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Remarks on fixed points of rotative Lipschitzian mappings

JAROSŁAW GÓRNICKI

Abstract. Let $C$ be a nonempty closed convex subset of a Banach space $E$ and $T : C \to C$ a $k$-Lipschitzian rotative mapping, i.e. such that $\|Tx - Ty\| \leq k \cdot \|x - y\|$ and $\|T^n x - x\| \leq a \cdot \|x - Tx\|$ for some real $k$, $a$ and an integer $n > a$. The paper concerns the existence of a fixed point of $T$ in $p$-uniformly convex Banach spaces, depending on $k$, $a$ and $n = 2, 3$.

Keywords: rotative mappings, fixed points

Classification: 47H09, 47H10

1. Introduction

Many authors discussed the problem concerning the existence of fixed points for different class of mappings defined on nonempty closed convex subsets $C$ of infinite dimensional Banach space $E$ and satisfying some metric conditions. The main problem was connected with establishing some conditions of geometrical nature implying the fixed point property for nonexpansive mappings $T : C \to C$ (i.e. mappings satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$). The usual assumptions are those of uniform convexity and normal structure.

In 1981, Goebel and Koter [6] defined the conditions of rotativeness (see below) and proved the following

**Theorem 1.** If $C$ is a nonempty closed convex subset of a Banach space $E$, then any nonexpansive rotative mapping $T : C \to C$ has a fixed point. \( \square \)

Note that this result does not require weak compactness or even boundedness of $C$, or any special geometric structure on $C$.

Further on, the authors studied the existence of fixed points for some class of $k$-Lipschitzian ($k > 1$) and rotative mappings in Banach spaces ([7], [13]).

In this note we extend Goebel and Koter’s results for a real $p$-uniformly convex Banach space and give an estimate for the function $\gamma_3$ in a Hilbert space.

2. Preliminaries

Let $C$ be a nonempty closed convex subset of a Banach space $E$. A mapping $T : C \to C$ is called $(n, a)$-rotative if there exists an integer $n \geq 2$ and a real number $0 \leq a < n$ such that for any $x \in C$, $\|x - T^n x\| \leq a \cdot \|x - Tx\|$.
The simplest examples of rotative mappings are contractions and rotation of the Euclidean space $\mathbb{R}^n$ or any periodic nonexpansive mappings (i.e. $T^n = I$ for some $n \in \mathbb{N}$, where $I$ means identity mapping) in any Banach space.

**Definition 1.** Denote by $\Phi(n, a, k, C)$ the class of all mappings $T : C \to C$ which are $(n, a)$-rotative and satisfy the following condition

$$\forall x, y \in C \quad \|Tx - Ty\| \leq k \cdot \|x - y\|.$$ 

A mapping $T \in \Phi(n, a, k, C)$ is said to be $k$-Lipschitzian $(n, a)$-rotative on $C$.

We shall now consider mappings of the family $\Phi(n, a, k, C)$ with $k > 1$. For fixed $n \in \mathbb{N}$ put

$$\gamma_n(a) = \inf \left\{ k > 1 : \text{there exists a set } C \text{ (closed convex) and} \right.$$ 

$$\left. \quad \text{a mapping } T \text{ such that } T \in \Phi(n, a, k, C) \right.$$ 

$$\left. \quad \text{and } F(T) = \emptyset \right\}$$

($F(T)$ denotes the set of all fixed points of $T$).

The definition of $\gamma_n(a)$ implies that for an arbitrary set $C$, if $T \in \Phi(n, a, k, C)$ and $k < \gamma_n(a)$, then $T$ has at least one fixed point. It was proved in [7] that for an arbitrary Banach space $E$ and for any $n \in \mathbb{N}$, we have $\gamma_n(a) > 1$ for all $a < n$. It is a qualitative result which raises a number of technical yet attractive questions concerning the precise values of $\gamma_n(a)$. Even the exact value of $\gamma_n(0)$ is of interest since it characterizes the fixed point behavior of mappings of period $n$ (see [11], [16] and [4], [8], [9], [10] for involutions, i.e. mappings $T$ for which $T^2 = I$).

