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Elliptic boundary value problem  
in Vanishing Mean Oscillation hypothesis  

MARIA ALESSANDRA RAGUSA  

Dedicated to the memory of Professor Filippo Chiarenza  

Abstract. In this note the well-posedness of the Dirichlet problem (1.2) below is proved in the class $H^{1,p}_0(\Omega)$ for all $1 < p < \infty$ and, as a consequence, the Hölder regularity of the solution $u$.

$L$ is an elliptic second order operator with discontinuous coefficients ($VMO$) and the lower order terms belong to suitable Lebesgue spaces.

Keywords: elliptic equations, Morrey spaces  

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1. Introduction  

Let us consider the Dirichlet problem for the equation

\begin{equation}
L u + b_i u x_i - (d_i u)_x_i + cu = (f_j)_x_j
\end{equation}

in an open bounded set $\Omega \subset \mathbb{R}^n, n \geq 3$, where we assume $L$ to be the elliptic second order operator in the divergence form

$L \equiv -\frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial}{\partial x_i} \right)$

with discontinuous coefficients $a_{ij}$ which belong to the Sarason class $VMO$ of the vanishing mean oscillation functions (see [23]). $VMO$ is the subspace of the John-Nirenberg’s space $BMO$ (see [14]) whose elements have norm on the balls vanishing as the radius of the ball approaches zero (see Section 2 for definitions). This hypothesis will be crucial to obtain our results. The lower order terms $b_i, c, d_i$ belong to suitable Lebesgue spaces $L^s(\Omega)$.

The aim of this note is to prove the well-posedness of the following Dirichlet problem

\begin{equation}
\begin{cases}
L u + b_i u x_i - (d_j u)_x_j + cu = (f_j)_x_j & \text{a.e. } x \in \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\end{equation}

in the class of weak solutions $u \in H^{1,p}_0(\Omega)$ for all $1 < p < \infty$. 

Then we extend the result contained in [8] in order to allow operators to have lower order terms.

In our treatment we will always assume the following

**Hypothesis I.**

\[
\begin{align*}
I_1 & \quad a_{ij}(x) \in VMO \cap L^\infty(\mathbb{R}^n) \quad \forall i, j = 1, \ldots, n, \\
I_2 & \quad a_{ij}(x) = a_{ji}(x) \quad \forall i, j = 1, \ldots, n, \text{ a.e. in } \Omega, \\
I_3 & \quad \exists \tau > 0 : \tau^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \tau|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,
\end{align*}
\]

and

\[
b_i, d_i \in L^r(\Omega) \quad \forall i = 1, \ldots, n \text{ with } \begin{cases} r = n & \text{if } 1 < p < n, \\
r > n & \text{if } p = n, \\
r = p & \text{if } p > n,
\end{cases}
\]

\[
c \in L^{\frac{r}{2}}(\Omega) \text{ where } r \text{ is defined as above.}
\]

We also make the following assumption

\[
c - (d_j)_{x_j} \geq c_0 > 0.
\]

We next enunciate the main results of this note, while for the precise meaning of the hypothesis

\[
a_{ij}(x) \in VMO, \quad \forall i, j = 1, \ldots, n
\]

we refer to Section 2.

**Theorem 1.1.** Let \(a_{ij}, b_i, c, d_i\) verify Hypothesis I, \(f \in [L^p(\Omega)]^n, 1 < p < \infty,\) and \(\partial \Omega \in C^{1,1}.\)

Then the Dirichlet problem (1.2) has a unique solution and there exists a constant \(k\) independent on \(u\) and \(f\) such that

\[
\|\nabla u\|_{L^p(\Omega)} \leq k\|f\|_{L^p(\Omega)}.
\]

**Theorem 1.2.** Let \(a_{ij}, b_i, c, d_i\) satisfy Hypothesis I, \(f \in [L^p(\Omega)]^n, p > n\) and \(\partial \Omega \in C^{1,1}.\)

The solution of (1.2) is Hölder regular in \(\overline{\Omega}\) and there exists a constant \(k\) independent on \(u\) and \(f\) such that

\[
\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq k\|f\|_{L^p(\Omega)}.
\]

The hypothesis \(a_{ij} \in VMO\) allows us to extend classical results obtained only for \(p = 2\) with hypothesis \(a_{ij} \in L^\infty\) (see e.g. [15], [12], [17]) to all \(p \in [1, +\infty[.\)

We also observe that the structure of the equation in the divergence form and the non existence of the derivatives of the coefficients \(a_{ij}\) leads us to examine
the weak and not the strong solutions even if there is a certain similarity in the
technique used to study both strong and weak solutions.

