

Elliot Tonkes

Solutions to a perturbed critical semilinear equation concerning the N -Laplacian in \mathbb{R}^N

Commentationes Mathematicae Universitatis Carolinae, Vol. 40 (1999), No. 4, 679--699

Persistent URL: <http://dml.cz/dmlcz/119123>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1999

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

Solutions to a perturbed critical semilinear equation concerning the N -Laplacian in \mathbb{R}^N

ELLIOT TONKES

Abstract. The aim of this paper is to study the existence of variational solutions to a nonhomogeneous elliptic equation involving the N -Laplacian

$$-\Delta_N u \equiv -\operatorname{div}(|\nabla u|^{N-2}\nabla u) = e(x, u) + h(x) \text{ in } \Omega$$

where $u \in W_0^{1,N}(\mathbb{R}^N)$, Ω is a bounded smooth domain in \mathbb{R}^N , $N \geq 2$, $e(x, u)$ is a critical nonlinearity in the sense of the Trudinger-Moser inequality and $h(x) \in (W_0^{1,N})^*$ is a small perturbation.

Keywords: variational methods, elliptic equations, critical growth

Classification: 35J20, 35J60, 35J65

1. Introduction

Let Ω be a smooth bounded set in \mathbb{R}^N , $N \geq 2$, and consider the problem

$$(1) \quad \begin{aligned} -\Delta_N u &= e(x, u) + h(x) \\ u &\in W_0^{1,N}(\Omega) \end{aligned}$$

where $e(x, u)$ is a critical function in terms of the Trudinger-Moser inequality and $h \in W^{-1,N'}$. Such a nonlinearity $e(x, u)$ possesses the maximal growth in u which permits a variational formulation of problem (1).

Solutions are sought in the Sobolev space $W_0^{1,N}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| \equiv (\int_\Omega |\nabla u|^N)^{\frac{1}{N}}$. The dual space is denoted $W^{-1,N'}$, where N' is the Hölder conjugate of N , and has the associated norm $\|\cdot\|_*$. Denote strong convergence by “ \rightarrow ”, weak convergence by “ \rightharpoonup ” and convergence in the sense of measure (or distributions) as “ \rightharpoonup^* ”. Unless otherwise denoted, integration is performed over the domain Ω . Specific constraints on $e(x, u)$ and $h(x)$ are described later, but we now present the main results:

Theorem 1.1. *Suppose $E(x, u)$ is a function of critical growth satisfying (7) to (11).*

- (i) *There exists $h^* > 0$ such that for each $h(x)$ with $0 < \|h\|_* < h^*$, problem (1) possesses a solution at negative energy.*

- (ii) If $e(x, u)$ further satisfies (12) then there exists a number $h^{**} > 0$, possibly smaller than h^* from (i), such that for each $h(x)$ with $0 < \|h\|_* < h^{**}$, there exists another solution to (1).

Theorem 1.2. *If the conditions of Theorem 1.1 hold and $h(x) \geq 0$ ($h(x) \leq 0$) almost everywhere, then the solutions in (i) and (ii) are nonnegative (nonpositive).*

Weak solutions of (1) correspond to critical points of the functional I :

$$(2) \quad I(u) = \frac{1}{N} \int_{\Omega} |\nabla u|^N dx - \int_{\Omega} E(x, u) dx - \int_{\Omega} hu dx$$

where $E(x, u) = \int_0^u e(x, t) dt$. At this stage we introduce an associated functional

$$I^+(u) = \frac{1}{N} \int |\nabla u|^N - \int E^+(x, u) - \int hu$$

where $E^+(x, u)$ corresponds with $E(x, u)$ when $u \geq 0$, but is otherwise set to zero. Critical points of I^+ correspond to solutions in $W_0^{1,N}(\Omega)$ of

$$(3) \quad -\Delta_N u = e^+(x, u) + h(x).$$

It can be shown ([14]) that both $I^+(u)$ and $I(u) \in C^1(W_0^{1,N}; \mathbb{R})$.

Publication [13] has considered problem (1) with $h(x) \equiv 0$. Much of the geometrical structure captured in this analysis still holds, and this paper includes useful convergence lemmas. The geometry of the functional allows application of the Mountain-Pass theorem of Ambrosetti-Rabinowitz, without the Palais-Smale condition.

In [13], to prove that Palais-Smale sequences expose solutions, a weakly convergent sequence is shown to converge to a nontrivial solution. This method elicits no further information.

In this paper, analogous arguments may be made. For the unperturbed problem, $u = 0$ is a local minimum. For small $\|h\|_*$, we anticipate a local minimum solution near zero, and this is located via a local minimisation technique. A perturbed solution close to the non-trivial solution derived in [13] is also expected. We derive a solution from a mountain pass technique, but the lack of a Palais-Smale condition means that strong convergence is not assured. Indeed, the lack of a (PS) condition resulting from a critical nonlinearity makes it difficult to prove that these two solutions are not identical. To distinguish two solutions, a distinction result is achieved based on the difference in sequence energies and P.L. Lions' theorem.

In a similarly perturbed problem, Deng and Li [7] show the existence of solutions without a Palais-Smale condition, and a distinction between solutions, but do not go so far as to allege strong convergence. The maximum principle is used to show positivity of solutions, but this technique fails for the N -Laplacian case.

The Trudinger-Moser [17], [12] inequality says that

$$(4) \quad \exp\left(\alpha|u|^{\frac{N}{N-1}}\right) \in L^1(\Omega) \quad \forall u \in W_0^{1,N}(\Omega), \quad \forall \alpha > 0$$

$$(5) \quad \sup_{\|u\| \leq 1} \int_{\Omega} \exp\left(\alpha|u|^{\frac{N}{N-1}}\right) \leq C(N) \in \mathbb{R} \quad \text{if } \alpha \leq \alpha_N$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the volume of the $(N - 1)$ dimensional surface of the unit sphere and $C(N)$ is a constant depending only on N .

This result is stronger than the Sobolev embedding, which expresses that $W_0^{1,N}(\Omega) \hookrightarrow L^t(\Omega)$ compactly for all $t \geq 1$, but *not* $L^\infty(\Omega)$.

Carleson and Chang [5] have shown that when Ω is a ball, the extremal function for this inequality is achieved in $W_0^{1,N}(\Omega)$. Recently Lin [10], has extended this result to general domains $\Omega \subset \mathbb{R}^N$. This contrasts with the case of critical functions for the embeddings $W_0^{1,2}$ into the space $L^{\frac{2N}{N-2}}$. The so-called Talenti extremal functions are scale and translation invariant and rely on an unbounded domain.

