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Generalized $n$–coherence

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Abstract. In this paper necessary and sufficient conditions for large subdirect products of $n$-flat modules from the category $\text{Gen}(Q)$ to be $n$-flat are given.

Keywords: relative finiteness conditions, relative coherence, large subdirect products of $n$-flat modules

Classification: 16D40

In what follows, $R$ stands for an associative ring with a unit element and $R$-$\text{Mod}$ ($\text{Mod}$-$R$) denotes the category of all unitary left (right) $R$-modules.

Let $\mathcal{F}$ be a filter on a set $I$ and $\{M_i; i \in I\}$ be a family of left $R$-modules. We define an equivalence relation $\sim$ on $\prod_{i \in I} M_i$ as follows: For $(m_i),(n_i) \in \prod_{i \in I} M_i$, $(m_i) \sim (n_i)$ if $\{i \in I; m_i = n_i\} \in \mathcal{F}$. The equivalence class of $(0,0,\ldots)$ is called the $\mathcal{F}$-product and it is denoted by $\prod_{i \in I}^\mathcal{F} M_i$. Clearly, $\prod_{i \in I}^\mathcal{F} M_i$ is a submodule of $\prod_{i \in I} M_i$. For a set $X$ let $|X|$ denote the cardinality of $X$ and for $m = (m_i)_{i \in I} \in \prod_{i \in I} M_i$ let $\text{supp}(m) = \{i \in I; m_i \neq 0\}$. For an infinite cardinal number $\aleph$ the $\aleph$-product is defined as $\prod_{i \in I}^{\aleph} M_i = \{m \in \prod_{i \in I} M_i; |\text{supp}(m)| < \aleph\}$. For an infinite cardinal number $\aleph$ let $\aleph^+$ be its immediate successor. Let $\mathcal{F}$ be a filter on an index set $I$ and let $\aleph$ be $\sup\{|I \setminus X|; X \in \mathcal{F}\}$. According to [9] we define $\text{sup}(\mathcal{F})$ to be $\aleph$ if the supremum is not attained and $\aleph^+$ if the supremum is attained. If $\aleph$ is an infinite cardinal number and $|I| \geq \aleph$ then $\mathcal{F} = \{X \subseteq I; |I \setminus X| < \aleph\}$ is a filter on $I$ with $\text{sup}(\mathcal{F}) = \aleph$ and $\prod_{i \in I}^{\aleph} M_i = \prod_{i \in I}^\mathcal{F} M_i$. If $|I| < \aleph$ then obviously $\prod_{i \in I}^{\aleph} M_i = \prod_{i \in I} M_i$. If $|I| = \aleph$, then we have $\sum_{i \in I}^{\oplus} M_i = \prod_{i \in I}^{\aleph} M_i \subseteq \prod_{i \in I}^{\aleph} M_i \subseteq \cdots \subseteq \prod_{i \in I}^{\aleph n} M_i \subseteq \prod_{i \in I}^{\aleph n+1} M_i = \prod_{i \in I} M_i$. The $\mathcal{F}$-products ($\aleph$-products) of flat and projective modules were investigated in [9] and [10] by P. Loustaunau.

Let $n$ be a nonnegative integer. A module $M \in \text{Mod}$-$R$ is called $n$-presented if there is a finite $n$-presentation of $M$ i.e. an exact sequence

$$F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

in which every $F_i$ is free of finite rank. A ring $R$ is said to be right $n$-coherent if every $n$-presented right module is $(n+1)$-presented. The following definition of $n$-flat and $n$-$FP$-injective module is due to J. Chen and N. Ding. Let $n$ be

\begin{thebibliography}{99}
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\end{thebibliography}
a positive integer. A left $R$-module $Q$ is called $n$-flat if $\text{Tor}^R_n(N, Q) = 0$ for all $n$-presented right $R$-modules $N$. A right $R$-module $M$ is said to be $n$-FP-injective if $\text{Ext}^n_R(N, M) = 0$ for all $n$-presented right $R$-modules $N$.