During this century the variational approach to the Dirichlet problem for linear
elliptic equations has been object of much research and has been developed by
many authors. Far from being complete we recall the research of Ladyzhenskaya,
Uralt’seva, see [15], and Stampacchia, see [26] and [27]. These authors derived the
Fredholm alternative but their existence and uniqueness results were restricted
by smallness or coercivity conditions.

The Dirichlet problem was also considered by Friedrichs in [10], [11] and Gard-
ing in [13].

Furthermore we wish to mention classical results by Miranda, see [19] and
[18] who deals with the case of strong solutions, with hypotheses \( a_{ij} \in H^{1,n} \),
\( b_i, c \in L^n \), \( d_i = 0 \).

Higher order differentiability theorems for weak solutions were proved by vari-
oun authors including Browder [2], Nirenberg in [21] and [22], Agmon in [1], Lax
in [16], Bers and Schechter in [3] and Friedman in [9].

We also recall the celebrated paper [7] by De Giorgi in which the author studies
local pointwise estimates. The global bound appears in the works of Ladyzhen-
skaya and Uralt’seva [15] and Stampacchia [26], [27] and is an extension of an
earlier version by Stampacchia [24], [25]. A priori bound is due to Trudinger
in [28].

The method used in this paper, following the idea of the papers [4], [5], is
based on explicit representation formulas for the first derivatives. It permits us
to obtain interior and boundary estimates for the solution of the Dirichlet problem
(1.2) (respectively Lemma 3.1 and 3.2). In the interior case the integral operators
appearing in the representation formula are Calderón-Zygmund singular integrals
and singular commutators like those used by Coifman, Rochberg and Weiss in [6].

The boundary estimates are similar because the representation formula ob-
tained using the half space Green function contains the same integral operators
as in the interior case and a second type which are less singular operators.

Finally, both interior and boundary estimates assuring the global regularity for
the first derivatives of a solution of (1.2) are used to prove in Theorem 1.1 the
well-posedness of (1.2). As a consequence the Hölder regularity of \( u \) is proved in
Theorem 1.2.

2. Definitions and preliminary results

**Definition 2.1** (see [14]). We say that a function \( f \in L^1_{loc}(\mathbb{R}^n) \) belongs to the
space \( BMO \) if

\[
\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \equiv \|f\|_* < \infty
\]

where \( B \) is a ball in \( \mathbb{R}^n \) and \( f_B \) is the average \( \frac{1}{|B|} \int_B f(x) \, dx \).
BMO is a Banach space with the norm \( \| f \|_* \) modulo constant functions, see [20].

Let \( f \in BMO \) and \( r > 0 \). We set

\[
\eta(r) = \sup_{\rho \leq r} \frac{1}{|B\rho|} \int_{B\rho} |f(x) - f_{B\rho}| \, dx
\]

where \( B\rho \) is a ball of radius \( \rho \) centered at the point \( x \in \mathbb{R}^n \).

**Definition 2.2.** We say that a function \( f \in BMO \) is in the space \( VMO \) if

\[
\lim_{r \to 0^+} \eta(r) = 0
\]

and we call \( \eta \) the \( VMO \) modulus of the function \( f \).

In the following we denote by \( \eta_{ij} \) the \( VMO \) modulus of \( a_{ij}, \ i,j = 1, \ldots, n \), and let \( \|a\|_* = \sum_{i,j=1}^n \eta_{ij} \).

**Definition 2.3.** Let \( k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \). We say that \( k(x) \) is a Calderón-Zygmund kernel (C-Z kernel) if

1. \( k \in C^\infty(\mathbb{R}^n \setminus \{0\}) \);
2. \( k(x) \) is homogeneous of degree \(-n\);
3. \( \int_{\Sigma} k(x) \, dx = 0 \), where \( \Sigma = \{ x \in \mathbb{R}^n : |x| = 1 \} \).