The Trudinger-Moser inequalities can be improved in a theorem by Lions [11]:

Theorem 1.3. *Let $\{u_n : \|u_n\| = 1\}$ be a sequence in $W_0^{1,N}$ converging weakly to a nonzero function u . Then, for every $p < (1 - \|u\|)^{\frac{1}{N-1}}$ we have*

$$(6) \quad \sup_n \int_{\Omega} \exp\left(p\alpha_N|u_n|^{\frac{N}{N-1}}\right) dx < \infty.$$

Theorem 1.3 improves the Trudinger-Moser inequality (5) by accounting for the possibility of concentration. Suppose $\{u_n\} \subset W_0^{1,N}$, $\|u_n\| = 1$, $u_n \rightharpoonup u_0 \neq 0$. If $u_n \not\rightarrow u_0$ strongly, then $\|u_n\|^N = \|u_0\|^N + \|v_n\|^N + o(1)$ where $v_n \rightharpoonup 0$ but $\lim_{n \rightarrow \infty} \|v_n\| > 0$ contains the concentrations. Consequently, $\|u_0\| < 1$ and $(1 - \|u_0\|)^{\frac{1}{N-1}} > 1$. Thus expression (6) improves (5) by allowing a larger exponent. If $u_n \rightarrow 0$ then the two results correspond. If $u_n \rightarrow u_0$ then $(1 - \|u_0\|)^{\frac{1}{N-1}} = \infty$ and $\lim_{n \rightarrow \infty} \int \exp(p\alpha_N|u_n|^{\frac{N}{N-1}}) < \infty$ for any $p > 0$.

1.1 Assumptions

The assumptions on the nonlinearity $e(x, u)$ will be altered slightly from the version in [13] to accommodate negative solutions. Essentially we impose symmetric constraints on $e(x, u)$. Of course, these can be lifted if we neglect interest in signs of solutions, and a remark to this effect is made later.

Make the following assumptions on $e(x, u)$:

Assume $e(x, u)$ is a critical function with exponent α_0 , so that

$$(7) \quad \begin{aligned} \lim_{|u| \rightarrow \infty} \frac{e(x, u)}{\exp(\alpha|u|^{\frac{N}{N-1}})} &= 0 \text{ for } \alpha > \alpha_0; \\ \lim_{|u| \rightarrow \infty} \frac{|e(x, u)|}{\exp(\alpha|u|^{\frac{N}{N-1}})} &= \infty \text{ for } \alpha < \alpha_0. \end{aligned}$$

Assume the continuity and sign restrictions:

$$(8) \quad e(x, u) \in C(\Omega \times \mathbb{R}; \mathbb{R});$$

$$(9) \quad e(x, u) \geq 0 \text{ on } \Omega \times [0, \infty), \quad e(x, u) \leq 0 \text{ on } \Omega \times (-\infty, 0].$$

Assume there exists $R > 0$ and $M > 0$ such that for all $|u| \geq R$ and $x \in \Omega$

$$(10) \quad 0 < E(x, u) \leq M|e(x, u)|.$$

Further, make the assumption on $E(x, u)$ that

$$(11) \quad \limsup_{u \rightarrow 0} \frac{NE(x, u)}{|u|^N} < \lambda_1$$

where λ_1 is the first eigenvalue of $-\Delta_N u = \lambda|u|^{N-2}u$ characterised by

$$\lambda_1 = \inf \left\{ \int |\nabla u|^N : u \in W_0^{1,N}, \int |u|^N = 1 \right\}.$$

As per [13], define

$$\mathcal{M} = \lim_{n \rightarrow \infty} n \int_0^1 \exp \left[n(t^{\frac{N}{N-1}} - t) \right] dt \geq 2.$$

Denote by d the inner radius of Ω . Introduce the condition that uniformly on Ω ,

$$(12) \quad \lim_{u \rightarrow \infty} ue(x, u) \exp \left(-\alpha_0|u|^{\frac{N}{N-1}} \right) \geq \beta_0 > \left(\frac{N}{d} \right)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}}.$$

Another condition which we shall find useful is that uniformly on Ω ,

$$(13) \quad \lim_{u \rightarrow \pm\infty} ue(x, u) \exp \left(-\alpha_0|u|^{\frac{N}{N-1}} \right) \geq \beta_0 > \left(\frac{N}{d} \right)^N \frac{1}{\mathcal{M}\alpha_0^{N-1}}.$$

In publications such as [1], [2] and [15] restrictions imposed on $e(x, u)$ are of the form

$$\frac{\partial e(x, t)}{\partial t} > \frac{e(x, t)}{t}.$$

Throughout this work, we discard this restriction in favour of the restraints posed in [13]. The definition of a critical function $e(x, u)$ (in (7)) compares $e(x, u)$ with $\exp(\alpha|u|^{\frac{N}{N-1}})$ at infinity when $\alpha < \alpha_0$ and $\alpha > \alpha_0$. Condition (12) fills in the gap by comparing $e(x, u)$ with $\exp(\alpha_0|u|^{\frac{N}{N-1}})$.

1.2 Direct results from assumptions

For a critical function $e(x, u)$, for any $\beta > \alpha_0$, there exists $C > 0$ such that

$$|e(x, u)| \leq C \exp\left(\beta|u|^{\frac{N}{N-1}}\right).$$

There is a $C > 0$ such that for $|u| \geq R$, and all $x \in \Omega$

$$(14) \quad E(x, u) \geq C \exp\left(\frac{1}{M}u\right).$$

There is $R_0 > 0$ and $\theta > N$ such that for $|u| \geq R_0$ and $x \in \Omega$,

$$(15) \quad \theta E(x, u) \leq ue(x, u).$$

From these, we can deduce that for fixed $q > N$, fixed $\lambda < \lambda_1(N)$ and fixed $\beta > \alpha_0$, there is some $C > 0$ such that

$$(16) \quad E(x, u) \leq \frac{1}{N}\lambda|u|^N + C|u|^q \exp\left(\beta|u|^{\frac{N}{N-1}}\right).$$

2. Geometry of the functional

Throughout, we assume that the $E(x, u)$ satisfies (7) to (11). From line to line, constants are denoted C but may assume different values. This section has the two-fold aim of analysing the geometry of I and I^+ .

Lemma 2.1. (i) *There exists a number $h^* > 0$ such that for each $h(x)$ with $\|h\|_* < h^*$, there exists $\rho_h > 0$ such that the functional I satisfies*

$$I(u) > 0 \quad \forall u \in W_0^{1,N}, \quad \|u\| = \rho_h.$$

Furthermore, ρ_h may be chosen such that $\rho_h \rightarrow 0$ as $\|h\|_* \rightarrow 0$.

(ii) *The same result holds for $I^+(u)$.*

PROOF: (i) To develop the mountain ridge we estimate $I(u)$ from below on a ρ -ball in $W_0^{1,N}$. Fix $\lambda < \lambda_1$ and $\beta > \alpha_0$. By implementing estimate (16),

$$\begin{aligned} I(u) &\geq \frac{1}{N} \int |\nabla u|^N - \frac{\lambda}{N} \int |u|^N - C \int \exp\left(\beta|u|^{\frac{N}{N-1}}\right) |u|^q - \|h\|_* \|u\| \\ &\geq \frac{1}{N} \left(1 - \frac{\lambda}{\lambda_1}\right) \int |\nabla u|^N - C \int \exp\left(\beta|u|^{\frac{N}{N-1}}\right) |u|^q - \|h\|_* \|u\|. \end{aligned}$$

Choose any $r > 1$. Then provided that

$$(17) \quad \|u\| < \left(\frac{\alpha_N}{\beta r}\right)^{\frac{N-1}{N}}$$

it follows from the Trudinger-Moser inequality that for $r^{-1} + s^{-1} = 1$,

$$\int \exp\left(\beta|u|^{\frac{N}{N-1}}\right) |u|^q \leq \left[\int \exp\left(\beta r \|u\|^{\frac{N}{N-1}} \left|\frac{u}{\|u\|}\right|^{\frac{N}{N-1}}\right)\right]^{\frac{1}{r}} \left(\int |u|^{sq}\right)^{\frac{1}{s}}$$

$$\leq C \left(\int |u|^{sq}\right)^{\frac{1}{s}}.$$

Hence,

$$I(u) \geq \frac{1}{N} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^N - C \|u\|_{sq}^q - \|h\|_* \|u\|.$$