In [3] J. Chen and N. Ding characterize right $n$-coherent rings as rings for which direct products of $n$-flat left $R$-modules are $n$-flat. In [5] $(\aleph, Q)$-coherent rings were introduced and they were characterized as rings for which $\aleph$-products of flat modules from the category $\text{Gen}(Q)$ are flat. These rings were also studied in [11]. The aim of this paper is to generalize results of J. Chen and N. Ding and the results in [5] to $\aleph$-products of $n$-flat modules from the category $\text{Gen}(Q)$ for a fixed flat module $Q$.

Throughout all the paper $RQ$ denotes a fixed flat left $R$-module and $\aleph$ denotes an infinite cardinal number.

The notions of $(\aleph, Q)$-finitely generated, $(\aleph, Q)$-finitely presented and $(\aleph, Q)$-coherent modules were introduced in [5]. In the following lemmas we summarize basic properties of these modules.

**Lemma 1.1.** Let $\{Q_i; i \in I\}$ be a set of left $R$-modules. Then

(i) if $\mathcal{F}$ is a filter on $I$ with $\text{sup}(\mathcal{F}) \leq \aleph$ then $\prod_{i \in I}^\mathcal{F} Q_i \subseteq \prod_{i \in I}^\aleph Q_i$;

(ii) let $\mathcal{F}$ be a filter on $I$ with $\text{sup}(\mathcal{F}) = \aleph$ and $q \in \prod_{i \in I}^\aleph Q_i$. If $S = \text{supp}(q)$ then there is $X \in \mathcal{F}$ and an injective map $f: S \to I \setminus X$. Since $X \subseteq I \setminus f(S)$ the element $\overline{q}$ defined by $\overline{q}_i = q_f^{-1}(i)$ for $i \in f(S)$ and $\overline{q}_i = 0$ for $i \in I \setminus f(S)$ belongs to $\prod_{i \in I}^\mathcal{F} Q_i$.

**Proof:** (i). If $q \in \prod_{i \in I}^\mathcal{F} Q_i$ then $|\text{sup}(q)| < \text{sup}(\mathcal{F}) \leq \aleph$ and consequently $q \in \prod_{i \in I}^\aleph Q_i$.

(ii). If $\text{sup}(\mathcal{F}) = \aleph$ and $|S| < \aleph$ then there is $X \in \mathcal{F}$ with $|S| \leq |I \setminus X|$. The rest is clear. $\square$

**Lemma 1.2.** Let $\mathcal{F}$ be a filter on $I$ with $\text{sup}(\mathcal{F}) = \aleph$, $\{Q_i; i \in I\}$ be a family of left $R$-modules and $M$ be a right $R$-module. Then the following conditions are equivalent:

(i) the natural homomorphism $\varphi_{\mathcal{F}}: M \otimes R \prod_{i \in I}^\mathcal{F} Q_i \to \prod_{i \in I}^\mathcal{F} (M \otimes R Q_i)$ defined via $\varphi_{\mathcal{F}}(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I}$ is an epimorphism;

(ii) the natural homomorphism $\varphi_{\aleph}: M \otimes R \prod_{i \in I}^\aleph Q_i \to \prod_{i \in I}^\aleph (M \otimes R Q_i)$ defined via $\varphi_{\aleph}(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I}$ is an epimorphism.