**Definition 2.4.** We set

\[
\Gamma(x, \zeta) = \frac{1}{n(2 - n)\omega_n \sqrt{\det \{ a_{ij}(x) \}}} \left( \sum_{i,j=1}^n A_{ij}(x) \zeta_i \zeta_j \right)^{(2-n)/2}
\]

for a.a. \( x \) and \( \forall \zeta \in \mathbb{R}^n \setminus \{0\} \), where \( A_{ij}(x) \) stand for the entries of the inverse matrix of the matrix \( \{ a_{ij}(x) \}_{i,j=1}^n \), and \( \omega_n \) is the measure of the unit ball in \( \mathbb{R}^n \). Also we denote

\[
\Gamma_i(x, \zeta) = \frac{\partial}{\partial \zeta_i} \Gamma(x, \zeta), \quad \Gamma_{ij}(x, \zeta) = \frac{\partial}{\partial \zeta_i \partial \zeta_j} \Gamma(x, \zeta).
\]

It is well known that \( \Gamma_{ij}(x, \zeta) \) are Calderón-Zygmund kernels in the \( \zeta \) variable.

**Theorem 2.5** (see [4, Theorem 2.10]). Let \( k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R} \) be such that

1. \( k(x,.) \) is a Calderón-Zygmund kernel for a.a. \( x \in \mathbb{R}^n \);
2. \( \max_{|j| \leq 2n} \left\| \frac{\partial^j}{\partial \zeta^j} k(x, z) \right\|_{L^\infty(\mathbb{R}^n \times \Sigma)} = M < +\infty \).

Let also \( f \in L^p(\mathbb{R}^n), 1 < p < \infty, a \in L^\infty(\mathbb{R}^n) \).

For any \( \varepsilon > 0 \) and \( x \in \mathbb{R}^n \) we set

\[
K_\varepsilon f(x) = \int_{|x-y| > \varepsilon} k(x, x-y) f(y) \, dy,
\]
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\[ C_\varepsilon(a, f)(x) = \int_{|x-y|>\varepsilon} k(x, x-y)(a(x) - a(y))f(y) \, dy. \]

Then, there exist \( Kf, C(a, f) \in L^P(\mathbb{R}^n) \) such that

\[ \lim_{\varepsilon \to 0} \|K_\varepsilon f - Kf\|_{L^P(\mathbb{R}^n)} = 0, \quad \lim_{\varepsilon \to 0} \|C_\varepsilon(a, f) - C(a, f)\|_{L^P(\mathbb{R}^n)} = 0 \]

and there exists a constant \( c = c(n, p, M) \) such that

\[ \|Kf\|_{L^P(\mathbb{R}^n)} \leq c\|f\|_{L^P(\mathbb{R}^n)}, \quad \|C(a, f)\|_{L^P(\mathbb{R}^n)} \leq c\|a\|_*\|f\|_{L^P(\mathbb{R}^n)}. \]

As in [4] the functions \( Kf \) and \( C(a, f) \) obtained by the above limiting process are called \textit{Principal Value} functions and the notations usually used to indicate that \( Kf \) and \( C(a, f) \) are such linear functionals, are

\[ Kf(x) = P.V. \int_{\mathbb{R}^n} k(x, x-y)f(y) \, dy \]

and

\[ C(a, f)(x) = a(Kf) - K(af). \]

The result we are going to mention follows from the above theorem.

**Theorem 2.6** (see [4, Theorem 2.13]). Let \( a \in VMO \cap L^\infty(\mathbb{R}^n) \) and \( k(x, z) \) satisfy the hypothesis of Theorem 2.5. Then for any \( \varepsilon > 0 \), there exists \( \rho_0 > 0 \) such that for any ball \( B_r \) of radius \( r \in \mathbb{R} \), \( \rho_0 \) and \( f \in L^P(B_r) \) with \( 1 < p < \infty \) we have

\[ \|C(a, f)\|_{L^P(B_r)} \leq c\|f\|_{L^P(B_r)}. \]

Let us define \( \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \equiv (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\} \) and for \( x \in \mathbb{R}^n \) let \( \tilde{x} = (x', -x_n) \).