Use the Sobolev embedding $W_0^{1,N} \hookrightarrow L^t \forall t \geq 1$, so $C \|u\| \geq \|u\|_{sq}$, to reveal

$$I(u) \geq \|u\| \left[\frac{1}{N} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^{N-1} - C \|u\|^{q-1} - \|h\|_* \right].$$

With $\|u\| = \rho$, the functional is estimated from below on a ball

$$(18) \quad I(u) \geq \rho \left[\frac{1}{N} \left(1 - \frac{\lambda}{\lambda_1}\right) \rho^{N-1} - C \rho^{q-1} - \|h\|_* \right].$$

Since $q > N$, it follows that if $\|h\|_*$ is sufficiently small then there exists some $\rho_h > 0$ such that $I(u) > 0$ on $B(0, \rho_h) \subset W_0^{1,N}$. As $\|h\|_*$ becomes smaller, expression (18) permits ρ_h to be chosen commensurately smaller. In particular, ρ_h is sufficiently small to ensure that restriction (17) is fulfilled.

(ii) The proof in the case of I^+ is identical. □

Lemma 2.2. (i) *There exists $\eta > 0$ sufficiently small such that*

$$\inf_{\|u\| \leq \eta} I(u) < 0.$$

Furthermore, there exists $\eta > 0$ and $u \in W_0^{1,N}$ with $\|u\| = 1$ such that $I(tu) < 0$ for all $0 < t < \eta$.

(ii) *The same result holds for $I^+(u)$.*

PROOF: (i) Choose $\tilde{u} \in W_0^{1,2}(\mathbb{R}^N)$ with $\|\tilde{u}\|_{W_0^{1,N}} = 1$ and $\int h\tilde{u} > 0$. For $t > 0$,

$$\frac{d}{dt} I(t\tilde{u}) = t^{N-1} \int |\nabla \tilde{u}|^N - \int e(x, t\tilde{u}) \tilde{u} - \int h\tilde{u}.$$

As $e(x, \cdot)$ is continuous with $e(x, 0) = 0$, it follows that there exists $\eta > 0$ such that $\frac{d}{dt} I(t\tilde{u}) < 0$ for all $t < \eta$. Since $I(0) = 0$, it must hold that $I(t\tilde{u}) < 0$ for all $0 < t < \eta$.

(ii) The method of proof for I^+ is identical. □

Lemma 2.3. (i) *There exists $u_b \in W_0^{1,N}$ with $\|u_b\| > \rho_h$ and*

$$I(u_b) < \inf_{\|u\|=\rho_h} I(u).$$

(ii) *The same result holds for I^+ .*

PROOF: (i) From (14), for $p > N$, there are positive constants C and d such that for all $u \geq 0$,

$$E(x, u) \geq Cu^p - d.$$

For any $u \in W_0^{1,N} \setminus \{0\}$, have

$$I(tu) \leq \frac{1}{N}t^N \int |\nabla u|^N - Ct^p \int |u|^p + t\|h\|_*\|u\| + d.$$

As $t \rightarrow \infty$, $I(tu) \rightarrow -\infty$ and the result follows.

(ii) For a nonlinearity $E^+(x, u)$, equation (14) becomes

$$E^+(x, u) \geq C \exp\left(\frac{1}{M}u\right)$$

for all $u \geq R$ and $x \in \Omega$. The method of proof for I^+ then follows in an identical manner. □

We define the sequence of Moser [12] functions $M_n(x, x_0, r)$ in $W_0^{1,2}(\Omega)$. This often utilised family provides a large value of $\int \exp(\alpha_N |M_n|^{\frac{N}{N-1}})$, while maintaining $\|M_n\| = 1$. In many applications, this particular sequences poses problems as it is weakly convergent to zero. In fact [10], M_n converges in the sense of distributions to a Dirac delta-function.

$$M_n(x, x_0, r) = \omega_{N-1}^{-\frac{1}{N}} \begin{cases} (\log n)^{\frac{N-1}{N}} & \text{if } 0 \leq |x - x_0| \leq \frac{r}{n} \\ \frac{\log\left|\frac{|x-x_0|}{r}\right|^{-1}}{(\log n)^{\frac{1}{N}}} & \text{if } \frac{r}{n} \leq |x - x_0| \leq r \\ 0 & \text{if } |x - x_0| \geq r. \end{cases}$$

For completeness, we reproduce Lemma 3 from [13].

Lemma 2.4. *Assume that $e(x, u)$ satisfies (12). Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t \geq 0} \frac{t^N}{N} \int |\nabla M_n|^N - \int E(x, tM_n) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Remark 2.5. *By construction, $M_n(x) \geq 0$ on Ω and so $E(x, tM_n(x)) \equiv E^+(x, tM_n(x))$ on Ω . Lemma 2.4 holds for E replaced with E^+ .*

The following is a simple consequence of taking the negative half of $e(x, u)$, the negative limit in (13) and implementing Lemma 2.4.

Corollary 2.6. *Suppose $e(x, u)$ satisfies (13). Then there exists $n \in \mathbb{N}$ such that*

$$\max_{t \geq 0} \frac{t^N}{N} \int |\nabla M_n|^N - \int E(x, -tM_n) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$

These results still hold with a perturbation. Using the known results from Lemma 2.4 and Corollary 2.6 upper bounds on energy levels for the functional I follow easily. This proof relies upon the limit in (13) tending to positive or negative infinity.

Lemma 2.7. (i) *If $e(x, u)$ satisfies (12) and $h(x) \geq 0$ almost everywhere then there exists $\tilde{u}(x) \in W_0^{1,N}$ such that*

$$I^+(t\tilde{u}), I(t\tilde{u}) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

for all $t \geq 0$.

(ii) *If $e(x, u)$ satisfies (13) and $h(x)$ is of any sign then*

$$I(t\tilde{u}) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

for all $t \geq 0$.

PROOF: (i) Let $n > 0$ be the integer from Lemma 2.4. Since $\int hM_n > 0$ then there is nothing further to prove as $E(x, tM_n) = E^+(x, tM_n)$ and

$$\begin{aligned} I^+(tM_n) &= I(tM_n) = \frac{t^N}{N} \|M_n\|^N - \int E(x, tM_n) - t \int hM_n \\ &\leq \frac{t^N}{N} \|M_n\|^N - \int E(x, tM_n) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}. \end{aligned}$$

(ii) Suppose that $\int hM_m < 0$ for all $m > n$. Consider the sequence

$$\frac{t^N}{N} \int |\nabla M_m|^N - \int E(x, -tM_m).$$

Using Corollary 2.6, for sufficiently large $m \in \mathbb{N}$,

$$\begin{aligned} \max_{t \geq 0} I(-tM_m) &= \max_{t \geq 0} \left\{ \frac{t^N}{N} \|tM_m\|^N - \int E(x, -tM_m) + t \int hM_m \right\} \\ &\leq \max_{t \geq 0} \left\{ \frac{t^N}{N} \|tM_m\|^N - \int E(x, -tM_m) \right\} \\ &< \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}. \end{aligned}$$

□

Remark 2.8. *It appears possible to surmount technical difficulties and marginally improve on Lemma 2.7 by demanding only that $e(x, u)$ satisfy (12) irrespective of the sign of $h(x)$. The key to this proposal is to use the fact that $M_n \rightarrow 0$ in $W_0^{1,N}$, and hence the perturbation term $\int hM_n$ becomes arbitrarily small with large n .*

By the continuity of $I(u)$, it follows from Lemmas 2.1 and 2.2 that

$$(19) \quad -\infty < c_0 \equiv \inf\{I(u) : u \in W^{1,N}, \|u\| \leq \rho_h\} < 0.$$

Later we prove that this infimum is achieved and elicits a solution.