**3. About the function $\gamma_2(a)$**

Now, we restrict our attention to the case $n = 2$. It was proved in [5] that for an arbitrary Banach space $E$

$$\gamma_2(a) \geq \gamma_B(a), \quad a \in [0, 2),$$

where

$$\gamma_B(a) = \max \left\{ \frac{1}{2} \left[ 2 - a + \sqrt{(2 - a)^2 + a^2} \right], \right.$$ 

$$\frac{1}{8} \left[ a^2 + 4 + \sqrt{(a^2 + 4)^2 - 64 \cdot (a - 1)} \right] \right\}.$$
Surprisingly, it is possible to show that the first term provides a better estimate if \( a \leq 2(\sqrt{2} - 1) \approx 0.828 \), while the second is better for \( a \in [2(\sqrt{2} - 1), 2) \).

No upper bound for \( \gamma_2(a) \) with \( a \in [0, 1] \) is known until now, while if \( a \in (1, 2) \) we have \( \gamma_2(a) \leq \frac{k_R(a+1)}{a-1} \), where \( k_R \) is the minimal Lipschitz constant of the retraction of the unit ball onto the unit sphere in \( E \) (see Example 1 in [13]). In general, the value of \( k_R \) is unknown, so that the bound given above shows only that \( \gamma_2(a) < +\infty \) for \( a \in (1, 2) \). It is however essential that this fact is true in an arbitrary Banach space. In \( C[0, 1] \) or \( L^1[0, 1] \), we have \( \gamma_2(a) \leq \frac{1}{a-1} \), \( a \in (1, 2) \) (see Examples 1, 2 in [7] and Example 17.2 in [5]).

These results are illustrated in Figure 1.

![Figure 1](image)

Denote

\[
D_1 = \{(a, k) \in [0, 2) \times [0, +\infty) : k < \gamma_2(a)\};
\]

\[
D_2 = \{(a, k) \in (1, 2) \times (1, +\infty) : k \geq \frac{k_R(a+1)}{a-1}\};
\]

\[
D_3 = \{(a, k) \in (1, 2) \times (1, +\infty) : k \geq \frac{1}{a-1}\};
\]

\[
D_4 = [0, 2) \times [0, +\infty) \setminus (D_1 \cup D_3).
\]

If \( T \) is \( k \)-Lipschitzian and \( (2, a) \)-rotative, where \( (a, k) \in D_1 \), then \( T \) has at least one fixed point. In other words: the graph of the function \( \gamma_2 \) for an arbitrary
space $E$ lies above the region $D_1$. On the other hand, it lies always below the curve which is the lower bound of the region $D_2$ (in some spaces even below the lower bound of $D_3$). The existence of fixed points for mappings $T \in \Phi (2, a, k, C)$, where $(a, k) \in D_4$, remains an open problem.

However, in some spaces one can slightly raise the lower bound of the region $D_4$. Koter [13] proved the following theorem (in spaces with known modulus of convexity, see [5]).

**Theorem 2.** Let $C$ be a nonempty closed convex subset of a Banach space $E$ with the modulus of convexity $\delta_E$. If $T \in \Phi (2, a, k, C)$ and

$$1 - \delta_E (2/k) \leq \frac{2 - a}{k},$$

then $T$ has at least one fixed point. \hfill $\Box$

Since in the space $L^p$ (or $\ell^p$), $p \in (2, +\infty)$, we have $\delta_p (\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$, routine calculations and the previous estimates (1) yield

**Corollary 1.** Let $C$ be a nonempty closed convex subset of the space $L^p$ (or $\ell^p$), $2 < p < +\infty$. If $T \in \Phi (2, a, k, C)$ and

$$k < \max \left\{ \gamma_B (a), [(2 - a)^p + 1]^{1/p} \right\}, \quad a \in [0, 2),$$

then $T$ has at least one fixed point. \hfill $\Box$

Hence, in the space $L^p$ (or $\ell^p$), $2 < p < +\infty$, we have

$$\gamma_2 (a) \geq \max \left\{ \gamma_B (a), [(2 - a)^p + 1]^{1/p} \right\}, \quad a \in [0, 2).$$

Komorowski [12] shows that for a real Hilbert space $H$ we have a better bound for $\gamma_2$, namely

$$\gamma_2 (a) \geq \sqrt{\frac{5}{a^2 + 1}} = \gamma_H (a), \quad a \in [0, 2)$$

(see Figure 2).

**4. The function $\gamma_2$ in $p$-uniformly convex spaces**

In this section we give some estimates of the function $\gamma_2$ by means of inequalities in Banach spaces.

Let $p > 1$ and denote by $\lambda$ a number in $[0, 1]$ and by $W_p (\lambda)$ the function $\lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda)$.