Analogous inequalities are proved in [5], as we recall in the next theorem, for the following operators

\[ \tilde{K}f(x) = \int_{\mathbb{R}^n_+} f(y) \frac{1}{|\tilde{x} - y|^n} \, dy \]

and

\[ \tilde{C}(a, f)(x) = \int_{\mathbb{R}^n_+} \frac{[a(x) - a(y)]}{|\tilde{x} - y|^n} f(y) \, dy, \]

where \( a \in VMO \cap L^\infty(\mathbb{R}^n) \) and \( f \in L^P(\mathbb{R}^n_+), 1 < p < \infty. \)
**Theorem 2.7.** Let \( f \in L^p(\mathbb{R}^n_+) \) with \( 1 < p < \infty \), and let \( \tilde{K} f \) and \( \tilde{C}(a, f)(x) \) be defined as above.

Then there exists a constant \( c \) independent of \( f \) and \( \phi \) such that

\[
\| \tilde{K} f \|_{L^p(\mathbb{R}^n_+)} \leq c \| f \|_{L^p(\mathbb{R}^n_+)}
\]

and

\[
\| \tilde{C}(a, f) \|_{L^p(\mathbb{R}^n_+)} \leq c \| a \|_{*} \| f \|_{L^p(\mathbb{R}^n_+)}.
\]

Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \), be an open bounded domain with \( \partial \Omega \in C^{1,1} \). Consider in \( \Omega \) the elliptic equation (1.1) or, equivalently,

\[
(2.1) \quad \mathcal{L}u = (f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu)
\]

and the associated Dirichlet problem

\[
(2.2) \quad \begin{cases}
\mathcal{L}u = (f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu), \\
u \in H^{1,p}_0(\Omega), \quad 1 < p < \infty.
\end{cases}
\]

In our treatment we assume that \( f = (f_1, \ldots, f_n) \in [L^p(\Omega)]^n \) with \( 1 < p < \infty \).

We shall say that \( u \in H^{1,p}_0(\Omega), 1 < p < \infty \), is a weak solution of the Dirichlet problem (1.2) if

\[
(2.3) \quad \int_{\Omega} (a_{ij} u_{x_i} \phi_{x_j} - b_i u_{x_i} \phi - cu \phi) \, dx = -\int_{\Omega} (f_j + d_j u) \phi_{x_j} \, dx, \quad \forall \phi \in C_0^\infty(\Omega).
\]

### 3. Proofs of Theorems 1.1 and 1.2

Now we shall make some preliminary observations.

Let \( \theta \) be a standard cut-off function, \( \theta \in C_0^\infty(\mathbb{R}) \), such that for fixed \( r \in \mathbb{R} \) and every \( s : 0 < s < r \)

\[
\theta(x) = \begin{cases} 
1 & x \in B_s, \\
0 & x \notin B_r.
\end{cases}
\]

Then if \( u \) is a solution of (1.2) we have

\[
\mathcal{L}(\theta u) = -(a_{ij} \theta (\theta u)_{x_i})_{x_j} = \mathcal{L}(\theta u) - \theta \mathcal{L}u + \theta((f_j + d_j u)_{x_j} - (b_i u_{x_i} + cu))
\]

\[
= - (a_{ij} \theta (x_i u + \theta u_{x_i}))_{x_j} - \theta((f_j + d_j u)_{x_j} - b_i u_{x_i} + cu)
\]

\[
= -(a_{ij} \theta x_i u - \theta(f_j + d_j u))_{x_j} - (a_{ij} \theta x_i u_{x_i} + \theta x_j (f_j + d_j u) + \theta b_i u_{x_i} + c\theta u).
\]

Then we write, for \( v = \theta u \),

\[
(3.1) \quad \mathcal{L}(v) \equiv \mathcal{L}(\theta u) = \text{div}(\Phi) + \Psi
\]
with \( \Phi, \Psi \) supported in \( B_r \) and defined by
\[
\Phi \equiv -\left( a_{ij} \theta x_i u - \theta (f_j + d_j u) \right)
\]
and
\[
\Psi \equiv -\left( a_{ij} \theta x_j u_{x_i} + \theta x_j (f_j + d_j u) + \theta b_i u_{x_i} + c \theta u \right).
\]
In the following we consider only \( p > 2 \) because the case \( p = 2 \) is classical and \( 1 < p < 2 \) will be obtained by duality.

Before proving Theorem 1.1 and Theorem 1.2 we need the following two lemmas.

**Lemma 3.1.** Let \( u \in C^\infty(\Omega) \) such that (2.3) is satisfied, let \( \theta \) and \( v \) be defined as above.