In order to invoke the forthcoming convergence results, we require a tighter bound on the maximal energy of the functional than the expressions in Lemma 2.7. To control the magnitude of $I(u)$ along a mountain-pass path, the size of $\|h\|_*$ is restricted.

Lemma 2.9. (i) *Assume that $e(x, u)$ satisfies (12) and $h(x) \geq 0$; or*
 (ii) *that $h(x)$ is indefinite in sign and $e(x, u)$ satisfies (13).*

*There exists $h^{**} > 0$ such that for all $0 \leq h(x) \in W^{-1,N'}$ with $0 < \|h\|_* < h^{**}$ there is some $\tilde{u}(x) \in W_0^{1,N}$ with the property that for (i)*

$$I^+(t\tilde{u}), I(t\tilde{u}) < c_0 + \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$$

for all $t \geq 0$, while for (ii),

$$I(t\tilde{u}) < c_0 + \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$

PROOF: When $h \equiv 0$, Lemma 2.1 shows that the origin forms a local minimum of the functional I . Let $\rho > 0$ be chosen such that $J(u) = \frac{1}{N}\|u\|^N - \int E(x, u) > \delta$ for all $\|u\| = \rho$. Perturbing the functional $J(u)$ by the term $-\int hu$, we see that $I(u)$ will remain positive for all $\|u\| = \rho$ if $\|h\|_* < \frac{\delta}{\rho}$.

It is possible to raise the lower bound for c_0 by reducing $\|h\|_*$. By Lemma 2.1, $\rho_h \rightarrow 0$ as $\|h\|_* \rightarrow 0$. Consequently, the infimum of $I(u)$ on $B(0, \rho_h)$ is increasing and $c_0 \rightarrow 0$ as $\|h\|_* \rightarrow 0$.

For part (i) use Lemma 2.7(i) and for part (ii) use Lemma 2.7(ii) to see that $I^+(t\tilde{u}), I(t\tilde{u}) < [\frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1} - \epsilon]$ for some $\epsilon > 0$. Defining h^{**} sufficiently small enforces $c_0 > -\epsilon$ and the results follow. □

3. Convergence properties of sequences

The development of the Palais-Smale levels leads to the construction of a sequence which is not necessarily strongly convergent in $W_0^{1,N}$. Instead, weak convergence to a nontrivial weak solution is achieved for certain energies.

The following lemma is attributed to de Figueiredo *et al* [8]:

Lemma 3.1. *Let $\{u_n\} \subset L^1(\Omega)$ such that $u_n \rightarrow u$ in $L^1(\Omega)$ and $e(x, u)$ be continuous. Then $e(x, u_n) \rightarrow e(x, u)$ in $L^1(\Omega)$ provided that $e(x, u_n) \in L^1(\Omega)$ for all n and $\int |e(x, u_n(x))u_n(x)| \leq C_1$.*

Lemma 3.2. *For a $(PS)_c$ sequence $\{u_n\}$ for I or I^+ at any level c , there is a subsequence relabelled $\{u_n\}$ and $u \in W_0^{1,N}$ such that*

$$(20) \quad e(x, u_n) \rightarrow e(x, u) \text{ in } L^1(\Omega);$$

$$(21) \quad E(x, u_n) \rightarrow E(x, u) \text{ in } L^1(\Omega);$$

$$(22) \quad |\nabla u_n|^{N-2} \nabla u_n \rightharpoonup |\nabla u|^{N-2} \nabla u \text{ weakly in } (L^{\frac{N}{N-1}}(\Omega))^N.$$

PROOF: This entire proof works in an identical manner for I and I^+ . The following details concern I , but replacing E and e with E^+ and e^+ provides the analogous result.

Suppose $\{u_n\}$ is a Palais-Smale sequence at level c , so

$$(23) \quad \frac{1}{N} \int |\nabla u_n|^N - \int E(x, u_n) - \int h u_n \rightarrow c$$

$$(24) \quad \int |\nabla u_n|^{N-2} \nabla u_n \cdot \nabla v - \int e(x, u_n) v - \int h v \rightarrow 0 \quad \forall v \in W_0^{1,N}.$$

Step 1: Show $\{u_n\}$ bounded in $W_0^{1,N}$.

From (23) and (24) have that

$$\begin{aligned} \left| \left(\frac{\theta}{N} - 1 \right) \|u_n\|^N - \int [\theta E(x, u_n) - u_n e(x, u_n)] - (\theta - 1) \int h u_n \right| \\ \leq C + \epsilon_n \|u_n\| \end{aligned}$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$(25) \quad \left| \left[\left(\frac{\theta}{N} - 1 \right) \|u_n\|^{N-1} - (\theta - 1) \|h\|_* \right] \|u_n\| - \int [\theta E(x, u_n) - u_n e(x, u_n)] \right| \leq C + \epsilon_n \|u_n\|.$$

Using (25) with (15), we have that $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$. From this,

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } W_0^{1,N} \\ u_n &\rightarrow u \text{ in } L^q \quad \forall q \geq 1 \\ u_n(x) &\rightarrow u(x) \text{ a.e. in } \Omega. \end{aligned}$$

Step 2: Claim $\{u_n\}$ has a subsequence such that (20) holds.

Justification of (20) follows from Lemma 3.1. To see the applicability of this lemma, note that $\|u_n\| \leq K$, so

$$-K\|h\|_* \leq \int hu_n \leq K\|h\|_*$$

then applying this to (23) and (24),

$$(26) \quad \int E(x, u_n) \leq C; \quad \text{and} \quad \left| \int e(x, u_n)u_n \right| \leq C.$$

Further, $e(x, u_n) \in L^1(\Omega)$ for all n by the Trudinger-Moser inequality:

$$|e(x, u_n)| \leq C \exp(\beta|u_n|^{\frac{N}{N-1}}) \in L^1(\Omega) \quad \text{for each } u_n \in W_0^{1,N}.$$

Step 3: Claim (21) holds.

By condition (10), there exists $\bar{E} > 0$ such that

$$E(x, u) \leq \bar{E} + Me(x, u).$$

Now,

$$\int E(x, u_n)u_n \leq \int \bar{E}u_n + M \int e(x, u_n)u_n.$$

The first term is bounded as $u_n \rightarrow u$ in $L^1(\Omega)$, and the second term is bounded by (26). Consequently, the requirements are satisfied to invoke Lemma 3.1 on $E(x, u)$ to prove (21).

Step 4: Claim $\{u_n\}$ has a subsequence such that (22) holds.