The functional $\| \cdot \|_p$ is said to be *uniformly convex* ([22]) on the Banach space if

\[(*) \text{ there exists a positive constant } c_p \text{ such that for all } \lambda \in [0, 1] \text{ and } x, y \in E \text{ the following inequality holds:} \]

$$\| \lambda \cdot x + (1 - \lambda) \cdot y \|_p \leq \lambda \cdot \| x \|_p + (1 - \lambda) \cdot \| y \|_p - c_p \cdot W_p (\lambda) \cdot \| x - y \|_p.$$
Xu [12] proved that the functional $\| \cdot \|^p$ is uniformly convex on the whole Banach space $E$ if and only if $E$ is $p$-uniformly convex, i.e. there exists constant $c > 0$ such that the modulus of convexity (see [5]) $\delta_E(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$. We note that a Hilbert space $H$ is 2-uniformly convex (indeed $\delta_H(\varepsilon) = 1 - \sqrt{1 - (\varepsilon/2)^2} \geq (1/8) \cdot \varepsilon^2$) and $L^p$ (or $\ell^p$) $(1 < p < +\infty)$ is max(2, $p$)-uniformly convex.

**Theorem 3.** Let $E$ be a Banach space with the norm satisfying $(\ast)$ for some $p > 1$, let $C$ be a nonempty closed convex subset of $E$. If $T \in \Phi(2,a,k,C)$ and

$$k < \max \left\{ 1, \left[ \frac{1 + 2^p}{2^{p-2} \cdot (1 + a^p)} \right]^{1/p} \right\} \quad \text{if} \quad c_p = 1,$$

or

$$k < \max \left\{ 1, \left[ \frac{c_p + 2^p}{2^{p-2} \cdot (2 - c_p)(1 + a^p)} \right]^{1/p}, \right.$$

$$\left. \left[ \frac{\sqrt{[2^{p-1} \cdot (1 + a^p)]^2 + 8 \cdot (1 - c_p) \cdot (2^p + c_p) - 2^{p-1} \cdot (1 + a^p)^2}}{2 \cdot (1 - c_p)} \right]^{1/p} \right\} \quad \text{if} \quad 0 < c_p < 1 \quad \text{and} \quad a \in [0, 2),$$

then $T$ has at least one fixed point.
PROOF: If $k < 1$, then the Banach Contraction Principle implies that $T$ has a fixed point. Thus we assume that $k \geq 1$. Let $x$ be an arbitrary point in the set $C$ and $\varepsilon$ an arbitrary real positive number. Suppose that

$$\|T^2x - Tx\|^p > (1 - \varepsilon) \cdot \|x - Tx\|^p$$

and put $z = (1/2)(Tx + T^2x)$. Then we have

$$\|z - Tz\|^p = \|(1/2) \cdot (Tx + T^2x) - Tz\|^p$$

$$= \|(1/2) \cdot (Tx - Tz) + (1/2) \cdot (T^2x - Tz)\|^p$$

$$\leq (1/2) \cdot \|Tx - Tz\|^p + (1/2) \cdot \|T^2x - Tz\|^p$$

$$- c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p$$

$$\leq (1/2) \cdot k^p \|1/2 \cdot (x - Tx) + (1/2) \cdot (x - T^2x)\|^p$$

$$+ (1/2) \cdot k^p \cdot \|(1/2) \cdot (Tx - T^2x)\|^p - c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p$$

$$\leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p\} \cdot \|x - Tx\|^p$$

$$+ (1/2)^{p+1} \cdot k^p \cdot (1 - c_p) \cdot \|T^2x - Tx\|^p - c_p \cdot (1/2)^p \cdot \|T^2x - Tx\|^p.$$  

If $c_p = 1$, then by last inequality we have

$$\|z - Tz\|^p \leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p\} \cdot \|x - Tx\|^p$$

$$- (1/2)^p \cdot \|T^2x - Tx\|^p$$

$$\leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p - (1/2)^p \cdot (1 - \varepsilon)\} \cdot \|x - Tx\|^p$$

$$= f(\varepsilon) \cdot \|x - Tx\|^p.$$  

Now, assume $0 < c_p < 1$.