Let also \( a_{ij} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), such that \( I_2 \) and \( I_3 \) of Hypothesis I are true. Let also \( f \in [C^\infty(\Omega)]^n \) and \( b_i, d_i, c \in C^\infty(\Omega) \), for every \( i, j = 1, \ldots, n \).

Then there exist \( r > 0 \) and \( C = C(n, p, \tau, \eta_{ij}, \text{dist}(B_r, \partial \Omega)) \) such that
\[
(3.2) \quad \| \nabla u \|_{L^p(B_s)} \leq C \left( \| \nabla u \|_{L^2(B_r)} + \| f \|_{L^p(B_r)} + \| u \|_{L^p(B_r)} \right)
\]
for every \( s \in [0, r] \).

**Proof:** Let us define
\[
\theta(x) = \begin{cases} 
1 & x \in B_{\rho r}, \quad 0 < \rho < 1, \\
0 & x \notin B_r.
\end{cases}
\]
Set \( \mathcal{L}(v) = \text{div}(\Phi) + \Psi \). If \( a_{ij} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), \( \Phi \in [C^\infty_0(B_r)]^n \), \( \Psi \in C^\infty_0(B_r) \)
we have (see [8]) the representation formula and the consequent estimate based on Theorem 2.5 and Theorem 2.6
\[
v_{x_i}(x) = \text{P.V.} \int_{B_r} \Gamma_{ij}(x, x - y) \{ (a_{kj}(x) - a_{kj}(y)) v_{x_k}(y) - \Phi_j(y) \} dy \\
+ c_{ij} \Phi_j(x) - \int_{B_r} \Psi(y) \Gamma_i(x, x - y) dy, \quad \forall x \in B_r
\]
with
\[
c_{ij} = \int_{|\xi|=1} \Gamma_i(x, \xi) \xi_j d\sigma_\xi,
\]
(3.3) \( \| \nabla v \|_{L^p(B_r)} \leq C \left( \| a \|_\| \nabla v \|_{L^p(B_r)} + \| \Phi \|_{L^p(B_r)} + \| \Psi \|_{L^{p_*}(B_r)} \right) \)
where \( C \geq 0 \) does not depend on \( v, \Phi, \Psi \) and \( p_* \) such that \( \frac{1}{p_*} = \frac{1}{p} + \frac{1}{n} \).

Fixing \( r > 0 \) so small that \( C \| a \|_\| \) is less than 1 it follows
\[
(3.4) \quad \| \nabla v \|_{L^p(B_r)} \leq C \left( \| \Phi \|_{L^p(B_r)} + \| \Psi \|_{L^{p_*}(B_r)} \right).
\]
From (3.4) we have
\[
\|\nabla (\theta u)\|_{L^p(B_r)} \leq C \left( \|a_{ij}\theta x_i u - \theta(f_j + d_j u)\|_{L^p(B_r)} + \|a_{ij}\theta x_j u + \theta x_i (f_j + d_j u) + \theta(b_i u x_i + cu)\|_{L^{p^*}(B_r)} \right)
\]
and then
\[
\|\nabla (\theta u)\|_{L^p(B_r)} \leq C \left( \|a_{ij}\theta x_i u\|_{L^p(B_r)} + \|\theta f_j\|_{L^p(B_r)} + \|\theta d_j u\|_{L^p(B_r)} + \|\theta x_i f_j\|_{L^{p^*}(B_r)} + \|\theta x_i d_j u\|_{L^{p^*}(B_r)} + \|\theta b_i u x_i\|_{L^{p^*}(B_r)} + \|c\theta u\|_{L^{p^*}(B_r)} \right).
\]

Let us suppose at the beginning $2 < p \leq 2^*$ where $2^*$ is such that $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$, then $p^* \leq 2$.