Note that u_n satisfies convergence weakly to a measure:

$$\begin{aligned} |\nabla u_n|^N &\rightharpoonup^* \mu \text{ in } \mathcal{D}(\Omega) \\ |\nabla u_n|^{N-2} \nabla u_n &\rightharpoonup V \text{ weakly in } (L^{\frac{N}{N-1}})^N \end{aligned}$$

where μ is a regular finite measure and $\mathcal{D}(\Omega)$ are the distributions on Ω .

Clearly $A_\sigma = \{x \in \bar{\Omega} : \mu(B_r(x) \cap \bar{\Omega}) \geq \sigma\}$ is a finite set, say $\{x_1, x_2, \dots, x_m\}$, for otherwise $\mu(A_\sigma) = \infty$ contradicting $\mu(A_\sigma) = \lim_{k \rightarrow \infty} \int_{A_\sigma} |\nabla u_n|^N \leq C$.

Assertion 1. If we choose $\sigma > 0$ such that $\sigma^{\frac{1}{N-1}}\beta < \alpha_N$, then $e(x, u_n)u_n \rightarrow e(x, u)u$ in $L^1(K)$ where $K \subset \bar{\Omega} \cap A_\sigma$ is compact.

To prove the assertion, let $x_0 \in K$ and $r_0 > 0$ be such that $\mu(B_{r_0} \cap \bar{\Omega}) < \sigma$. Define a function $\phi \in C^\infty(\bar{\Omega})$ which assumes the value zero when $x \in \bar{\Omega} \setminus B_{r_0}$ and the value one when $x \in B_{r_0/2} \cap \bar{\Omega}$ and has range $[0, 1]$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{B_{r_0}(x_0) \cap \bar{\Omega}} |\nabla u_n|^N \phi &= \int_{B_{r_0}(x_0) \cap \bar{\Omega}} \phi d\mu \\ &\leq \mu(B_{r_0}(x_0) \cap \bar{\Omega}) \leq \sigma. \end{aligned}$$

By the assumptions on σ , there exists some $q > 1$ such that

$$q\beta\sigma^{\frac{1}{N-1}} < \alpha_N$$

and hence

$$\int_{B_{r_0/2}(x_0) \cap \overline{\Omega}} |e(x, u_n(x))|^q \leq C.$$

Consequently, with all integration performed over the domain $B_{r_0/2}(x_0) \cap \overline{\Omega}$,

$$\int e(x, u_n)u_n - e(x, u)u = \int (e(x, u_n) - e(x, u))u + \int e(x, u_n)(u_n - u).$$

We know that $e(x, u_n) \rightarrow e(x, u)$ in $L^1(\Omega)$ and so the first term on the right hand side tends to zero. Apply Hölder’s inequality to the second term to reveal

$$\left| \int e(x, u_n)(u_n - u) \right| \leq \|e(x, u_n)\|_q \|u_n - u\|_{q'} \rightarrow 0.$$

Consequently $\int_{B_{r_0}(x_0) \cap \overline{\Omega}} (e(x, u_n)u_n - e(x, u)u) \rightarrow 0$. Since K is a compact set, repeating the same procedure over a finite covering of balls gives the result.

Assertion 2.

$$\int_{\Omega_\epsilon} (|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u) (\nabla u_n - \nabla u) \rightarrow 0 \text{ as } n \rightarrow \infty$$

where ϵ is sufficiently small that $B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset$ for $i \neq j$ and

$$\Omega_\epsilon = \{x \in \overline{\Omega} : \|x - x_j\| \geq \epsilon\}.$$

That is, Ω_ϵ consists of $\overline{\Omega}$ except for m ϵ -balls around $\{x_1, \dots, x_m\}$.

Let $0 \leq \psi_\epsilon(x) \leq 1$ be a $C^\infty(\Omega)$ function set to be 1 on Ω_ϵ and 0 on $\bigcup_{i=1}^m B(x_i, \epsilon/2)$.

From the Palais-Smale sequence (24):

$$\left| \int |\nabla u_n|^{N-2} \nabla u_n \nabla v - \int e(x, u_n)v - \int hv \right| \rightarrow 0.$$

Successively substitute $v = \psi_\epsilon u_n$ and $v = \psi_\epsilon u$, then

$$\begin{aligned} & \left[\int |\nabla u_n|^N \psi_\epsilon + |\nabla u_n|^{N-2} (\nabla u_n \cdot \nabla \psi_\epsilon) u_n - \int e(x, u_n) u_n \psi_\epsilon - \int h u_n \psi_\epsilon \right] \\ & \leq \epsilon_n \|\psi_\epsilon u_n\|; \\ & \left[- \int |\nabla u_n|^{N-2} (\nabla u_n \cdot \nabla u) \psi_\epsilon - |\nabla u_n|^{N-2} (\nabla u_n \cdot \nabla \psi_\epsilon) u \right. \\ & \quad \left. + \int e(x, u_n) u \psi_\epsilon + \int h u \psi_\epsilon \right] \leq \epsilon_n \|\psi_\epsilon u\|. \end{aligned}$$

Combine these by addition, then

$$\begin{aligned}
 \int \psi_\epsilon |\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u - \nabla u_n) &\leq \int |\nabla u_n|^{N-2} (\nabla u_n \cdot \nabla \psi_\epsilon) (u - u_n) \\
 (27) \qquad \qquad \qquad &+ \int \psi_\epsilon |\nabla u|^{N-2} \nabla u (\nabla u - \nabla u_n) \\
 &+ \int \psi_\epsilon e(x, u_n) (u_n - u) + \int h \psi_\epsilon (u_n - u) \\
 &+ \epsilon_n \|\psi_\epsilon u_n\| + \epsilon_n \|\psi_\epsilon u\|.
 \end{aligned}$$

Make use of the convexity of the map $v \mapsto |v|^N$ to establish that

$$\left(|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u \right) (\nabla u_n - \nabla u) \geq 0$$

and to prove that the left hand side in (27) is nonnegative. Estimate each of the integrals in (27) using the information that $u_n \rightharpoonup u_0$ in $W_0^{1,N}$ to show that

$$\int_{\Omega_\epsilon} \left(|\nabla u_n|^{N-2} \nabla u_n - |\nabla u|^{N-2} \nabla u \right) (\nabla u_n - \nabla u) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since ϵ is arbitrary,

$$\nabla u_n(x) \rightarrow \nabla u(x) \text{ a.e. in } \Omega$$

and using the boundedness of $(|\nabla u_n|^{N-2} \nabla u_n)$ in $(L^{\frac{N}{N-1}})^N$, we have for a subsequence

$$|\nabla u_n|^{N-2} \nabla u_n \rightharpoonup |\nabla u|^{N-2} \nabla u \text{ in } \left(L^{\frac{N}{N-1}}(\Omega) \right)^N.$$

□

Corollary 3.3. *It follows from Lemma 3.2 that any Palais-Smale sequence for I or I^+ is bounded and weakly convergent to a weak solution of (1) or (3) respectively.*

PROOF: Again the proof is identical for I and I^+ .