**Case I.** By the estimate

$$\|T^2x - Tx\|^p \leq \left(\|T^2x - x\| + \|x - Tx\|\right)^p$$

$$\leq 2^{p-1} \cdot \left(\|T^2x - x\|^p + \|x - Tx\|^p\right)$$

$$\leq 2^{p-1} \cdot (a^p + 1) \|x - Tx\|^p,$$

we have

$$\|z - Tz\|^p \leq \{(1/4) \cdot k^p + (1/4) \cdot k^p \cdot a^p$$

$$+ (1/2)^{p+1} \cdot k^p \cdot (1 - c_p) \cdot 2^{p-1} \cdot (a^p + 1)$$

$$- (1/2)^p \cdot c_p (1 - \varepsilon)\} \cdot \|x - Tx\|^p$$

$$= g(\varepsilon) \cdot \|x - Tx\|^p.$$
Case II. By the estimate
\[ \|T^2 x - T x\|^p \leq k^p \cdot \|T x - x\|^p \]
we have
\[
\|z - Tz\|^p \leq \left\{ \left(1/4\right) \cdot k^p + \left(1/4\right) \cdot k^p \cdot a^p + (1/2)^{p+1} \cdot k^{2p} \cdot (1 - c_p) \
- (1/2)^p \cdot c_p \cdot (1 - \varepsilon) \right\} \cdot \|x - T x\|^p \\
= h(\varepsilon) \cdot \|x - T x\|^p.
\]

If the assumptions of the theorem are satisfied, then there exists \( \varepsilon > 0 \) such that
\[
\max\{f(\varepsilon), g(\varepsilon), h(\varepsilon)\} < 1,
\]
and we may consider the following sequence
\[
x_1 = x, \\
x_{n+1} = T x_n \text{ if } \|T^2 x_n - T x_n\|^p \leq (1 - \varepsilon) \cdot \|T x_n - x_n\|^p,
\]
or
\[
x_{n+1} = (1/2) (T x_n + T^2 x_n) \text{ if } \|T^2 x_n - T x_n\|^p > (1 - \varepsilon) \cdot \|T x_n - x_n\|^p
\]
for \( n = 1, 2, \ldots \).

Now, we show the convergence of the sequence \( \{x_n\} \). Indeed,
\[
\|T x_{n+1} - x_{n+1}\|^p \leq A \cdot \|T x_n - x_n\|^p, \text{ for } n \in \mathbb{N},
\]
where \( A = \max\{f(\varepsilon), g(\varepsilon), h(\varepsilon), 1 - \varepsilon\} < 1 \). Thus
\[
\|T x_{n+1} - x_{n+1}\|^p \leq A^n \cdot \|T x_1 - x_1\|^p \to 0,
\]
as \( n \to +\infty \), which shows that \( \{x_n\} \) is a Cauchy sequence. Let \( y = \lim_{n \to \infty} x_n \).
Since \( \|T x_{n+1} - x_{n+1}\|^p \to 0 \) as \( n \to +\infty \), we have \( Ty - y = 0 \), and \( Ty = y \).  

5. Applications

Note that in a Hilbert space \( \mathcal{H} \) we have the identity
\[
\|\lambda \cdot x + (1 - \lambda) \cdot y\|^2 = \lambda \cdot \|x\|^2 + (1 - \lambda) \cdot \|y\|^2 - \lambda \cdot (1 - \lambda) \cdot \|x - y\|^2
\]
for all \( x, y \) in \( C \) and \( 0 \leq \lambda \leq 1 \). In this case \( p = 2 \) and \( c_2 = 1 \). Thus by Theorem 3, we have the following corollary.
Corollary 2 ([12]). Let $\mathcal{H}$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(2, a, k, C)$ and

$$k < \sqrt{\frac{5}{a^2 + 1}}, \quad a \in [0, 2),$$

then $T$ has at least one fixed point. \hfill \Box

If $1 < p < 2$, then we have for all $x, y$ in $L^p$ (or $\ell^p$) and $\lambda \in [0, 1]$,

$$\| \lambda \cdot x + (1 - \lambda) \cdot y \|^2 \leq \lambda \cdot \| x \|^2 + (1 - \lambda) \cdot \| y \|^2 - (p - 1) \cdot \lambda \cdot (1 - \lambda) \cdot \| x - y \|^2,$$

(see [20], [14]). Thus by Theorem 3 we have the following estimate for $k$ in $L^p$ (or $\ell^p$) spaces ($1 < p < 2$):

$$k < \max \left\{ 1, \frac{3 + 2}{(1 + a^2)(3 - p)}, \sqrt{\frac{4(1 + a^2)^2 + 8(2 - p)(3 + p) - 2(1 + a^2)}{2(2 - p)}} \right\} = f_p(a), \quad a \in [0, 2).$$

If $p \to 2_+$, then $f_p(a) \to f_2(a) = \gamma_H(a)$. Moreover, $f_p(0) > 2$ for $2 > p > 9/5$.