Majorizing each term we have
\[
\|a_{ij}\theta x_i u\|_{L^p(B_r)} \leq C_1 \|u\|_{L^p(B_r)},
\]
\[
\|\theta f_j\|_{L^p(B_r)} + \|\theta d_j u\|_{L^p(B_r)} \leq \|f\|_{L^p(B_r)} + \|d_j\|_{L^r(B_r)} \|\theta u\|_{L^{p^*}(B_r)} \leq \|f\|_{L^p(B_r)} + S \|d_j\|_{L^r(B_r)} \|\nabla (\theta u)\|_{L^p(B_r)}
\]
where $p_S = \frac{pm}{n-p}$ and $S$ is Sobolev constant,
\[
\|a_{ij}\theta x_j u x_i\|_{L^{p^*}(B_r)} \leq C_2 \|\nabla u\|_{L^{p^*}(B_r)} \leq C_2 \|\nabla u\|_{L^2(B_r)}
\]
\[
\|\theta x_i f_j\|_{L^{p^*}(B_r)} \leq C_3 \|f\|_{L^p(B_r)}
\]

and, using Hölder inequality,
\[
\|\theta x_i d_j u\|_{L^{p^*}(B_r)} \leq \|d_j\|_{L^r(B_r)} \|u\|_{L^p(B_r)}.
\]

Moreover,
\[
\|\theta b_i u x_i\|_{L^{p^*}(B_r)} = \|b_i[(\theta u)x_i - \theta x_i u]\|_{L^{p^*}(B_r)} \leq \|b_i(\theta u)x_i\|_{L^{p^*}(B_r)} + \|b_i\theta x_i u\|_{L^{p^*}(B_r)} \leq \|b_i\|_{L^r(B_r)} \|\nabla (\theta u)\|_{L^p(B_r)} + C_4 \|b_i\|_{L^r(B_r)} \|u\|_{L^p(B_r)},
\]
\[
\|c\theta u\|_{L^{p^*}(B_r)} \leq \|c\|_{L^\frac{p}{2}(B_r)} \|\theta u\|_{L^{p^*}(B_r)} \leq S \|c\|_{L^\frac{p}{2}(B_r)} \|\nabla (\theta u)\|_{L^p(B_r)}.
\]

Fix $\tilde{r} > 0$ so small that
\[
\left[ \|b_i\|_{L^r(B_{\tilde{r}})} + S \left( \|d_j\|_{L^r(B_{\tilde{r}})} + \|c\|_{L^\frac{p}{2}(B_{\tilde{r}})} \right) \right] \leq \frac{1}{3C},
\]
then for every $r \in ]0, \tilde{r}[$ we have proved

\[ \| \nabla u \|_{L^p(B_r)} \leq \| \nabla (\theta u) \|_{L^p(B_r)} \]

(3.5)

\[ \leq C \left( \| u \|_{L^p(B_r)} + \| f \|_{L^p(B_r)} + \| \nabla u \|_{L^{p^*}(B_r)} \right), \ \forall s \in ]0, r[ . \]

Let us now prove (3.2) if $2 < p \leq 2^*$, choosing

\[ \theta(x) = \begin{cases} 
1 & x \in B_{pr}, \ 0 < \rho < 1, \\
0 & x \notin B_r. 
\end{cases} \]

From (3.5) we obtain

\[ \| \nabla u \|_{L^p(B_{pr})} \leq C \left( \| u \|_{L^p(B_r)} + \| f \|_{L^p(B_r)} + \| \nabla u \|_{L^{p^*}(B_r)} \right) \]

(3.6)

\[ \leq C \left( \| u \|_{L^p(B_r)} + \| f \|_{L^p(B_r)} + \| \nabla u \|_{L^{2^*}(B_r)} \right) \]

because $p^* \leq 2$, and then we get (3.2) choosing $\rho = \frac{s}{r}$.

Let us define $2^{**}$ such that $\frac{1}{2^{**}} = \frac{1}{2^r} - \frac{1}{n}$.

Set $2^* < p \leq 2^{**}$ (observe that we put formally $2^{**} = \infty$ and take $2^* < p < \infty$ provided $2^* \geq n$); then $p^* \leq 2^*$, and

\[ \theta(x) = \begin{cases} 
1 & x \in B_{p^2 r}, \ 0 < \rho < 1, \\
0 & x \notin B_{pr}. 
\end{cases} \]

Using again (3.4) we have

\[ \| \nabla (\theta u) \|_{L^p(B_{p^2 r})} \leq C \left( \| u \|_{L^p(B_{pr})} + \| f \|_{L^p(B_{pr})} + \| \nabla u \|_{L^{p^*}(B_{pr})} + \| f \|_{L^{p^*}(B_{pr})} \right) \]

\[ \leq C \left( \| u \|_{L^p(B_r)} + \| f \|_{L^p(B_r)} + \| \nabla u \|_{L^{2^*}(B_{pr})} \right) \]

and, majorizing the last term with (3.6) for $p = 2^*$,

\[ \| \nabla u \|_{L^p(B_{p^2 r})} \leq C \left( \| u \|_{L^p(B_r)} + \| f \|_{L^p(B_r)} + \| \nabla u \|_{L^{2^*}(B_{pr})} \right). \]

We obtain again (3.2) choosing $\rho = \left( \frac{s}{r} \right)^{\frac{1}{2^*}}$.