Suppose $\{u_n\}$ is a Palais-Smale sequence. Using the previous lemma and equation (24),

$$\int |\nabla u_0|^{N-2} \nabla u_0 \cdot \nabla w - \int e(x, u_0) w - \int h w = 0 \text{ for all } w \in \mathcal{D}(\Omega)$$

and thus u_0 is a weak solution. Since $h(x) \not\equiv 0$, $u_0 \not\equiv 0$. □

Remark 3.4. *The case of $h \equiv 0$ is covered in [13]. There, convergence to a nontrivial solution relies upon the energy of the sequence remaining below a forbidden level,*

$$\frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$

Lemma 3.5. *If $\{u_n\}$ is a (PS)-sequence for I or I^+ at any level with*

$$\liminf_{n \rightarrow \infty} \|u_n\| < \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}$$

then a subsequence converges strongly to a solution u_0 .

PROOF: Again the proof for I and I^+ is constructed in an identical manner. Let $\{u_n\}$ be such a sequence. Extract a subsequence, again relabelled as u_n such that $\lim_{n \rightarrow \infty} \|u_n\| = \liminf_{n \rightarrow \infty} \|u_n\|$. Corollary 3.3 establishes that u_n converges weakly to u_0 , a solution of (1). Let $u_n = u_0 + w_n$. Then $w_n \rightarrow 0$ in $W_0^{1,N}$ and $w_n \rightarrow 0$ in L^t for all $t \geq 1$. By the Brezis-Lieb lemma [4], $\|u_n\|^N = \|u_0\|^N + \|w_n\|^N + o(1)$. Since $u_0 \in W_0^{1,N}$, Lemma 3.2 implies that $\int e(x, u_n)u_0 \rightarrow \int e(x, u_0)u_0$. This gives that

$$(I'(u_n), u_n) = (I'(u_0), u_0) + \|w_n\|^N - \int e(x, u_n)w_n + o(1).$$

Since $\lim \|u_n\| < \left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}}$, choose $q > 1$ such that

$$\lim_{n \rightarrow \infty} q\alpha_0 \|u_n\|^{\frac{N}{N-1}} < \alpha_N.$$

Now,

$$\begin{aligned} \int |e(x, u_n)|^q &\leq C \int \exp\left(q\alpha_0 |u_n|^{\frac{N}{N-1}}\right) \\ &= C \int \exp\left(q\alpha_0 \|u_n\|^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right) \\ &\leq C. \end{aligned}$$

Thus $\int e(x, u_n)w_n \leq \|e(x, u_n)\|_q \|w_n\|_{q'} \rightarrow 0$. Consequently $\|w_n\| \rightarrow 0$ and the result follows. □

A local semicontinuity result is expressed below.

Lemma 3.6. *For any fixed $\epsilon > 0$, let B be the ball in $W_0^{1,N}(\Omega)$ centred at the origin with radius $\left(\frac{\alpha_N}{\alpha_0}\right)^{\frac{N-1}{N}} - \epsilon$. The functionals $I(u)$ and $I^+(u)$ are lower semicontinuous on B .*

PROOF: Again the method of proof is the same for I and I^+ .

Let $\{u_n\} \subset B$. Then $u_n \rightarrow u_0 \in B$ and

$$(28) \quad I(u_n) - I(u_0) = \frac{1}{N}\|u_n\|^N - \frac{1}{N}\|u_0\|^N - \int E(x, u_n) + \int E(x, u_0) + o(1).$$

We now show that $E(x, u_n) \rightarrow E(x, u_0)$ in L^1 by invoking Lemma 3.1.

Equation (28) establishes that $\lim_{n \rightarrow \infty} \int E(x, u_n) < \infty$. From condition (10), there exists $\bar{E} > 0$ such that $E(x, u) \leq \bar{E} + Me(x, u)$. Consequently,

$$\int E(x, u_n)u_n \leq \int \bar{E}u_n + M \int e(x, u_n)u_n.$$

The first term is bounded as $u_n \rightarrow u_0$ in $L^1(\Omega)$.

Since u_n is bounded, $I'(u_n)$ must be bounded. Consequently

$$(I'(u_n), u_n) = \|u_n\|^N - \int e(x, u_n)u_n - \int hu_n \leq C.$$

But $\|u_n\|^N$ and $\int hu_n \leq \|h\|_* \|u_n\|$ are both bounded and subsequently $\int e(x, u_n)u_n \leq C$. Thus, Lemma 3.1 proves that $\int E(x, u_n) \rightarrow \int E(x, u_0)$. As a consequence, $I(u_n) - I(u_0) \geq 0$ by lower semicontinuity of norms. □

4. Generation of solutions

A solution can be obtained by local minimisation near the origin in $W_0^{1,N}(\Omega)$. To show the existence of this solution, we use Ekeland’s variational principle ([9]). The number c_0 is defined in (19).

Lemma 4.1. *For $h(x) \in W^{-1,N'}$ with $0 < \|h\|_* \leq h^*$, there exists a minimum type solution, u_0 , to (1) at energy $c_0 < 0$. As $\|h\|_* \rightarrow 0$, $\|u_0\| \rightarrow 0$.*

PROOF: Let ρ_h be the radius of the ball from Lemma 2.1. Consulting (17) in the proof of Lemma 2.1, we see that $h^* > 0$ has been chosen sufficiently small that $\rho_h < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. It follows that $B(0, \rho_h) \subset W_0^{1,N}(\Omega)$ forms a complete metric space with metric $d(u_1, u_2) = \|u_1 - u_2\|$. On this set, I is lower semicontinuous and bounded below. Ekeland’s principle ensures the existence of a (PS) $_{c_0}$ -sequence from a minimising sequence $\{u_n\}$ for I in $B(0, \rho_h)$ in the same way as [6]. Each element of the sequence minimises

$$(29) \quad \inf \left\{ I(u) + \delta_n \|u_n - u\|_{W_0^{1,N}} : u \in \overline{B(0, \rho_h)} \right\}$$

for some $0 < \delta_n \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2.2, $I(u^*) < 0$ for some $u^* \in B(0, \rho_h)$ so $I(u_n) \rightarrow c_0 < 0$. Condition (29) implies $I'(u_n) \rightarrow 0$ in $W^{-1,N'}$ providing the Palais-Smale sequence. Lemma 3.5 guarantees that this sequence converges strongly to the minimiser which must be a solution. □

Lemma 4.2. *Suppose that (i) $h(x) \geq 0$ almost everywhere and $e(x, u)$ satisfies (12), or (ii) $h(x)$ is indefinite in sign and $e(x, u)$ satisfies (13). There exists a number $h^{**} > 0$ such that the mountain pass geometry reveals a solution, u_M , to (1) when $\|h\|_* \leq h^{**}$.*

PROOF: Lemmas 2.3 and 2.7 verify that there exists some $\tilde{u} \in W_0^{1,N}(\Omega)$ such that $I(t\tilde{u}) < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$ for all $t \geq 0$ and $I(\bar{t}\tilde{u}) < 0$ for some large $\bar{t} > 0$. Lemma 2.1 guarantees that a mountain ridge exists.

Invoking the mountain pass theorem without a Palais-Smale condition ([3]) provides a Palais-Smale sequence. Although not required for this lemma, we incidentally remark that for suitably small h^{**} , the energy of the sequence lies below $c_0 + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$.

