$ in such a way that this restriction influences the whole map $F$ and we can ultimately find clopen copies $X_\nu$ of $X$ in $M_\nu$ for all $\nu < \mu$ such that $f$ restricted to $\prod\{X_\nu : \nu < \mu\}$ is a homeomorphism onto a closed subset of $Z$. Now suppose that $X^\mu$ is not normal (paracompact, etc.). Then $Z$ is not normal (paracompact, etc.) either. This means that $M^\mu$ cannot be condensed onto a normal (paracompact, etc.) space.
The fact that $M$ is good itself when $X$ is so follows from Lemma 2. Hence $M$ is the desired example.

**Lemma 2.** Let $X$ be a Tychonoff space and let $cX$ be a compactification of $X$. Let $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$ for some cardinals $\tau_i$, $i = 1, 2, 3, 4$. Then $M$ is normal ($\tau$-paracompact, realcompact) iff $X$ is so and $M^\mu$ is pseudocompact iff $X^\mu$ is so.

Let a property $\mathcal{P}$ be invariant of continuous mappings, of inverse perfect mappings and suppose $\mathcal{P}$ is inherited by clopen subsets. Then $M^\mu$ satisfies $\mathcal{P}$ iff so does $X^\mu$. In particular, $l(M^\mu) = \tau$ $(M^\mu$ is $\tau$-initially compact, $\sigma$-compact, $\tau$ is regular and $M^\mu$ is $\tau$-compact, respectively) iff the same is true for $X^\mu$.

**Proof:** $K = \{(\tau_1 + 1) \times \tau_2 \} \bigoplus \{(\tau_3 + 1) \times \tau_4 \}$ is compact and any neighborhood of $\tau$ combination $(\varphi, \gamma)$ of the lemma. Therefore $\alpha = \varphi + \gamma$ and $\alpha$ is so. The fact that $M$ is so follows from Lemma 2. Hence $M^\mu$ is pseudocompact iff so is $X^\mu$.

The space $M/(K \times cX)$ is obtained from $M$ by identifying a closed subset $K \times cX$ to a single point (see [5]). $K \times cX$ is compact, so the corresponding quotient map $q : M \rightarrow M/(K \times cX)$ is perfect. Let $p$ be a restriction of $q$ to $K_1 \times X$, then $p(K_1 \times X) = q(M)$. Let $p_\alpha$, $q_\alpha$ be the $\alpha$-th “copies” of $p$, $q$, $\alpha < \mu$ and $p = \Delta\{p_\alpha : \alpha < \mu\}$, $q = \Delta\{q_\alpha : \alpha < \mu\}$, then $M^\mu = q^{-1}(p((K_1 \times X)^\mu))$. \qed

**Theorem 3.** Let $X^\mu$ be $\tau$-compact and let $\tau, \tau_i$ be regular cardinals, $i = 1, 2, 3, 4$, such that $\tau_1 > \tau_2 > \tau_3 > \tau_4 > \max\{|cX|, \tau\}$. Then for $M = M(X, cX, \tau_1, \tau_2) \bigoplus M(X, cX, \tau_3, \tau_4)$, $Y = M^\mu$ and any condensation $f : Y \rightarrow Z$ there is a closed subset $F$ of $Y$ homeomorphic to $X^\mu$ such that $f|_F$ is a homeomorphism onto a closed subset of $Z$. Also, any continuous function on $f(F)$ that can be extended to a function on $(cX)^\mu$ (when $f(F)$ is naturally embedded in $(cX)^\mu$) can be extended on $Z$. In particular, if $X^\mu$ is pseudocompact and $cX = \beta X$, then $f(F)$ is $C$-embedded in $Z$.

**Proof:** Assume that $cf(\mu) \neq \tau_1, \tau_2$. Let $Y = \prod\{Y_\alpha : \alpha < \mu\}$, where each $Y_\alpha$ is homeomorphic to $M$. We denote $\tilde{Y} = \beta Y$, $\tilde{Z} = \beta Z$; $\tilde{f}$ is a continuous extension of $f$ from $\tilde{Y}$ to $\tilde{Z}$. For any $\alpha < \mu$, let $\varpi_\alpha : Y \rightarrow Y_\alpha$ be a projection and let $\tilde{\varpi}_\alpha$ be its extension from $\tilde{Y}$ onto $\tilde{Y}_\alpha = \beta Y_\alpha$. For $y \in \tilde{Y}_\alpha$ and $i = 1, 2, 3$, $\phi_i(y)$ is a projection onto $(\tau_1 + 1)$, $(\tau_2 + 1)$ or $cX$ respectively if $y \in \overline{M(X, cX, \tau_1, \tau_2)\tilde{Y}_\alpha}$ or onto $(\tau_3 + 1)$, $(\tau_4 + 1)$ or $cX$ respectively if $y \in \overline{M(X, cX, \tau_3, \tau_4)\tilde{Y}_\alpha}$. For $\alpha < \mu$ and $i = 1, 2, 3$, we denote $\psi_{\alpha,i} = \phi_i \circ \tilde{\varpi}_\alpha$ and $\psi_3 = \Delta\{\psi_{\alpha,3} : \alpha < \mu\}$. For any combination $i, j$ of indexes $1, 2, 3$, let $\varphi_{ij} = \phi_i \Delta \phi_j$ and $\psi_{\alpha,ij} = \phi_{ij} \circ \tilde{\varpi}_\alpha$. For $(\alpha, \beta) \in \tau_1 \times \tau_2$, let $Y_{\alpha\beta} = \{y \in \tilde{Y} : \text{if } \psi_{\gamma,3}(y) \in cX \setminus X \text{ for some } \gamma < \mu, \text{ then } \psi_{\gamma,12}(y) = (\alpha, \beta)\}$. If $\gamma < \mu$ then let $Y_{\gamma,12} = \{y \in Y_{\alpha\beta} : \psi_{\gamma,3}(y) \in cX \setminus X\}$. 