**Proof:** (i) implies (ii). Let $\varphi_{\mathcal{F}}$ be an epimorphism, $q \in \prod_{i \in I}^\aleph (M \otimes Q_i)$, $S = \text{supp}(q)$ and $\overline{q} \in \prod_{i \in I}^\mathcal{F} (M \otimes Q_i)$ be the element defined in Lemma 1.1(ii). Then there is an element $m_1 \otimes q_1 + \cdots + m_r \otimes q_r \in M \otimes \prod_{i \in I}^\mathcal{F} Q_i$ with $(m_1 \otimes q_{1i} + \cdots + m_r \otimes q_{ri})_{i \in I} = \overline{q}$. We can assume without loss of generality that $q_{ij} = 0$ for $j \in I \setminus f(S)$ and $i = 1, \ldots, r$. Let $p_j \in \prod_{i \in I}^\aleph Q_i$ such that $p_{j_1} f = 0$ for $t \in I \setminus S$ and $p_{j_s} = q_{j f(s)}$ for $s \in S$, $j = 1, \ldots, r$. Hence $q_s = \overline{q}_{j f(s)} = m_1 \otimes q_{1 f(s)} + \cdots + m_r \otimes q_{r f(s)} = m_1 \otimes p_{1 s} + \cdots + m_r \otimes p_{r s}$ for $s \in S$ and consequently $\varphi_{\aleph}$ is an epimorphism.
(ii) implies (i). If \( \varphi_N \) is an epimorphism and \( q \in \prod_{i \in I}^F (M \otimes Q_i) \) then there is an element \( m_1 \otimes q_1 + \cdots + m_r \otimes q_r \in M \otimes \prod_{i \in I}^N Q_i \) with \( (m_1 \otimes q_1 + \cdots + m_r \otimes q_r)_{i \in I} = q \). If \( S = \text{supp}(q) \) then \( I \setminus S \in \mathcal{F} \). Without loss of generality we can take \( q_i \) such that \( q_{ij} = 0 \) for \( j \in I \setminus S \) and \( i = 1, \ldots, r \). Thus \( q_i \in \prod_{i \in I}^F Q_i \) for \( i = 1, \ldots, r \) and consequently \( \varphi_F \) is an epimorphism.

The following definition is motivated by the definition of R.R. Colby and E.A. Rutter of the \( Q \)-finitely generated module in [4] and the definition of P. Lous-taunau of the \( \aleph \)-finitely generated module in [9].

**Definition 1.3.** A right \( R \)-module \( M \) is said to be \((\aleph, Q)\)-finitely generated if every subset \( T \) of \( M \otimes_R Q \) with \( |T| < \aleph \) is contained in \( N \otimes_R Q \) for some finitely generated submodule \( N \) of a module \( M \).

**Lemma 1.4.** Let \( M \) be a right \( R \)-module. Then the following conditions are equivalent:

(i) \( M \) is \((\aleph, Q)\)-finitely generated;

(ii) if \( I \) is a set and \( Q_i \in \text{Gen}(Q) \), \( i \in I \) then the natural homomorphism \( \varphi: M \otimes_R \prod_{i \in I}^N Q_i \rightarrow \prod_{i \in I}^F (M \otimes_R Q_i) \) defined via \( \varphi(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I} \) is an epimorphism;

(iii) if \( I \) is a set then the natural homomorphism \( \varphi: M \otimes_R \prod_{i \in I}^N Q \rightarrow \prod_{i \in I}^F (M \otimes_R Q) \) defined via \( \varphi(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I} \) is an epimorphism.

**Proof:** (i) implies (ii). Let \( u \in \prod_{i \in I}^N (M \otimes Q_i) \), \( T = \text{supp}(u) \) and \( f_i: Q(J_i) \rightarrow Q_i \), \( i \in I \) be epimorphisms. Then \( |T| < \aleph \) and \( \text{id}_M \otimes f_i: M \otimes Q(J_i) \rightarrow M \otimes Q_i \), \( i \in I \) are epimorphisms. Hence \( u_i = \sum_{j=1}^{n_i} m_{ij} \otimes f_i(q_{ij}) \), where \( m_{ij} \in M \), \( q_{ij} \in Q(J_i) \), \( i \in I \) and \( j = 1, \ldots, n_i \). Now \( q_{ij} = \sum_{k=1}^{t_{ij}} q_{ijk} \), where \( q_{ijk} \in Q \), \( k = 1, \ldots, t_{ij} \). Let \( S = \{m_{ij} \otimes q_{ijk} \mid i \in T, j = 1, \ldots, n_i, k = 1, \ldots, t_{ij} \} \). Then \( |S| < \aleph \) and \( S \subseteq M \otimes Q \). Thus \( S \subseteq N \otimes Q \) for some finitely generated submodule \( N = \sum_{p=1}^{l} n_p R \) of \( M \). Hence \( m_{ij} \otimes q_{ijk} = \sum_{p=1}^{l} n_p \otimes q_{ijkp} \) for some \( q_{ijkp} \in Q \). Put \( v_p = \sum_{i=1}^{n_i} \sum_{k=1}^{t_{ij}} q_{ijkp} \) for \( i \in T \) and \( v_p = 0 \) for \( i \in I \setminus T, p = 1, \ldots, l \). Then \( w_p = (f_i(v_p))_{i \in I} \in \prod_{i \in I}^N Q_i \), \( p = 1, \ldots, l \) and \( u_i = \sum_{p=1}^{l} n_p \otimes f_i(\sum_{j=1}^{n_j} \sum_{k=1}^{t_{ij}} q_{ijkp}) = \sum_{p=1}^{l} n_p \otimes f_i(v_p) \), \( i \in I \). Thus \( \varphi(\sum_{p=1}^{l} n_p \otimes w_p) = (\sum_{p=1}^{l} n_p \otimes f_i(v_p))_{i \in I} = u \) and consequently \( \varphi \) is an epimorphism.