The case $p = 3/2$ is illustrated by means of computer graphic in Figure 3.
Thus in $L^p$ (or $\ell^p$), $1 < p < 2$, we have the following

**Corollary 3.** Let $C$ be a nonempty closed convex subset of $L^p$ (or $\ell^p$), $1 < p < 2$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max\left\{ \gamma_B(a), \sqrt{\frac{3+2}{(1+a^2)(3-p)}}, \sqrt{\frac{4(1+a^2)^2 + 8(2-p)(3+p)-2(1+a^2)}{2(2-p)}} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point. □

For all $x, y$ in $L^p$ (or $\ell^p$) spaces, $2 < p < +\infty$, and all $\lambda \in [0, 1]$, we have

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^p \leq \lambda \cdot \|x\|^p + (1 - \lambda) \cdot \|y\|^p - c_p \cdot W_p(\lambda) \cdot \|x - y\|^p,$$

where $c_p = (p - 1) \cdot (1 - t_p)^{2-p}$, and $t_p$ is the unique zero of the function $j(x) = -x^{p-1} + (p - 1) \cdot x + (p - 2)$ on the interval $(1, +\infty)$, see for example [18], [14].

By numerical approximation we obtain $c_{2.1} \approx 0.948917$ and the case $p = 2.1$ is illustrated in Figure 4.
Thus by Corollary 1 and Theorem 3 we have

**Corollary 4.** Let $C$ be a nonempty closed convex subset of $L^p$ (or $\ell^p$), $2 < p < +\infty$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max \left\{ \gamma_B(a), \left[ (2 - a)^p + 1 \right]^{1/p}, \left[ \frac{c_p + 2^p}{2^{p-2} \cdot (2 - c_p)(1 + a^p)} \right]^{1/p}, \left[ \frac{\sqrt{2^{p-1} \cdot (1 + a^p) + 8 \cdot (1 - c_p) \cdot (2^p + c_p) - 2^{p-1} \cdot (1 + a^p)}}{2 \cdot (1 - c_p)} \right]^{1/p} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point.

Using the result of Prus, Smarzewski ([17], [19]) we obtain from Theorem 3 a fixed point theorem, for example, for Hardy and Sobolev spaces.

Let $H^p$, $1 < p < +\infty$, denote the Hardy space ([3]) of all functions $x$ analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\Theta})|^p \, d\Theta \right)^{1/p} < +\infty.$$ 

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $W^{r,p}(\Omega)$, $r \geq 0$, $1 < p < +\infty$, the Sobolev space ([1, p. 149]) of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leq k$ equipped with the norm

$$\|x\| = \left( \int_{\Omega} \left| D^\alpha x(\omega) \right|^p \, d\omega \right)^{1/p}.$$ 

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < +\infty$, $q = \max(2, p)$ ([15]), the linear space of all sequences

$$x = \{ x_\alpha \in X_\alpha : \alpha \in \Lambda \}$$

equipped with the norm

$$\|x\| = \left[ \sum_{\alpha \in \Lambda} (\|x_\alpha\|_{p,\alpha})^q \right]^{1/q},$$

where $\| \cdot \|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < +\infty$, $q = \max(2, p)$ and $(S_i, \Sigma_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach space ([2, III.2.10]) of all measurable $L^p$-valued functions $x$ on $S_2$ with the norm

$$\|x\| = \left( \int_{S_2} (\|x(s)\|_{p})^q \, \mu_2(ds) \right)^{1/q}.$$
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These spaces are $q$-uniform convex with $q = \max(2, p)$ ([17], [19]) and the norm in these spaces satisfies

$$\|\lambda \cdot x + (1 - \lambda) \cdot y\|^q \leq \lambda \cdot \|x\|^q + (1 - \lambda) \cdot \|y\|^q - d \cdot W_q(\lambda) \cdot \|x - y\|^q$$

with a constant

$$d = d_p = \frac{p - 1}{8} \quad \text{for} \quad 1 < p \leq 2 \quad \text{and} \quad d = d_p = \frac{1}{p \cdot 2^p} \quad \text{for} \quad 2 < p < +\infty.$$ 

