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A remark on associative copulas

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Abstract. A method for producing associative copulas from a binary operation and a convex function on an interval is described.

Keywords: copulas, associative copulas, Archimedean copulas

Classification: 60E05, 62E10

Let I denote the unit interval $[0, 1]$. Copulas are cumulative distribution functions on I^2 with uniform marginals; more precisely, a *copula* is a function $C(x, y)$ on I^2 that satisfies

(1) (Boundary Conditions)

$$C(x, 0) = C(0, y) = 0, C(x, 1) = x \text{ and } C(1, y) = y \text{ for all } x, y \in I,$$

and

(2) (Monotonicity)

$$C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) + C(x_1, y_1) \geq 0,$$

if $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$.

For φ , a continuous, strictly decreasing function from I to $[0, \infty]$ such that $\varphi(1) = 0$, we define the *pseudo-inverse* of φ to be the function $\varphi^{[-1]} : [0, \infty] \rightarrow I$ defined by

$$(1) \quad \varphi^{[-1]}(x) = \begin{cases} \varphi^{-1}(x), & \text{if } 0 \leq x \leq \varphi(0), \\ 0, & \text{if } \varphi(0) \leq x \leq \infty. \end{cases}$$

We say that C is an *Archimedean* copula with additive generator φ provided that it is a copula and that there exists a function φ of the type described here such that

$$(2) \quad C(x, y) = \varphi^{[-1]}(\varphi(x) + \varphi(y)).$$

To quote from [2], “These copulas find a wide range of applications for a number of reasons: (1) The ease with which they can be constructed; (2) The great variety of families of copulas which belong to this class; and (3) The many nice properties possessed by the members of this class.” One of the most salient of these properties is that C is associative, that is,

$$C(x, C(y, z)) = C(C(x, y), z).$$

We have the following characterization of Archimedean copulas:

Theorem 1. *Let φ be a continuous, strictly decreasing function from I to $[0, \infty]$ such that $\varphi(1) = 0$. Then the function C defined by (2) is a copula if and only if φ is convex.*

Proof of this theorem can be found in [1] and [2]. Discussion of related Archimedean binary operations can be found in [3].

We show this theorem can be generalized in a simple and elegant fashion in which, instead of dealing with pseudo-inverses, we extend the notion of convexity.

Let \oplus be a continuous associative operation in $[0, a]$, $a \in [0, \infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0, a]$.

Example 2. Let $a = \infty$ and \oplus by the ordinary addition extended to $[0, \infty]$ in the obvious way. Clearly, the above conditions are satisfied.

Example 3. Let $a \in [0, \infty]$ be arbitrary and let \oplus be defined by $s \oplus t = \max(s, t)$. Again, it is easy to check that the above conditions are satisfied.

Example 4. Let $a \in [0, \infty]$ be arbitrary and let \oplus be defined by $s \oplus t = \min(s + t, a)$. Simple argument shows that the above conditions are satisfied.

A function $\psi : [0, a] \rightarrow \mathbb{R}$ is called \oplus -convex if

$$(3) \quad \psi(r \oplus t) - \psi(r) \leq \psi(s \oplus t) - \psi(s)$$

for every $r \leq s$ and any t .

Lemma 5. *If \oplus is ordinary addition and ψ is continuous, then ψ satisfies (3) if and only if ψ is convex.*

Lemma 6. *If $s \oplus t = \max(s, t)$, then ψ satisfies is \oplus -convex if and only if ψ is decreasing.*

PROOF: In order to show that (3) implies that ψ is decreasing it suffices to take $t = s$.

Now consider $r \leq s$. We consider three cases. If $t \leq r$, then (3) becomes

$$\psi(r) - \psi(r) \leq \psi(s) - \psi(s),$$

which is always true. If $r \leq t \leq s$, then (3) reduces to

$$\psi(t) - \psi(r) \leq \psi(s) - \psi(s) = 0,$$

which is true since ψ is decreasing. Finally, if $s \leq t$, then (3) becomes

$$\psi(t) - \psi(r) \leq \psi(t) - \psi(s),$$

or

$$\psi(s) \leq \psi(r),$$

which is again true since ψ is decreasing. □

Now we prove the main theorem of this note.

Theorem 7. Let \oplus be a continuous associative operation in $[0, a]$, $a \in [0, \infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0, a]$. Let $\varphi : [0, 1] \rightarrow [0, a]$ be a strictly decreasing continuous surjection. Define

$$(4) \quad C(x, y) = \varphi^{-1}(\varphi(x) \oplus \varphi(y)).$$

Then

- (a) $C(0, z) = C(z, 0) = 0$ and $C(1, z) = C(z, 1) = z$ for all $z \in [0, 1]$,
- (b) C is associative,
- (c) C is a copula if and only if φ^{-1} is \oplus -convex.