Finally the estimate (3.2) is obtained for every $p > 2$ iterating this method a finite number of times. More precisely it is always possible to get $m \in \mathbb{N}$ such that

$p_{m-1} < p \leq p_m$ with $p_{m-1} = 2^{**} \ldots 2^{**}$, $p_m = 2^{**} \ldots 2^{**}$, then setting $\rho = \left( \frac{s}{r} \right)^{\frac{1}{m}}$ the result is obtained.

The technique used here is similar to that in [8].

Let us define $B^+_r = \{ x = (x_1, \ldots, x_n) \equiv (x', x_n) \in B_r : x_n > 0 \}$.\]
Lemma 3.2. There exists a positive number $r$ such that if

(i) $a_{ij}(x) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\forall i, j = 1, \ldots, n$ such that $I_2$ and $I_3$ are true;
(ii) $u \in C^\infty(B_r^+)$ is a solution of (1.2) in $B_r^+$, $u$ vanishing on $\{x_n = 0\} \cap B_r^+$;
(iii) $f \in [C_0^\infty(B_r^+)]^n$;
(iv) $b_i, c, d_i \in C^\infty(B_r^+)$, $\forall i = 1, \ldots, n$;

then

$$\|\nabla u\|_{L^p(B_r^+)} \leq C \left( \|u\|_{L^p(B_r^+)} + \|f\|_{L^p(B_r^+)} + \|\nabla u\|_{L^2(B_r^+)} \right)$$

where $C = C(n, p, \tau, \eta_{ij}, \text{dist}(B_r^+, \partial \Omega))$.

**Proof:** Let $\theta \in C_0^\infty(B_r^+)$ and let $v = \theta u$ be a solution of (3.1). It is easy to show that the representation formula for the first derivatives of $v$ is

$$v_{x_i}(x) = \text{P.V.} \int_{B_r^+} \Gamma_{ij}(x, x - y) \{(a_{kj}(x) - a_{kj}(y))v_{x_k}(y)$$

$$- (a_{ij}\theta_{x_i}u - \theta(f_j + d_j u))j(y)\} dy + \int_{B_r^+} (a_{ij}\theta_{x_j}u + \theta_{x_j}(f_j + d_j u) + \theta_{x_i}u_{x_i} + c\theta u)(y)\Gamma_{ij}(x, x - y) dy$$

$$+ c_{ij}(a_{ij}\theta_{x_i}u - \theta(f_j + d_j u))j(x) + I_i(x), \quad \forall x \in B_r^+, \quad \forall i = 1, \ldots, n$$

where $c_{ij}$ is defined as above and

$$I_i(x) = \int_{B_r^+} \Gamma_{ij}(x, T(x) - y) \{(a_{kj}(x) - a_{kj}(y))v_{x_k}(y)$$

$$- (a_{ij}\theta_{x_i}u - \theta(f_j + d_j u))j(y)\} dy, \quad \text{for } 1 \leq i < n;$$

$$I_n(x) = \int_{B_r^+} \Gamma_{kj}(x, T(x) - y)A_k(x) \{(a_{kj}(x) - a_{kj}(y))v_{x_h}(y)$$

$$- (a_{ij}\theta_{x_i}u - \theta(f_j + d_j u))j(y)\} dy,$$

where $A(y) = (A_1(y), \ldots, A_n(y)) = T(e_n, y) \equiv T((0, \ldots, 0, 1), y)$ and $T$ is defined by

$$T(x, y) = x - \frac{2x_n}{a_{nn}(y)}a_n(y), \quad T(x) \equiv T(x, x),$$

and $a_n(y) = (a_{in}(y))_{i=1,\ldots,n}$ is the last row (column) of the matrix $a(y) = \{a_{ij}(y)\}_{i,j=1,\ldots,n}$.