Hence it follows from Theorem 3 the following

**Corollary 5.** Let $C$ be a nonempty closed convex subset of the space $X$, where $X = H^p$ or $X = W^{r,p}(\Omega)$ or $X = L_{q,p}$ or $X = L_q(L_p)$ and $1 < p < +\infty$, $q = \max(2, p)$, $r \geq 0$. If $T \in \Phi(2, a, k, C)$ and

$$k < \max \left\{ \gamma_B(a), \left[ \frac{d_p + 2^q}{2^{q-2} \cdot (2 - d_p)(1 + a^q)} \right]^{1/q}, \right.$$

$$\left. \left[ \frac{\sqrt{2^{q-1} \cdot (1 + a^q) + 8 \cdot (1 - d_p) \cdot (2^q + d_p) - 2^{q-1} \cdot (1 + a^q)}}{2 \cdot (1 - d_p)} \right]^{1/q} \right\}$$

for $a \in [0, 2)$, then $T$ has at least one fixed point. \hfill \square

**6. $\gamma_3$ in a Hilbert space**

We mentioned that the function $\gamma_n$ may have different form in different spaces. Now we want to establish an evaluation of the function $\gamma_3$ in a Hilbert space.

**Theorem 4.** Let $\mathcal{H}$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $\mathcal{H}$. If $T \in \Phi(3, a, k, C)$ and

$$k < \max \left\{ \sqrt{(1/2) \cdot \left[ \sqrt{9a^4 + 2a^2 + 41 - 3 \cdot a^2 + 1} \right]}, \right.$$

$$\sqrt{(1/2) \cdot \left[ \sqrt{(1 + a^2)^2 + 40 - (1 + a^2)} \right]} \right\}, \quad a \in [0, 3),$$

then $T$ has at least one fixed point.

(Note that it is possible to show that the second term provides a better estimate if $\sqrt{2} < a < \sqrt{(1/2)(\sqrt{29} + 7)} \approx 2.48849$.)

**Proof:** Let $x$ be an arbitrary point in the set $C$ and $\varepsilon$ an arbitrary real positive number. Suppose that

$$\|Tx - T^3x\|^2 + \|T^2x - T^3x\|^2 > (1 - \varepsilon) \cdot \|x - Tx\|^2$$
and put
\[ z = (1/3)(T x + T^2 x + T^3 x) = (1/3) \cdot T x + (2/3) \cdot [(1/2)(T^2 x + T^3 x)]. \]

Then we have
\[
\|z - T z\|^2 = \|(1/3) \cdot T x + (2/3) \cdot [(1/2)(T^2 x + T^3 x)] - T z\|^2 \\
= \|(1/3) \cdot (T x - T z) + (2/3) \cdot [(1/2)(T^2 x + T^3 x) - T z]\|^2 \\
= (1/3) \cdot \|T x - T z\|^2 + (2/3) \cdot \|(1/2)(T^2 x + T^3 x) - T z\|^2 \\
- (2/9) \cdot \|T x - (1/2)(T^2 x + T^3 x)\|^2 \\
\leq (1/3) \cdot k^2 \cdot \|x - z\|^2 + (2/3) \cdot \|(1/2)(T^2 x - T z) + (1/2) \cdot (T^3 x - T z)\|^2 \\
- (2/9) \cdot \|(1/2) \cdot (T x - T^2 x) + (1/2) \cdot (T x - T^3 x)\|^2 \\
\leq (1/3) \cdot k^2 \cdot \|x - (1/3) \cdot T x - (2/3) \cdot [(1/2)(T^2 x + T^3 x)]\|^2 \\
+ (2/3) \left\{ (1/2) \cdot k^2 \cdot \|T x - z\|^2 + (1/2) \cdot k^2 \cdot \|T^2 x - z\|^2 \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
- (2/9) \cdot \left\{ (1/2) \cdot \|T x - T^2 x\|^2 + (1/2) \cdot \|T x - T^3 x\|^2 \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
= (1/3) \cdot k^2 \cdot \left\{ (1/3) \cdot \|x - T x\|^2 + (2/3) \cdot \|x - (1/2)(T^2 x - T^3 x)\|^2 \\
- (2/9) \cdot \|T x - (1/2)(T^2 x - T^3 x)\|^2 \right\} \\
+ (2/3) \left\{ (1/2) \cdot k^2 \cdot \|(2/3)[T x - (1/2)(T^2 x + T^3 x)]\|^2 \\
+ (1/2) \cdot k^2 \cdot \|(1/3)(T^2 x - T x) + (2/3)[T^2 x - (1/2)(T^2 x + T^3 x)]\|^2 \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
- (2/9) \cdot \left\{ (1/2) \cdot \|T x - T^2 x\|^2 + (1/2) \cdot \|T x - T^3 x\|^2 \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
= (1/9) \cdot k^2 \cdot \|x - T x\|^2 + (2/9) \cdot k^2 \cdot \left\{ (1/2) \cdot \|x - T^2 x\|^2 \\
+ (1/2) \cdot \|x - T^3 x\|^2 - (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
- (2/27) \cdot k^2 \cdot \|T x - (1/2)(T^2 x - T^3 x)\|^2 \\
+ (4/27) \cdot k^2 \cdot \|T x - (1/2)(T^2 x - T^3 x)\|^2 \\
+ (1/3) \cdot k^2 \cdot \left\{ (1/3) \cdot \|T^2 x - T x\|^2 + (2/3) \cdot \|T^2 x - (1/2)(T^2 x + T^3 x)\|^2 \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
+ (2/9) \cdot \left\{ (1/2) \cdot \|T^2 x - T^3 x\|^2 + (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\} \\
- (1/4) \cdot \|T^2 x - T^3 x\|^2 \right\}.
- (2/9) \cdot \|Tx - (1/2)(T^2x - T^3x)\|^2 \right\} - (1/6) \cdot \|T^2x - T^3x\|^2
\leq (reduction)
\leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \|x - T^2x\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \|x - T^2x\|^2
+ [(1/9) \cdot k^2 - (1/9)] \cdot \|x - T^2x\|^2
- (1/9) \cdot \left\{ \|Tx - T^3x\|^2 + \|T^2x - T^3x\|^2 \right\}.