Parts (a) and (b) are easy. Before we prove part (c) we prove the following lemma.

Lemma 8. Let \oplus , φ , and C be as in Theorem 7. Then C is monotonic if and only if

$$(5) \quad C(u_2, v) - C(u_1, v) \leq u_2 - u_1 \text{ whenever } u_1 \leq u_2.$$

PROOF: Since every copula satisfies (5), it suffices to prove that (5) implies that C is monotonic.

Assume (5) and consider $v_1 \leq v_2$. Since φ and \oplus are continuous and $\varphi(v_2) \leq \varphi(v_1)$, there exists $t \in [0, 1]$ such that

$$\varphi(v_2) \oplus \varphi(t) = \varphi(v_1).$$

Hence

$$\begin{aligned} C(u_2, v_1) - C(u_1, v_1) &= \varphi^{-1}(\varphi(u_2) \oplus \varphi(v_1)) - \varphi^{-1}(\varphi(u_1) \oplus \varphi(v_1)) \\ &= \varphi^{-1}(\varphi(u_2) \oplus (\varphi(v_2) \oplus \varphi(t))) - \varphi^{-1}(\varphi(u_1) \oplus (\varphi(v_2) \oplus \varphi(t))) \\ &= \varphi^{-1}((\varphi(u_2) \oplus \varphi(v_2)) \oplus \varphi(t)) - \varphi^{-1}((\varphi(u_1) \oplus \varphi(v_2)) \oplus \varphi(t)) \\ &= C(C(u_2, v_2), t) - C(C(u_1, v_2), t) \\ &\leq C(u_2, v_2) - C(u_1, v_2), \end{aligned}$$

which proves that C is monotonic. □

PROOF OF PART (c) IN THEOREM 7: Suppose C is monotonic. Let $r \leq s$ and $t > 0$. Let $u_1 = \varphi^{-1}(s)$, $u_2 = \varphi^{-1}(r)$, and $v = \varphi^{-1}(t)$. Since $u_1 \leq u_2$, by Lemma 8, we have (5) and consequently

$$\varphi^{-1}(r \oplus t) - \varphi^{-1}(s \oplus t) \leq \varphi^{-1}(r) - \varphi^{-1}(s),$$

which proves that φ^{-1} is \oplus -convex.

Now suppose φ^{-1} is \oplus -convex. Let $u_1 \leq u_2$ and v be arbitrary. Define $r = \varphi(u_2)$, $s = \varphi(u_1)$, and $t = \varphi(v)$. Then (3) implies (5), which proves that C is monotonic by Lemma 8. □

The following simple theorem shows that every associative copula can be obtained in the way described in Theorem 7.

Theorem 9. For every associative copula C there exist \oplus and φ as in Theorem 7, with $a = 1$, such that

$$C(x, y) = \varphi^{-1}(\varphi(x) \oplus \varphi(y)).$$

PROOF: It suffices to define

$$r \oplus s = 1 - C(1 - r, 1 - s)$$

and

$$\varphi(t) = 1 - t.$$

□

The copula defined by (2) is commutative, because $+$ is a commutative operation. Since, in Theorem 7, commutativity plays no role, one may expect that by using a noncommutative \oplus we can construct noncommutative associative copulas. However, this is not possible. In [1] the following theorem is proved.

Theorem 10. Let $T : I^2 \rightarrow I$ be a continuous mapping such that

$$T(x, 0) = T(0, y) = 0 \text{ and } T(x, 1) = T(1, x) = x \text{ for all } x \in I,$$

and

$$T(T(x, y), z) = T(x, T(y, z)) \text{ for all } x, y, z \in I.$$

Then

$$T(x, y) = T(y, x) \text{ for all } x \in I.$$

From this result we easily obtain the following property of the operation \oplus .

Theorem 11. A continuous associative operation \oplus in $[0, a]$, $a \in [0, \infty]$, such that $t \oplus 0 = 0 \oplus t = t$ and $t \oplus a = a \oplus t = a$ for all $t \in [0, a]$, is commutative.

PROOF: If $a < \infty$, then define

$$T(x, y) = 1 - \frac{1}{a} ((a - ax) \oplus (a - ay)).$$

If $a = \infty$, then define

$$T(x, y) = \frac{1}{1 + \frac{1-x}{x} \oplus \frac{1-y}{y}}.$$

In both cases T satisfies the assumptions of Theorem 10, and thus $T(x, y) = T(y, x)$ for all $x, y \in I$. Now commutativity of \oplus follows easily. □

Note that although every decreasing function is \oplus -convex with respect to the operation \oplus defined in Example 3, we do not get a variety of copulas this way. Indeed,

$$\varphi^{-1}(\max(\varphi(x), \varphi(y))) = \min(x, y),$$

whenever φ is decreasing.

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