(ii) implies (iii). Obvious.

(iii) implies (i). Let \( S \subseteq M \otimes Q \) with \( |S| < \aleph \) and \( I \) be a set such that \( |S| \leq |I| \) (e.g. \( I = M \otimes Q \) or \( I \) is a set with \( |I| \geq \aleph \)). Then there is an injective map \( f: S \rightarrow I \). Let us consider \( u \in \prod_{i \in I}^N (M \otimes Q) \) defined by \( u_i = f^{-1}(i) \) for \( i \in f(S) \) and \( u_i = 0 \) for \( i \in I \setminus f(S) \). Then by assumption there is \( \sum_{j=1}^{r} m_j \otimes q_j \in M \otimes \prod_{i \in I}^N Q \) such that \( (\sum_{j=1}^{r} m_j \otimes q_j)_{i \in I} = u \). Now if \( s \in S \) then \( s = f^{-1}(i) = \sum_{j=1}^{r} m_j \otimes q_{ji} \) for some \( i \in f(S) \) and therefore \( S \subseteq N \otimes Q \), where \( N = \sum_{j=1}^{r} m_j R \) is a finitely generated submodule of \( M \). \( \square \)
Corollary 1.5. The class of all $(\aleph, Q)$-finitely generated modules is closed under extensions, homomorphic images, finite direct sums, direct summands and contains the class of all finitely generated modules.

Proof: It follows immediately from Lemma 1.4(ii) and the definition of $(\aleph, Q)$-finitely generated module.

Definition 1.6. A right $R$-module $M$ is said to be $(\aleph, Q)$-finitely presented if there is a short exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $F$ free of finite rank and $K$ $(\aleph, Q)$-finitely generated.

Lemma 1.7. Let $M$ be a finitely generated right $R$-module. Then the following conditions are equivalent:

(i) $M$ is $(\aleph, Q)$-finitely presented;

(ii) if $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ is a projective presentation with $P$ finitely generated then $K$ is $(\aleph, Q)$-finitely generated;

(iii) if $I$ is a set and $Q_i \in Gen(Q), i \in I$ then the natural homomorphism $\varphi: M \otimes_R \prod_{i \in I}^\aleph Q_i \rightarrow \prod_{i \in I}^\aleph (M \otimes_R Q_i)$ defined via $\varphi(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I}$ is an isomorphism;

(iv) if $I$ is a set then the natural homomorphism $\varphi: M \otimes_R \prod_{i \in I}^\aleph Q_i \rightarrow \prod_{i \in I}^\aleph (M \otimes_R Q_i)$ defined via $\varphi(m \otimes (q_i)_{i \in I}) = (m \otimes q_i)_{i \in I}$ is an isomorphism.

Proof: (i) implies (ii). Let $0 \rightarrow K_i \rightarrow P_i \rightarrow M \rightarrow 0, i = 1, 2$ be two projective presentations of $M$. By Schanuel’s Lemma we have $P_1 \oplus K_2 \cong P_2 \oplus K_1$. Now if $P_1, P_2$ are finitely generated and $K_1$ is $(\aleph, Q)$-finitely generated then $K_2$ is $(\aleph, Q)$-finitely generated by Corollary 1.5.