Now let $\gamma < \mu$ be fixed. For any $\beta' \in \tilde{\tau}_2$, let $A_{\beta'} = \{y \in Y^\gamma_{\alpha \beta'} : \alpha \in \tilde{\tau}_1$ and there is $y' \in Y^\gamma_{\alpha \beta'} \cup Y$ such that $\psi_{\gamma, 3}(y) \neq \psi_{\gamma, 3}(y')$ and $\tilde{f}(y) = \tilde{f}(y')\}$. Let $\tau' = \max\{\tau, |cX|\}^+$, we claim that $|\{\psi_{\gamma, 1}(A_{\beta'})\}| < \tau'$. For, assume the contrary. Then there is a monotonically increasing mapping $\phi$ from $\tau'$ in $\tilde{\tau}_1$, a point $c \in cX \setminus X$, sets $A = \{y_\delta : \delta < \tau'\}$ and $A' = \{y'_\delta : \delta < \tau'\}$ and a neighborhood $U$ of $c$ in $\tau_2 \times cX$ such that for any $\delta < \tau'$, $y_\delta \in Y^\gamma_{\phi(\delta)\beta'} \cup Y$, $\psi_{\gamma, 23}(y_\delta) = c$, $\psi_{\gamma, 23}(y'_\delta) \notin U$, and $\tilde{f}(y_\delta) = \tilde{f}(y'_\delta)$ (it’s all possible because $\psi_{\gamma, 23}(A_{\beta'}) \subset \{\beta'\} \times cX$ and $\{\beta'\} \times cX$ is open in $\tau_2 \times cX$, so $\psi_{\gamma, 23}(A_{\beta'})$ has a base of cardinality $\leq cX < \tau'$ in $\tau_2 \times cX$). For any $y_\delta \in A$, let $\tilde{y}_\delta$ be such a point from $Y$ that for any $\nu < \mu$, $\pi_\nu(\tilde{y}_\delta) = \tilde{\pi}_\nu(y_\delta)$ if $\tilde{\pi}_\nu(y_\delta) \in Y_\nu$, otherwise let $\psi_{\nu, 23}(\tilde{y}_\delta) = \psi_{\nu, 23}(y_\delta)$ and $\psi_{\nu, 1}(\tilde{y}_\delta) = \psi_{\nu, 1}(y_\delta) + \omega$. Let $A = \{\tilde{y}_\delta : \delta < \tau'\}$. In the same way the set $A' = \{y'_\delta : \delta < \tau'\}$ is defined. The set $\{(\tilde{y}_\delta, y'_\delta) \in Y \times \tilde{Y} : \delta < \tau'\}$ has a complete accumulation point $(a, a')$ in $Y \times \tilde{Y}$ ($Y \times \tilde{Y}$ is $\tau$-compact). From the constructions of $\tilde{A}$ and $\tilde{A}'$ from $A$ and $A'$, $(a, a')$ is also a complete accumulation point of $\{(y_\delta, y'_\delta) \in \tilde{X} \times \tilde{X} : \delta < \tau'\}$, so from the continuity of $f = f(a) = f(a')$. But $\psi_{\gamma, 23}(a) \notin U$, so $a \neq a'$ — contradiction to the fact that $f$ is a condensation. So $|\psi_{\gamma, 1}(A_{\beta'})| \leq \tau \times |cX| < \tau_1$ and, since $\tau_2 < \tau_1$, there is an ordinal $\nu_\gamma < \tau_1$ such that $\psi_{\gamma, 1}(A_{\beta'}) \subset \psi_{\gamma, 1}(A_{\beta'})$ for any $\beta' \in \tilde{\tau}_2$.

