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$C_p(I)$ is not subsequential
Abstract. If a separable dense in itself metric space is not a union of countably many nowhere dense subsets, then its $C_p$-space is not subsequential.

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0. Introduction

A subspace of a sequential space is called subsequential. Some time ago A.V. Arhangel’skii asked if $C_p(I)$ is subsequential. In [2] the author gave an example of a countable space which is not subsequential but can be embedded as a subspace in $C_p(2^\omega)$. In this note we prove several general propositions concerning non subsequentiality of $C_p$-spaces. We also give two simple examples of nonsubsequential subspaces of $C_p(2^\omega)$.

Recall that $C_p(X)$ denotes the space of real-valued continuous functions on $X$ with pointwise convergence topology, $I$ denotes the usual segment $[0,1]$. It is well known that $C_p(I)$ is not sequential (see, for example [1]).

The following proposition is in fact due to E.G. Pytke’ev [3].

Proposition 0.1. Let $X$ be subsequential, $x \notin A$, $x \in \overline{A}$. Then there exists a countable $\pi$-network at $x$ of infinite subsets of $A$, i.e. there exists at $x$ a countable family $A$ of infinite subsets of $A$ such that each neighbourhood of $x$ contains an element of $A$.

1. Propositions

Here we prove that very often a $C_p$-space is not subsequential.

Proposition 1.1. Let $X$ be a separable metric space and $\mathcal{P}$ a countable family of infinite subsets of $X$. Then there exists an open $\omega$-cover $\mathcal{V}$ of $X$ with the property

$(\mathcal{P}_s)$. Suppose $\mathcal{K}$ is an infinite subfamily of $\mathcal{V}$, then $\bigcap \{\overline{\mathcal{V}} : \mathcal{V} \in \mathcal{K}\}$ does not contain any element of $\mathcal{P}$.

Proof: We can assume that the metric $d$ of $X$ is totally bounded, i.e. for every $\delta > 0$ there exists a finite cover of balls of diameter less than $\delta$. Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$. Now we need a very simple
Lemma 1.2. Suppose $M$ is an infinite subset of a metric space. Then for every $n \in \omega$ there exists $\delta > 0$ such that $M$ cannot be covered by a union of $n$ balls of diameter less than $\delta$.

Proof of Lemma: Let $d$ be the metric on the space under consideration. As $M$ is infinite, we can find an $N \subset M$, $|N| = n + 1$. Then $\delta = \min\{d(x, y) : x, y \in N, x \neq y\}$ is the desired number. \hfill \Box

Further, using this lemma we can construct a decreasing sequence of positive reals $\delta_i, i \in \omega$, such that for every $i \in \omega$ every $P_k, k \leq i$, cannot be covered by a union of $i$ closed balls of diameter less than $\delta_i$. Now we find a sequence of finite open covers $W_i, i \in \omega$, of balls of diameter less than $\delta_i$. Further let $V_i = \{\cup T : T \subset W_i, |T| \leq i\}$. It is clear that $V_i$ is finite. Let $V = \bigcup\{V_i : i \in \omega\}$. Let us prove that $V$ is an open $\omega$-cover of $X$ with property $(\mathcal{P}_s)$. Let $Z$ be a finite subset of $X$. Let us take an $i \in \omega$, $i \geq |Z|$. There is an element of $V_i$ that covers $Z$. We proved that $V$ is an $\omega$-cover of $X$. Now let us finish the proof of Proposition 1.1. Let $P_k \in \mathcal{P}$. If $T \in V$ and $T \supset P_k$, then $T \in V_i$ with $i \leq k$. So, there are only finitely many elements of $V$ that contain the given $P_k$. The proof of 1.1 is complete. \hfill \Box

Proposition 1.3. Let $X$ be a separable metric space. Let $\mathcal{P}$ be a countable family of infinite subsets of $X$. Then $C_p(X)$ has an infinite subspace $F$, $1 \notin F$, $1 \in \overline{F}$ with the property $(\mathcal{P}_c)$. Suppose $K$ is an infinite subset of $F$, then $\bigcap\{f^{-1}[1/2, 3/2] : f \in K\}$ does not contain any element of $\mathcal{P}$.

Proof: Let $\{P_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{P}$ and let $\{V_i : i \in \omega\}$ be an enumeration of elements of $\mathcal{V}$ from Proposition 1.1. It is clear from the proof of Proposition 1.1 that there is a function $f : \omega \to \omega$ such that $P_k \not\subset \overline{V_i}$ if $i \geq f(k)$. For every $i \in \omega$ we can easily construct a real-valued continuous function $f_i$ such that $f_i^{-1}(1) \supset V_i$ and $P_k \not\subset f_i^{-1}[1/2, 3/2]$ for every $i \geq f(k)$. \hfill \Box

Now it remains to check that $F = \{f_i : i \in \omega\}$ is the desired subset of $C_p(X)$.

Proposition 1.4. Let $X$ be a space which is not a union of countably many nowhere dense subsets, let $X$ have a countable $\pi$-network $\mathcal{N}$ of infinite subsets. If $C_p(X)$ has a subspace $F$ from Proposition 1.3 with the property $(\mathcal{N}_c)$, then $C_p(X)$ is not subsequential.