(ii) implies (iii). Let $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence, where $F$ is free of finite rank and $Q_i \in Gen(Q), i \in I$. Consider the following commutative diagram

$$
\begin{array}{cccccc}
K \otimes \prod_{i \in I}^\aleph Q_i & \rightarrow & F \otimes \prod_{i \in I}^\aleph Q_i & \rightarrow & M \otimes \prod_{i \in I}^\aleph Q_i & \rightarrow & 0 \\
\downarrow \varphi_K & & \downarrow \varphi_F & & \downarrow \varphi_M & & \\
\prod_{i \in I}^\aleph (K \otimes Q_i) & \rightarrow & \prod_{i \in I}^\aleph (F \otimes Q_i) & \rightarrow & \prod_{i \in I}^\aleph (M \otimes Q_i) & \rightarrow & 0 
\end{array}
$$

Then $\varphi_F$ is obviously an isomorphism since $F$ is free of finite rank and $\varphi_K$ is an epimorphism since $K$ is $(\aleph, Q)$-finitely generated. Hence $\varphi_M$ is an isomorphism.

(iii) implies (iv). Obvious.

(iv) implies (i). Let $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence with $F$ free of finite rank. Consider the following commutative diagram

$$
\begin{array}{cccccc}
K \otimes \prod_{i \in I}^\aleph Q & \rightarrow & F \otimes \prod_{i \in I}^\aleph Q & \rightarrow & M \otimes \prod_{i \in I}^\aleph Q & \rightarrow & 0 \\
\downarrow \varphi_K & & \downarrow \varphi_F & & \downarrow \varphi_M & & \\
0 & \rightarrow & \prod_{i \in I}^\aleph (K \otimes Q) & \rightarrow & \prod_{i \in I}^\aleph (F \otimes Q) & \rightarrow & \prod_{i \in I}^\aleph (M \otimes Q) & \rightarrow & 0
\end{array}
$$
Now \( \varphi_F \) and \( \varphi_M \) are isomorphisms. Hence \( \varphi_K \) is an epimorphism and \( K \) is \((\aleph, Q)\)-finitely generated by Lemma 1.4.

**Remark 1.8.** As it follows from Lemma 1.2 and the proof of Lemma 1.7 every \( \aleph \)-product \( \prod_{i \in I}^{\aleph} \) in Lemma 1.4 and Lemma 1.7 can be replaced by \( F \)-product \( \prod_{i \in I}^{F} \) for a filter \( F \) on \( I \) with \( \sup(F) = \aleph \).

**Definition 1.9.** Let \( n \) be a nonnegative integer. A right \( R \)-module \( M \) is called \( n-(\aleph, Q) \)-presented if there is a finite \( n-(\aleph, Q) \)-presentation of \( M \) i.e. an exact sequence

\[
0 \to K_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to M \to 0
\]

in which every \( F_i \) is free of finite rank and \( K_n \) is \((\aleph, Q)\)-finitely generated.

**Definition 1.10.** Let \( n \) be a nonnegative integer. A ring \( R \) is said to be right \( n-(\aleph, Q) \)-coherent if every \( n \)-presented right \( R \)-module is \((n + 1)-(\aleph, Q) \)-presented.

**Lemma 1.11.** Let \( n \) be a positive integer, \( N \) be an \( n-(\aleph, Q) \)-presented right \( R \)-module and \( \{ Q_i; i \in I \} \) be a family of left \( R \)-modules from \( \text{Gen}(Q) \). Then:

(i) there is an epimorphism \( \text{Tor}^R_n(N, \prod_{i \in I}^{\aleph} Q_i) \rightarrow \prod_{i \in I}^{\aleph} \text{Tor}^R_n(N, Q_i) \);

(ii) there is an isomorphism \( \text{Tor}^R_{n-1}(N, \prod_{i \in I}^{\aleph} Q_i) \cong \prod_{i \in I}^{\aleph} \text{Tor}^R_{n-1}(N, Q_i) \).

