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## Totality of product completions

JIŘÍ ADÁMEK, LURDES SOUSA, WALTER THOLEN

*Abstract.* Categories whose Yoneda embedding has a left adjoint are known as total categories and are characterized by a strong cocompleteness property. We introduce the notion of multitotal category  $\mathcal{A}$  by asking the Yoneda embedding  $\mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathit{Set}]$  to be right multiadjoint and prove that this property is equivalent to totality of the formal product completion  $\Pi\mathcal{A}$  of  $\mathcal{A}$ . We also characterize multitotal categories with various types of generators; in particular, the existence of dense generators is inherited by the formal product completion iff measurable cardinals cannot be arbitrarily large.

*Keywords:* multitotal category, multisolid functor, formal product completion

*Classification:* 18A05, 18A22, 18A40

### 1. Introduction

The concept of totality, introduced by Street and Walters [15], is a strong property of categories (implying completeness and cocompleteness — and more, see [14], [11]) which, nevertheless, most “current” categories enjoy. Recall that a category  $\mathcal{A}$  is called *total* if its Yoneda embedding  $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathit{Set}]$  is right adjoint. Since solid functors (i.e., “good” faithful, right adjoint functors)  $\mathcal{A} \rightarrow \mathcal{X}$  lift totality from  $\mathcal{X}$  to  $\mathcal{A}$  (see [17]), totality of  $\mathit{Set}$  and of its small-indexed powers is responsible for the totality of many important types of categories. For example, it allows us to conclude that all cocomplete, cowellpowered categories with a generator are total. These include locally presentable categories and monadic or topological categories over  $\mathit{Set}$ .

In the present paper we investigate the totality of the free product completion  $\Pi\mathcal{A}$  of a category  $\mathcal{A}$  (dual to the free coproduct completion  $\mathit{Fam}\mathcal{A}^{op}$ ). The motivation is to describe the appropriate strong property of categories which are not cocomplete, but only *multicocomplete*, i.e., every small diagram has a multicolimit (as introduced by Y. Diers); examples are the category of linearly ordered sets, the category of fields, the category of local rings, all locally multipresentable categories in the sense of Diers [8], etc. Usually, “multi-concepts” for  $\mathcal{A}$  are easily seen to be equivalent to the corresponding concepts for  $\Pi\mathcal{A}$ , e.g., a category  $\mathcal{A}$  is

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multicocomplete iff  $\Pi\mathcal{A}$  is cocomplete, and a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is *right multiadjoint* iff  $\Pi F : \Pi\mathcal{A} \rightarrow \Pi\mathcal{B}$  is right adjoint, see [7]. We call a category *multitotal* if its Yoneda embedding  $Y_{\mathcal{A}}$  is right multiadjoint; the question naturally arising is whether  $\mathcal{A}$  is multitotal iff  $\Pi\mathcal{A}$  is total. The affirmative answer in Theorem 3.6 of this paper unexpectedly turns out to have a somewhat involved proof. This theorem enables us to establish quite easily analogues of the results of [17], [3] in the “multi” context. Namely, we are able to characterize multitotal categories with various types of generators; in particular the above mentioned examples of multicocomplete categories are all multitotal. In fact, any multicocomplete, cocomplete category with a generator is multitotal. While it is easy to see that the existence of a (strong or regular) generator in a category with a multi-initial object gives the same for its product completion, the corresponding property for dense generator turns out to be more involved. In fact, the question of whether a dense generator exists in  $\Pi(\mathcal{A})$  whenever it exists in  $\mathcal{A}$  depends on the set-theoretical assumption (M) that measurable cardinals are not arbitrarily large. One direction follows from Isbell’s result in [10] that  $Set^{op}$  has a dense generator iff (M) holds; the converse direction is more difficult.

Analogously to the role of solid functors for totality, multisolid functors play an essential role in detecting multitotal categories. Multisolid functors  $U : \mathcal{A} \rightarrow \mathcal{X}$  were already introduced (under a different name) and characterized in [18] by the property that the product-preserving extension  $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\mathcal{X}$  is solid. For  $\mathcal{X}$  multicocomplete and  $\mathcal{A}$  cocomplete, they are precisely the faithful right multiadjoint functors for which  $\mathcal{A}$  is multicocomplete, as shown more recently in [13].

## 2. Review of total categories and solid functors

**2.1.** A diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  in a category  $\mathcal{A}$  is said to be *small-partitioned* [12] if for all  $A \in \mathcal{A}$  the comma category  $(A \downarrow H)$  has only a small set of connected components. Thus, every small diagram is small-partitioned. An example of a (generally large) small-partitioned diagram is the diagram of elements of any functor from  $\mathcal{A}$  to  $Set$ . Recall from [11] that the following conditions are equivalent for  $\mathcal{A}$ :

- (i)  $\mathcal{A}$  is total, i.e., the Yoneda embedding  $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, Set]$  has a left adjoint;
- (ii)  $\text{colim } H$  exists in  $\mathcal{A}$  whenever the diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  is small-partitioned;
- (iii)  $\text{colim } H$  exists in  $\mathcal{A}$  whenever, for all  $A \in \mathcal{A}$ ,  $\text{colim } \mathcal{A}(A, H-)$  exists in  $Set$ .

**2.2.** Total categories are trivially cocomplete (i.e., have colimits of all small diagrams), but they are also complete — indeed, they are “as complete as a category with small hom-sets can possibly be”. In fact, recall that a category  $\mathcal{A}$  is called *hypercomplete* [5] if every diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  for which  $\text{lim } \mathcal{A}(A, H-)$  exists in  $Set$  for all  $A \in \mathcal{A}$ , has a limit in  $\mathcal{A}$ . The following conditions are equivalent ([4]):

- (i)  $\mathcal{A}$  is hypercomplete;

- (ii)  $\lim H$  exists in  $\mathcal{A}$  for every diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  with the property that for all  $A \in \mathcal{A}$ , there is only a small set of cones  $\Delta A \rightarrow H$  in  $\mathcal{A}$ .