Proof: We have $1 \in \overline{F}$. Let us prove that 1 has no countable $\pi$-network of infinite subsets of $F$. Let us suppose the contrary and let $\{P_j : j \in \omega\}$ be such a $\pi$-net. Let $O_x[1, \epsilon]$ denote a basic neighbourhood of 1 in $C_p(X)$, i.e. $O_x[1, \epsilon] = \{f \in C_p(X) : |f(x) - 1| < \epsilon\}$. Then for every $x \in X$ there is a $j_x \in \omega$ such that $P_{j_x} \subset O_x[1/2, 3/2]$. As $X$ is not a union of countably many nowhere dense subsets, there exist $m \in \omega$ and $X_m \subset X$ such that $X_m$ is not nowhere dense and $m = j_x$ for each $x \in X_m$. It is clear that $\overline{X_m} \subset f^{-1}[1/2, 3/2]$ for every $f \in P_m$, hence $\overline{X_m} \subset \bigcap\{f^{-1}[1/2, 3/2] : f \in P_m\}$. But $\text{Int}(\overline{X_m}) \neq \emptyset$, hence
$X_m$ contains some element $N' \in N$. Then $N' \subset (\bigcap \{f^{-1}[1/2, 3/2] : f \in P_m\})$. A contradiction is obtained.

Combining Propositions 1.1, 1.3, 1.4 we obtain

**Theorem 1.5.** If a separable dense in itself metric space is not a union of countably many nowhere dense subsets then its $C_p$-space is not subsequential.

**Proof:** It is enough to mention that a separable dense in itself metric space has a countable $\pi$-network of infinite subsets (moreover it has a countable base of nonempty open subsets which are infinite).

**Corollary 1.6.** $C_p(I)$ is not subsequential, $C_p(Y)$ is not subsequential for a non-scattered compactum $Y$.

**Proof:** A compactum $Y$ is not scattered iff it maps continuously onto $I$. But in this case $C_p(I) \subset C_p(Y)$.

**Proposition 1.7** (Compact dixotomy). A $C_p$-space over a compactum either is sequential or is not subsequential.

**Corollary 1.8.** If a metric space contains a copy of $2^\omega$ then its $C_p$-space is not subsequential. In particular, the $C_p$-space over an uncountable $A$-set in a metric space is not subsequential.

Because in these cases $C_p$-space contains a copy of $C_p(2^\omega)$.

Theorem 1.5 and Corollary 1.8 allow us to raise the following conjecture:

**Hypothesis 1.9** (General dixotomy). A $C_p$-space either is sequential or is not subsequential.

2. Two concrete examples

Here we give two examples of nonsubsequential subspaces of $C_p(2^\omega)$.

2.1. The first example. It is the space $Z$ introduced in [4]. We describe it here. Let $\{K_n : n \in \omega\}$ be disjoint finite subsets, $K = \bigcup \{K_n : n \in \omega\}$ and $* \notin K$. Let $Z = \{*\} \cup K$. Let all points of $K$ be isolated and a typical neighbourhood of $*$ be a set $\{*\} \cup (K \setminus L)$ where $|L \cap K_n| \leq m$ with the same $m$ for every $k \in \omega$. In [2] it is proved that $*$ has no countable $\pi$-net of infinite subsets of $K$ and it is proved that $Z$ can be embedded as a subspace in $C_p(2^\omega)$.

2.2. The second example. We will work in $2^\omega$. Let us follow the general way described in Propositions 1.1, 1.3, 1.4. Let $\Omega_n$ denote the set of functions $f : n \to 2$ and let $\Omega = \bigcup \{\Omega_n : n \in \omega\}$. For every $f \in \Omega$ the subset $O(f) = \{x \in 2^\omega : x \supset f\}$ is a basic clopen subset in $2^\omega$.

For every $n \in \omega$, let $S_n$ be the family $\{O(f) : f \in \Omega_{2^n}\}$ and $V_n = \{\bigcup T : T \subset S_n, |T| = n\}$.

Further, let $\mathcal{V} = \bigcup \{V_n : n \in \omega\}$. A little later we will prove that $\mathcal{V}$ is a clopen $\omega$-cover of $2^\omega$ with the following property:
Suppose $\mathcal{K}$ is an infinite subfamily of $\mathcal{V}$, then $Int(\bigcap \mathcal{K}) = \emptyset$.

It implies that a subspace $F$ of characteristic functions of elements of this cover $\mathcal{V}$ is the same as in Proposition 1.3. Hence this subspace demonstrates nonsubsequentiality of $C_p(2^\omega)$.

Now the desired proof. Let $Z$ be a finite subset of $2^\omega$. Let us take some $n \geq |Z|$. As $\mathcal{S}_n$ covers $2^\omega$, there is an element of $\mathcal{V}_n$ that contains $Z$. Now let $W$ be a clopen subset of $2^\omega$. For our goal we can assume that $W = O(f)$ for some $f \in \Omega_n$. We see that $m(O(f)) = 2^{-n}$ and $m(W) = i \cdot 2^{-2^i}$ for a $W \in \mathcal{V}_i$. Here $m$ denotes Lebesgue measure on $2^\omega$. Therefore if $W \supset O(f)$ then $i \leq n$, i.e. only finitely many elements of $\mathcal{V}$ contain $(f)$.

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References


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