Corollary 3.3 assures that this (PS) sequence converges weakly to a weak solution of (1). □

Lemma 4.3. *For suitably small h^{**} the solutions derived in Lemmas 4.1 and 4.2 are distinct.*

PROOF: Let $\{u_n\}$ be the minimising sequence and $\{v_n\}$ be the mountain pass sequence, so that

$$\begin{aligned} u_n \rightharpoonup u_0 & \quad \text{and} \quad v_n \rightharpoonup u_M \\ I(u_n) \rightarrow c_0 < 0 & \quad \text{and} \quad I(v_n) \rightarrow c_M > 0 \\ (I'(u_n), u_n) \rightarrow 0 & \quad \text{and} \quad (I'(v_n), v_n) \rightarrow 0. \end{aligned}$$

Suppose that $u_0 = u_M$. Then from Lemma 3.2

$$\begin{aligned} I(u_n) &= \frac{1}{N} \|u_n\|^N - \int E(x, u_0) - \int hu_0 + o(1) \rightarrow c_0 \\ I(v_n) &= \frac{1}{N} \|v_n\|^N - \int E(x, u_0) - \int hu_0 + o(1) \rightarrow c_M \end{aligned}$$

and subtracting one from the other, we have

$$(30) \quad \|u_n\|^N - \|v_n\|^N \rightarrow N(c_0 - c_M) < 0 \quad \text{as } n \rightarrow \infty.$$

Since u_n and v_n are both Palais-Smale sequences,

$$\begin{aligned} (I'(u_n), u_n) &= \int |\nabla u_n|^N - \int e(x, u_n)u_n - \int hu_n \rightarrow 0 \\ (I'(v_n), v_n) &= \int |\nabla v_n|^N - \int e(x, v_n)v_n - \int hv_n \rightarrow 0 \end{aligned}$$

to give

$$\begin{aligned} (\|u_n\|^N - \|v_n\|^N) &- \int [e(x, u_n)u_n - e(x, u_n)v_n + e(x, u_n)v_n - e(x, v_n)v_n] \\ (31) \quad &- \int [h(u_n - u_0) - h(v_n - u_0)] \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $h \in W^{-1, N'}$ and $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup u_0$, the last term tends to zero.

The second term may be written:

$$\int e(x, u_n)(u_n - v_n) + \int [e(x, u_n) - e(x, v_n)] v_n.$$

We have derived that for $\|h\|_*$ in the range $(0, h^*)$, the minimising sequence $\{u_n\}$ must satisfy $\|u_n\| < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. Letting q be slightly larger than 1 it follows that

$$\begin{aligned} \int |e(x, u_n)|^q &\leq C \int \exp\left(q\alpha_0|u_n|^{\frac{N}{N-1}}\right) \\ &= C \int \exp\left(q\alpha_0\|u_n\|^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right) \\ &\leq C. \end{aligned}$$

By the fact that $(u_n - v_n) \rightarrow 0$ in $W_0^{1,N}$,

$$\int e(x, u_n)(u_n - v_n) \leq \|e(x, u_n)\|_q \|u_n - v_n\|_{q'} \leq C \|u_n - v_n\|_{q'} \rightarrow 0.$$

It remains to show that

$$(32) \quad \int [e(x, u_n) - e(x, v_n)] v_n \rightarrow 0.$$

Let $v_n = u_0 + w_n$, so $w_n \rightarrow 0$. However, since v_n is a mountain pass sequence, $v_n \not\rightarrow u_0$. Consequently, v_n must concentrate and $\lim \|w_n\| > 0$. Now, (32) may be expressed as

$$\int [e(x, u_n) - e(x, v_n)] u_0 + \int [e(x, u_n) - e(x, v_n)] w_n \rightarrow 0.$$

Lemma 3.2 establishes that $e(x, u_n)$ and $e(x, v_n)$ both converge in $L^1(\Omega)$ to $e(x, u_0)$ and so the first term vanishes. Considering the second of these terms,

$$\int [e(x, u_n) - e(x, v_n)] w_n = \int e(x, u_n)w_n - \int e(x, v_n)w_n.$$

The minimising sequence u_n has the property that $\|u_n\| < (\frac{\alpha_N}{\alpha_0})^{\frac{N-1}{N}}$. Consequently

$$\begin{aligned} \int e(x, u_n)w_n &\leq \|e(x, u_n)\|_q \|w_n\|_{q'} \\ &\leq \left[C \int \exp\left(q\alpha_0\|u_n\|^{\frac{N}{N-1}} \left|\frac{u_n}{\|u_n\|}\right|^{\frac{N}{N-1}}\right) \right]^{\frac{1}{q}} \|w_n\|_{q'} \\ &\leq C \|w_n\|_{q'} \rightarrow 0. \end{aligned}$$

We are now left with only the term $\int e(x, v_n)w_n$.

By Lemma 2.9, the value of h^{**} is sufficiently small that we are guaranteed that for large n ,

$$\begin{aligned} c_M - c_0 &= I(v_n) - I(u_n) + o(1) = \frac{1}{N} \|v_n\|^N - \frac{1}{N} \|u_n\|^N + o(1) \\ &= \frac{1}{N} \|v_n\|^N - \frac{1}{N} \|u_0\|^N + o(1) < \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{\frac{N-1}{N}}. \end{aligned}$$

Thus there exists $q > 1$ such that for large n , $\|v_n\|^N - \|u_0\|^N < (\frac{\alpha_N}{q\alpha_0})^{N-1}$. As a direct implication,

$$\begin{aligned} (33) \quad q^{N-1} \left(\frac{\alpha_0}{\alpha_N} \right)^{N-1} &< \frac{1}{\|v_n\|^N - \|u_0\|^N} \\ \Rightarrow q\alpha_0 \|v_n\|^{\frac{N}{N-1}} &< \alpha_N \left[1 - \left\| \frac{u_0}{\|v_n\|} \right\|^N \right]^{\frac{-1}{N-1}}. \end{aligned}$$

Define $U_n = \frac{v_n}{\|v_n\|}$. Thus $\|U_n\| = 1$ and $U_n \rightharpoonup U_0 = \frac{u_0}{\lim \|v_n\|}$. We have deduced that v_n concentrates and hence $\|U_0\| < 1$.

Now,

$$\int e(x, v_n)w_n \leq \|e(x, v_n)\|_q \|w_n\|_{q'}$$

but

$$\int |e(x, v_n)|^q \leq C \int \exp \left(q\alpha_0 \|v_n\|^{\frac{N}{N-1}} \left| \frac{v_n}{\|v_n\|} \right|^{\frac{N}{N-1}} \right).$$

The right-hand side may be expressed as

$$C \int \exp \left(p\alpha_N |U_n|^{\frac{N}{N-1}} \right)$$

where (33) exposes that p is within the range demanded by Theorem 1.3. As a consequence, $\|e(x, v_n)\|_q$ is bounded. Using the information that $\|w_n\|_{q'} \rightarrow 0$, it follows that $\int e(x, u_n)u_n - e(x, v_n)v_n \rightarrow 0$. Hence expression (31) gives that $\|u_n\|^N - \|v_n\|^N \rightarrow 0$. But this contradicts (30), and thus $u_0 \neq u_M$ and the solutions are distinct. □

The proof to Theorem 1.1 now follows from Lemmas 4.1, 4.2 and 4.3.

5. Signs of solutions

Tarantello’s results [16] deduced that for a similar problem, a positive perturbation gives rise to positive solutions. The technique from [15] will be implemented to attain a similar result.

Publication [13] claims that for $h \equiv 0$, solutions to (1) are nonnegative, but this is not proven. We confirm this result here, and as a consequence discover that a negative solution exists also. This occurs despite the fact that the nonlinearity $E(x, u)$ is not necessarily even in u .