\textbf{Case I.} By the estimate
\|x - T^2x\|^2 \leq 2 \cdot \left( \|x - T^3x\|^2 + \|T^2x - T^3x\|^2 \right)
\leq 2 \cdot (a^2 + k^2) \cdot \|x - T^2x\|^2,
we have
\|z - Tz\|^2 \leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \|x - T^2x\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \|x - T^2x\|^2
+ [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (a^2 + k^2) \cdot \|x - T^2x\|^2
- (1/9) \cdot \left\{ \|Tx - T^3x\|^2 + \|T^2x - T^3x\|^2 \right\}
\leq \left\{ (1/9) \cdot k^4 + [(3/9) \cdot a^2 - (1/9)] \cdot k^2 - (2/9) \cdot a^2
- (1/9) \cdot (1 - \varepsilon) \right\} \cdot \|x - T^2x\|^2
= G(\varepsilon) \cdot \|x - T^2x\|^2.

\textbf{Case II.} By the estimate
\|x - T^2x\|^2 \leq 2 \cdot \left( \|x - T^2x\|^2 + \|Tx - T^2x\|^2 \right)
\leq 2 \cdot (1 + k^2) \cdot \|x - T^2x\|^2,
we have
\|z - Tz\|^2 \leq [(1/9) \cdot k^4 + (1/9) \cdot k^2] \cdot \|x - T^2x\|^2 + (1/9) \cdot k^2 \cdot a^2 \cdot \|x - T^2x\|^2
+ [(1/9) \cdot k^2 - (1/9)] \cdot 2 \cdot (1 + k^2) \cdot \|x - T^2x\|^2
- (1/9) \cdot \left\{ \|Tx - T^3x\|^2 + \|T^2x - T^3x\|^2 \right\}
\leq \left\{ (1/9) \cdot k^4 + (1/9)(1 + a^2) \cdot k^2 - (1/9) \cdot (1 - \varepsilon) \right\} \cdot \|x - T^2x\|^2
= H(\varepsilon) \cdot \|x - T^2x\|^2.
If the assumptions of the theorem are satisfied, then there exists $\varepsilon > 0$ such that $\max\{G(\varepsilon), H(\varepsilon)\} < 1$, and we may consider the following sequence

$$x_1 = x,$$

$$x_{n+1} = T^2x_n \quad \text{if} \quad \|Tx_n - T^3x_n\|^2 + \|T^2x_n - T^3x_n\|^2 \leq (1 - \varepsilon) \cdot \|x_n - Tx_n\|^2,$$

or

$$x_{n+1} = (1/3)(Tx_n + T^2x_n + T^3x_n) \quad \text{if} \quad \|Tx_n - T^3x_n\|^2 + \|T^2x_n - T^3x_n\|^2 > (1 - \varepsilon) \cdot \|x_n - Tx_n\|^2,$$