**Proof:** Let

\[
0 \to K_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to N \to 0
\]

be the finite \( n-(\aleph, Q) \)-presentation of \( N \) and \( K_i = \text{Ker}(F_{i-1} \to F_{i-2}) \) for \( i = 2, \ldots, n \). Then the short exact sequence \( 0 \to K_i \to F_{i-1} \to K_{i-1} \to 0 \) induces the commutative diagram

\[
\begin{array}{c}
0 \to \text{Tor}^R_1(K_{i-1}, \prod_{i \in I}^{\aleph} Q_i) \rightarrow K_i \otimes \prod_{i \in I}^{\aleph} Q_i \rightarrow F_{i-1} \otimes \prod_{i \in I}^{\aleph} Q_i \\
\rightarrow \downarrow f_{i-1} \quad \downarrow \varphi_{K_i} \quad \downarrow \varphi_{F_{i-1}} \\
0 \to \prod_{i \in I}^{\aleph} \text{Tor}^R_1(K_{i-1}, Q_i) \rightarrow \prod_{i \in I}^{\aleph} (K_i \otimes Q_i) \rightarrow \prod_{i \in I}^{\aleph} (F_{i-1} \otimes Q_i)
\end{array}
\]

Then \( f_{n-1} \) is an epimorphism since \( K_n \) is \((\aleph, Q)\)-finitely generated and \( f_{n-2} \) is an isomorphism since \( K_{n-1} \) is \((\aleph, Q)\)-finitely presented, \( K_i \) being finitely presented for \( i < n - 1 \). Now our lemma follows from the fact that \( \text{Tor}^R_{n-1}(N, -) \cong \text{Tor}^R_1(K_{n-2}, -) \) and \( \text{Tor}^R_n(N, -) \cong \text{Tor}^R_1(K_{n-1}, -) \).

**Theorem 1.12.** Let \( n \) be a nonnegative integer. Then the following conditions are equivalent:

(i) \( \prod_{i \in I}^{\aleph} Q \) is \( n \)-flat for every index set \( I \);

(ii) \( \prod_{i \in I}^{\aleph} Q_i \) is \( n \)-flat for every index set \( I \) and any family of \( n \)-flat modules \( Q_i \in \text{Gen}(Q) \);
(iii) $R$ is right $n$-$(\mathcal{R}, Q)$-coherent.

(iv) 
\[
\text{Tor}_n^R(N, \prod_{i \in I} \mathcal{R}Q_i) \cong \prod_{i \in I} \text{Tor}_n^R(N, Q_i)
\]
for every $n$-presented right $R$-module $N$ and any family of left $R$-modules $Q_i \in \text{Gen}(Q)$.

PROOF: (ii) implies (i). Obvious.

(i) implies (iii). Suppose that $N$ is an $n$-presented right $R$-module,
\[
F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to N \to 0
\]
is a finite $n$-presentation of $N$ and $K_i = \text{Ker}(F_{i-1} \to F_{i-2})$ for $i = 2, \ldots, n$. Then the exact sequence $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ induces the following commutative diagram
\[
\begin{array}{cccc}
0 & \to & K_n \otimes \prod_{i \in I} \mathcal{R}Q & \to & F_{n-1} \otimes \prod_{i \in I} \mathcal{R}Q & \to & K_{n-1} \otimes \prod_{i \in I} \mathcal{R}Q & \to & 0 \\
& & \downarrow \varphi_{K_n} & & \downarrow \varphi_{F_{n-1}} & & \downarrow \varphi_{K_{n-1}} & & \\
0 & \to & \prod_{i \in I} \mathcal{R}(K_n \otimes Q) & \to & \prod_{i \in I} (F_{n-1} \otimes Q) & \to & \prod_{i \in I} (K_{n-1} \otimes Q) & \to & 0
\end{array}
\]
Then $\text{Tor}_1^R(K_{n-1}, \prod_{i \in I} \mathcal{R}Q) \cong \text{Tor}_n^R(N, \prod_{i \in I} \mathcal{R}Q) = 0$ by assumption and the upper row is exact. The lower row is exact since $Q$ is flat. Now $\varphi_{F_{n-1}}, \varphi_{K_{n-1}}$ are isomorphisms and consequently $\varphi_{K_n}$ is an isomorphism. Thus $K_n$ is $(\mathcal{R}, Q)$-finitely presented by Lemma 1.7. Hence $N$ is $(n+1)$-$(\mathcal{R}, Q)$-presented.

(iii) implies (iv). It follows immediately from Lemma 1.11(ii).

(iv) implies (ii). Obvious. □

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