We also have, using Theorem 2.7, that there exists a positive number $r > 0$ and a positive constant $\overline{C}$ such that

$$\|\nabla v\|_{L^p(B_r^+)} \leq \overline{C} \left( \|\Phi\|_{L^p(B_r^+)} + \|\Psi\|_{L^{p^*}(B_r^+)} \right),$$
where $C$ is independent of the functions $v$, $\Phi$ and $\Psi$. Then similarly to Lemma 3.1 we get the conclusion. \hfill $\square$

We are now ready to establish the main result of the paper.

**Proof of Theorem 1.1**

We first observe that it is possible to find subsequences $\{(a_{ij})_h\}_{h \in \mathbb{N}}$, $\{(b_i)_h\}_{h \in \mathbb{N}}$, $\{(d_i)_h\}_{h \in \mathbb{N}}$, $\{f_h\}_{h \in \mathbb{N}}$, with $(a_{ij})_h \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $f_h \in [C^\infty(\Omega)]^n$, $(b_i)_h, (c)_h, (d_i)_h \in C^\infty(\Omega)$, $\forall i, j = 1, \ldots, n$, such that $\{(a_{ij})_h\}$ converges in the $\ast-$norm to $a_{ij}$, $\{f_h\}$ converges to $f$ in $[L^p(\Omega)]^n$ and $\{(b_i)_h\}$, $\{(c)_h\}$, $\{(d_i)_h\}$ are respectively converging to $b_i$, $c$, $d_i$ in $L^r(\Omega)$, $\forall i = 1, \ldots, n$.

We first prove the theorem with smooth hypothesis on the coefficients and the known term, then in the second step with the assumption requested.

**FIRST STEP.**

Let $a_{ij} \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $f \in [C^\infty(\Omega)]^n$, $b_i, c, d_i \in C^\infty(\Omega)$, $\forall i, j = 1, \ldots, n$.

From Lemma 3.1 and Lemma 3.2 by a covering and flattering argument (see [5, Theorem 4.2])

\begin{equation}
\|\nabla u\|_{L^p(\Omega)} \leq C \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|\nabla u\|_{L^2(\Omega)}\right), \quad \forall p > 2. \tag{3.7}
\end{equation}

Let $2 < p \leq 2^* (p_* \leq 2)$. From Sobolev theorem

\[ \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}. \]

Then by (3.7) and the well known $L^2$-results obtained by Miranda (see [18])

\begin{equation}
\|\nabla u\|_{L^p(\Omega)} \leq C \left(\|\nabla u\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\right) \leq C \|f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \tag{3.8}
\end{equation}

Let us suppose now $2^* < p \leq 2^{**}$; then it follows $p_* \leq 2^*$. From Sobolev theorem

\begin{equation}
\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^{p_*}(\Omega)} \leq C \|\nabla u\|_{L^{2^*(\Omega)}}. \tag{3.9}
\end{equation}

Applying (3.8) with $p = 2^*$, from (3.9) we have

\[ \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^{2^*(\Omega)}} \leq C \|f\|_{L^{2^*(\Omega)}} \leq C \|f\|_{L^p(\Omega)}. \]

Then using the above inequality, (3.7) and the $L^2$-results mentioned above, we obtain

\[ \|\nabla u\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}; \quad \text{for } p \leq 2^{**}. \]
The last inequality for every \( p > 2 \) can be obtained iterating this method.

SECOND STEP.

Let us consider the above sequences of smooth functions; and \( u_h, \forall h \in \mathbb{N} \), the solution of the associated Dirichlet problem.

Then there exists a constant \( C \) independent of \( h \) such that

\[
\|\nabla u_h\|_{L^p(\Omega)} \leq C\|f_h\|_{L^p(\Omega)}, \quad \forall h \in \mathbb{N}.
\]

Using the above inequality we have that \( \exists u \in H^{1,p}_0(\Omega) \) verifying

\[
\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}
\]

where \( u \) is the solution of (1.2).

This completes the proof of Theorem 1.1 with the constant \( k = k(n, p, \tau, \eta_{ij}, \partial \Omega) \).

□

Proof of Theorem 1.2

It is easy to see that it is a consequence of Theorem 1.1 and of the Sobolev imbedding theorem.

□

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References


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