$n = 1, 2, \ldots$

It is easy to see that this sequence is convergent. Indeed,

$$\|Tx_{n+1} - x_{n+1}\|^2 \leq A \cdot \|Tx_n - x_n\|^2,$$

for $n \in \mathbb{N}$, where $A = \max\{G(\varepsilon), H(\varepsilon), 1 - \varepsilon\} < 1$. Thus

$$\|Tx_{n+1} - x_{n+1}\|^2 \leq A^n \cdot \|x_1 - x_1\|^2 \to 0$$

as $n \to +\infty$, which proves that $\{x_n\}$ is a Cauchy sequence. Let $y = \lim_{n \to \infty} x_n$. Since $\|Tx_{n+1} - x_{n+1}\|^2 \to 0$ as $n \to +\infty$, we have $\|Ty - y\| = 0$ and $Ty = y$. □

Kirk [11] showed that a mapping $T : C \to C$ ($C$ is a nonempty closed convex bounded subset of a reflexive Banach space with the normal structure) for which $T^n = I$ ($n > 1$) has a fixed point if $\|T^ix - T^iy\| \leq k \cdot \|x - y\|, \, x, y \in C$, $i = 1, 2, \ldots, n - 1$, where $k$ satisfies

$$(n - 1)(n - 2) \cdot k^2 + 2(n - 1) \cdot k < n^2.$$ 

Thus a $k$-Lipschitzian mapping satisfying $T^n = I$ ($n > 1$) has fixed point if

$$(n - 1)(n - 2) \cdot k^{2(n-1)} + 2(n - 1) \cdot k^{n-1} < n^2.$$ 

For $n = 3$, we have the estimate $k < (1/2) \cdot \sqrt{\sqrt{88} - 4} \approx 1.1598$. Linhart [16] showed (in an arbitrary Banach space) that this mapping has a fixed point if

$$\frac{1}{n} \cdot \sum_{i=n-1}^{2n-3} k^i < 1.$$ 

Hence, for $n = 3$ we have the estimate for $k < k_0 \approx 1.174$.

By Theorem 4 a $k$-Lipschitzian involution $T$ of order $n = 3$ in a Hilbert space (i.e. $T \in \Phi(3, 0, k, C)$) has fixed points if $k < \sqrt{(1/2)(\sqrt{41} + 1)} \approx 1.92394$. 
Theorem 5. Let $C$ be a nonempty closed convex bounded subset of a Hilbert space $\mathcal{H}$. If $T : C \to C$ is $k$-Lipschitzian with $k < \sqrt{(1/2)(\sqrt{41}+1)}$ and $\|T^3 x - T^3 y\| \leq \|x - y\|$ for $x, y$ in $C$, then there exists a fixed point of $T$.

Proof: According to Browder-Göhde-Kirk’s fixed point theorem [5] the set $C^* = \{x \in C : x = T^3 x\}$ is nonempty. The strict convexity of $\mathcal{H}$ implies that $C^*$ is convex. Obviously, we have $T(C^*) = C^*$ and $T^3 = I$ on $C^*$. Hence, by Theorem 4, we obtain our result. □

7. Open problems

The main problem of rather technical nature is whether $\gamma_n$ is continuous. Other questions concern the evaluation of $\gamma_n(a)$. The evaluation given in Theorem 3 seem, in my opinion, to be not exact (for example, $k$-Lipschitzian involutions defined on a nonempty closed convex subset of a Hilbert space have a fixed point if $k < (1/2)(\pi + \sqrt{\pi^2 - 4}) \approx 2.78215$, see [13]). We do not even know whether there exist $a \in [0, 1]$ such that $\gamma_2(a) < +\infty$ (in any Banach space), i.e. whether there exist $T \in \Phi(2, a, k, C)$, $0 \leq a \leq 1$, without fixed points. The same question can be stated for the whole interval $[0, 2)$ in the case of a Hilbert space. Analogous questions can be formulated for the function $\gamma_3$.

REFERENCES

Department of Mathematics, Rzeszów Institute of Technology, P.O. Box 85, 35-959 Rzeszów, Poland
E-mail: gornicki@prz.rzeszow.pl

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