In the same way, for any $\gamma < \mu$ and $\alpha' < \tau_1$ there is an ordinal $\beta_{\alpha'} \gamma < \tau_2$ such that $\psi_{\gamma, 2}(A_{\alpha'}) \subset \beta_{\alpha'}^\gamma$, where $A_{\alpha'} = \{y \in Y^\gamma_{\alpha' \beta} : \beta \in \tilde{\tau}_2$ and there is $y' \in Y^\gamma_{\alpha' \beta} \cup Y$ such that $\psi_{\gamma, 3}(y) \neq \psi_{\gamma, 3}(y')$ and $\tilde{f}(y) = f(y')\}.$

Since $cf(\mu) \neq \tau_1$, there is $\tilde{\alpha} < \tau_1$ and $\Gamma_1 \subset \mu$ such that $|\Gamma_1| = \mu$ and for any $\gamma \in \Gamma_1, \nu_\gamma \leq \tilde{\alpha}$. Since also $cf(\mu) \neq \tau_2$, there is $\tilde{\beta} < \tau_2$ and $\Gamma_2 \subset \Gamma_1$ such that $|\Gamma_2| = \mu$ and for any $\gamma \in \Gamma_2, \beta_{\alpha + 1}^\gamma \leq \tilde{\beta}$. Now let $y \in Y$; for any $\gamma \in \Gamma_2$ we define $F_\gamma = (\tilde{\alpha} + 1) \times (\tilde{\beta} + 1) \times X$ and for any $\gamma \in \mu \setminus \Gamma_2, F_\gamma = \pi_\gamma(y)$. The set $F = \prod\{F_\gamma : \gamma \in \mu\}$ is homeomorphic to $X^\mu$ and $f|_F$ is a homeomorphism onto a closed subset $f(F)$ of $Z$. Let $g$ be a continuous function on $(cX)^\mu$ and let $h$ be a map from $\tilde{F}^Y$ onto $(cX)^\mu$ such that $h(y) = \{\psi_{\gamma, 3}(y) : \gamma \in \Gamma_2\}, y \in \tilde{F}^Y$. Then $h \circ f^{-1}|_{f(F)}$ is a natural embedding of $f(F)$ in $X^\mu \subset (cX)^\mu$ by the properties of $f|_F$. Since $\tilde{f}(h^{-1}(x_1)) \cap \tilde{f}(h^{-1}(x_2)) = \emptyset$ for $x_1 \neq x_2, x_1, x_2 \in (cX)^\mu$ by the choice of $F$, $h \circ f^{-1}$ is a continuous function from $\tilde{F}(F)^\tilde{Z}$ onto $(cX)^\mu$. Therefore $g$ can be lifted to a continuous function on $\tilde{F}(F)^\tilde{Z}$ and extended to a function on $\tilde{Z}$.

If $cf(\mu) = \tau_1$ or $cf(\mu) = \tau_2$, all the preceding arguments remain valid if $\tau_1$ and $\tau_2$ are replaced everywhere with $\tau_3$ and $\tau_4$ respectively. 

**Corollary 1. a.** For any Tychonoff space $X$ and any cardinal $\nu$ there is a larger space $M$ which preserves many properties of $X$ listed in Lemma 2 and
such that for any $\mu \leq \nu$ and a condensation $f : M^\mu \to Z$, $Z$ contains a closed subset homeomorphic to $X^\mu$; if $X^\mu$ is pseudocompact, then this subset is also $C$-embedded in $Z$. In particular, $M^\mu$ cannot be condensed onto a normal (Lindelöf, $\sigma$-compact, etc.) space if $X^\mu$ is not normal (Lindelöf, $\sigma$-compact, etc.).

b. If $X$ is countably compact in all powers or if there is a $|X|$-measurable cardinal, then $M$ satisfies the above properties for all $\nu$.

Proof: a. Let $\tau = |\beta X^\nu|^+$ and $\tau_1 = \tau^+; \tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. Clearly, $X^\mu$ is $\tau$-compact for any $\mu \leq \nu$, so $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is a required space.

b. If $X$ is countably compact in all powers, let $\tau = |\beta X|^+$, $\tau_1 = \tau^+$, and for $i = 1, 2, 3$, $\tau_{i+1} = \tau_i^+$. Then $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is as desired. If $\tau$ is the first $|X|$-measurable cardinal, then all powers of $X$ are $\tau$-compact, hence for $\tau_1 = \tau^+$, $\tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$, $M = M(X, \beta X, \tau_1, \tau_2) \bigoplus M(X, \beta X, \tau_3, \tau_4)$ is as required.

Corollary 2. For any infinite compactum $K$ there is a normal space $X$ such that $X \times K$ cannot be condensed onto a normal space.

Proof: Let $Y$ be a Dowker space and $\tau = max\{|\beta Y|, |K|\}^+$, $\tau_1 = \tau^+, \tau_{i+1} = \tau_i^+$, $i = 1, 2, 3$. The space $X = M(Y, \beta Y, \tau_1, \tau_2) \bigoplus M(Y, \beta Y, \tau_3, \tau_4)$ is normal by Lemma 2. $X \times K$ cannot be condensed onto a normal space by Theorem 3 since $X \times K = M(Y \times K, \beta Y \times K, \tau_1, \tau_2) \bigoplus M(Y \times K, \beta Y \times K, \tau_3, \tau_4)$.

From Theorem 1 and Corollary 1 we derive the following

Corollary 3. The following are equivalent:

1. for any Tychonoff non-pseudocompact space $X$ there is $\mu$ such that $X^\mu$ can be condensed onto a normal space;

2. for any Tychonoff non-pseudocompact space $X$ there is $\mu$ such that $X^\mu$ can be condensed onto a regular $\sigma$-compact space;

3. there is no measurable cardinal.

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References


Condensations of Cartesian products


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