Theorem 5.1. *Suppose $h \equiv 0$ and (13) holds. There exist at least one nonnegative and one nonpositive solution to (1).*

PROOF: Take the positive half of $E(x, u)$ and symmetrise. Define

$$\bar{E}(x, u) = \begin{cases} E(x, u) & \text{if } u \geq 0 \\ E(x, -u) & \text{if } u < 0. \end{cases}$$

Define \bar{e} accordingly, and construct $\bar{I} = \frac{1}{N}\|u\|^N - \int \bar{E}(x, u)$. The even functional \bar{I} satisfies the required geometry and convergence properties (Lemmas 2.1, 2.3, 2.4 and Remark 3.4), and so the mountain pass lemma without the (PS) condition exposes a nontrivial solution \bar{u} . It follows easily that $\bar{I}(\bar{u}) = \bar{I}(|\bar{u}|)$, and we may assume $\bar{u} = |\bar{u}| \geq 0$ is a solution. Since I and \bar{I} correspond when $\bar{u} \geq 0$, we have that \bar{u} solves (1).

To locate a negative solution, define

$$(34) \quad \underline{E}(x, u) = \begin{cases} E(x, -u) & \text{if } u > 0 \\ E(x, u) & \text{if } u \leq 0. \end{cases}$$

With \underline{e} and \underline{I} defined accordingly, the pertinent geometry and convergence properties hold and a nontrivial solution \underline{u} results. By the remarks above, $\underline{u} \geq 0$ in Ω . For any $v \in C_0^\infty(\Omega)$,

$$(\underline{I}'(\underline{u}), v) = \int |\nabla \underline{u}|^{N-2} \nabla \underline{u} \nabla v - \int \underline{e}(x, \underline{u})v$$

but since $\underline{u} \geq 0$, $\underline{e}(x, \underline{u}) = -e(x, -\underline{u})$. Consequently,

$$\int |\nabla(-\underline{u})|^{N-2} \nabla(-\underline{u}) \nabla v - \int e(x, -\underline{u})v = 0$$

and so $-\underline{u}$ is a nonpositive solution. Since solutions are of opposing signs and neither is trivial they must be distinct. □

Lemma 5.2. *If (12) holds and nonzero $h(x) \geq 0$, then the two derived solutions are positive.*

PROOF: We have shown that the functional $I^+(u)$ satisfies the pertinent geometry and convergence properties. Lemmas 4.1, 4.2 and 4.3 are applicable to $I(u)$ as well as to $I^+(u)$ and subsequently the functional I^+ elicits two critical points.

Let u be a critical point of I^+ . Decompose u as $u = u^+ - u^-$ where $u^+ \geq 0$ and $u^- \geq 0$. Then

$$((I^+)'(u), u^-) = \int |\nabla u|^{N-2} \nabla u \cdot \nabla u^- - \int e^+(x, u)u^- - \int hu^-.$$

However, $e^+(x, u)u^-(x) = 0$ almost everywhere and hence

$$-\|u^-\|^N - \int hu^- = 0.$$

But $h(x)u^-(x) \geq 0$ almost everywhere, and thus $\|u^-\| = 0$. Consequently $u(x) \geq 0$ on Ω . □

Lemma 5.3. *If (13) holds and nonzero $h(x) \leq 0$ on Ω , then there exist at least two negative solutions.*

PROOF: Assume the definition for \underline{E} from (34). Define $\underline{I}(u) = \frac{1}{N}\|u\|^N - \int \underline{E}(x, u) + \int hu$. Since $-h(x) \geq 0$, the conditions are fulfilled to implement Lemma 5.2 exposing two nonnegative solutions to $\underline{I}'(u) = 0$. Considering one such solution, \underline{u} , and recalling the construction of $\underline{E}(x, u)$, we have that $\underline{e}(x, \underline{u}) = -e(x, -\underline{u})$ and so

$$\begin{aligned} -(\underline{I}'(\underline{u}), v) &= -\int |\nabla \underline{u}|^{N-2} \nabla \underline{u} \nabla v + \int \underline{e}(x, \underline{u})v - \int hv \\ &= \int |-\nabla \underline{u}|^{N-2} (-\nabla \underline{u}) \nabla v - \int e(x, -\underline{u})v - \int hv \\ &= (I'(-\underline{u}), v) = 0. \end{aligned} \quad \square$$

Remark 5.4. *For the development of positive solutions with $h \geq 0$ in Theorem 5.1 and Lemma 5.2, condition (12) will suffice in place of (13). Further, (10) and (11) may be relaxed to:*

There exists $R > 0$ and $M > 0$ such that for all $u \geq R$ and $x \in \Omega$,

$$\begin{aligned} 0 < E(x, u) &\leq Me(x, u) \quad \text{and} \\ \limsup_{u \rightarrow 0^+} \frac{NE(x, u)}{|u|^N} &< \lambda_1 \end{aligned}$$

respectively.

Acknowledgments. The author would like to thank his advisor Jan Chabrowski for guidance, the referee for helpful remarks and the ARC for provision of funding through an APA.

REFERENCES

- [1] Adimurthi, *Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n -Laplacian*, Ann. Sc. Norm. Sup. Pisa, Series 4 **17** (1990), 393–413.
- [2] Adimurthi, *Some remarks on the Dirichlet problem with critical growth for the n -Laplacian*, Houston J. Math. **17** (2) (1991), 285–298.
- [3] Ambrosetti A., Rabinowitz P.H., *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
- [4] Brezis H., Lieb E., *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (3) (1983), 486–490.
- [5] Carleson L., Chang S-Y., *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), 113–127.
- [6] Chabrowski J., *On multiple solutions for the nonhomogeneous p -Laplacian with a critical Sobolev exponent*, Differential Integral Equations **8** (4) (1995), 705–716.
- [7] Yinbin Deng, Yi Li, *Existence and bifurcation of the positive solutions of a semilinear equation with critical exponent*, J. Differential Equations **130** (1996), 179–200.

- [8] de Figueiredo D.G., Miyagaki O.H., Ruf B., *Elliptic equations in \mathbb{R}^2 with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (2) (1995), 139–153.
- [9] Ekeland I., *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.
- [10] Kai-Ching Lin, *Extremal functions for Moser's inequality*, Trans. Amer. Math. Soc. **348** (7) (1996), 2663–2671.
- [11] Lions P.L., *The Concentration Compactness Principle in the Calculus of Variations, part I*, Rev. Mat. Iberoamericana **1** (1985), 185–201.
- [12] Moser J., *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (11) (1971), 1077–1092.
- [13] Do Ó J.M.B., *Semilinear Dirichlet problems for the N -Laplacian in \mathbb{R}^N with nonlinearities in the critical growth range*, Differential Integral Equations **9** (5) (1996), 967–979.
- [14] Rabinowitz P.H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS, No. 65, AMS, 1986.
- [15] Panda R., *On semilinear Neumann problems with critical growth for the n -Laplacian*, Nonlinear Anal. **26** (1996), 1347–1366.
- [16] Tarantello G., *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (3) (1992), 281–304.
- [17] Trudinger N.S., *On imbeddings into Orlicz spaces and some applications*, Journal of Mathematics and Mechanics **17** (5) (1967), 473–483.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QUEENSLAND, QUEENSLAND 4072,
AUSTRALIA

(Received October 16, 1998)