Every total category is hypercomplete, see [4].

**2.3.** Since solid functors detect totality, we briefly recall this notion. For a functor  $U : \mathcal{A} \rightarrow \mathcal{X}$ , a  $U$ -sink  $\sigma$  with codomain  $X \in \mathcal{X}$  is a (possibly large) family of  $\mathcal{A}$ -objects  $A_i$  and of  $\mathcal{X}$ -morphisms  $x_i : UA_i \rightarrow X$  ( $i \in I$ ). Let  $(X \downarrow U)_\sigma$  be the full subcategory of  $(X \downarrow U)$  of all objects  $y : X \rightarrow UB$  such that for every  $i \in I$  there exists a morphism  $f_i : A_i \rightarrow B$  in  $\mathcal{A}$  with  $Uf_i = y \cdot x_i$ . The functor  $U$  is *solid* (formerly semi-topological [16]) if  $U$  is faithful and if for every  $U$ -sink  $\sigma$  the category  $(X \downarrow U)_\sigma$  has an initial object. The following conditions are equivalent for every functor  $U : \mathcal{A} \rightarrow \mathcal{X}$  (see [16], [4], [11]):

- (i)  $U$  is solid;
- (ii)  $U$  has a left adjoint, and there is a class  $\mathcal{E}$  of morphisms in  $\mathcal{A}$  containing all isomorphisms and being closed under composition with them, such that
  1. the counits of  $U$  belong to  $\mathcal{E}$ ;
  2.  $\mathcal{A}$  is  $\mathcal{E}$ -cocomplete, that is: a. the pushout of a morphism in  $\mathcal{E}$  along any morphism exists in  $\mathcal{A}$  and belongs to  $\mathcal{E}$ , and b. the cointersection of a (possibly large) family of morphisms in  $\mathcal{E}$  with common domain exists in  $\mathcal{A}$  and belongs to  $\mathcal{E}$ .

In particular, every faithful right adjoint functor  $U : \mathcal{A} \rightarrow \mathcal{X}$  defined on a cocomplete and cowellpowered category  $\mathcal{A}$  is solid. The connection with totality is given by the following

**Theorem.** *Let  $U : \mathcal{A} \rightarrow \mathcal{X}$  be a functor. Then:*

- (1) *If  $\mathcal{X}$  is total and  $U$  solid, then also  $\mathcal{A}$  is total (see [17]).*
- (2) *If  $\mathcal{A}$  is total and  $U$  faithful and right adjoint, then  $U$  is solid (see [3]).*

Recall that a *generator* in a category  $\mathcal{A}$  is a set  $\mathcal{G}$  of objects such that the canonical functor  $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$ ,  $A \mapsto (\mathcal{A}(G, A))_{G \in \mathcal{G}}$ , is faithful. Since *Set* and all its small-indexed powers are total, the above theorem shows in particular:

**Corollary** ([4]). *Every cocomplete, cowellpowered category with a generator is total.*

**2.4.** Recall that the *free product completion*  $\Pi\mathcal{A}$  of a category  $\mathcal{A}$  has objects  $A = (A_i)_{i \in I}$  given by small-indexed families of  $\mathcal{A}$ -objects  $A_i$ , and a morphism  $f : A \rightarrow B = (B_j)_{j \in J}$  in  $\Pi\mathcal{A}$  is given by a function  $\varphi : J \rightarrow I$  and a family  $f_j : A_{\varphi(j)} \rightarrow B_j$  ( $j \in J$ ) of  $\mathcal{A}$ -morphisms (with the obvious composition and identity maps), see [7]. Writing  $SA = I$  and  $Sf = \varphi$ , one has a functor

$$S : (\Pi\mathcal{A})^{op} \longrightarrow \text{Set}.$$

Whenever necessary we write  $S_{\mathcal{A}}$  instead of  $S$  for distinction. In the terminology of [6],  $\Pi\mathcal{A} = (\text{Fam}(\mathcal{A}^{op}))^{op}$ . We denote by

$$J_{\mathcal{A}} : \mathcal{A} \longrightarrow \Pi\mathcal{A}$$

the canonical embedding which identifies objects of  $\mathcal{A}$  with singleton families. Every functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  into a category with products has a product-preserving extension  $\overline{F} : \Pi\mathcal{A} \rightarrow \mathcal{B}$  which is unique up to natural isomorphism; hence,  $[\mathcal{A}, \mathcal{B}]$  is equivalent to the full subcategory of product-preserving functors in  $[\Pi\mathcal{A}, \mathcal{B}]$ .

**2.5.** A *multicolimit* of a diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  in  $\mathcal{A}$  is given by a colimit of  $J_{\mathcal{A}}H$  in  $\Pi\mathcal{A}$ ; hence, it is a small-indexed family  $(L_i)_{i \in I}$  of  $\mathcal{A}$ -objects together with cocones  $\lambda_i : H \rightarrow \Delta L_i$ , such that every cocone  $H \rightarrow \Delta B$  in  $\mathcal{A}$  factors uniquely through  $\lambda_i$ , for a unique  $i \in I$ . Already Diers [7] proved that every small diagram in  $\mathcal{A}$  has a multicolimit if and only if  $\Pi\mathcal{A}$  is cocomplete. As we shall deal with large diagrams, we need a precise analysis of this result. For  $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$  first assume that the limit  $K = \lim SH^{op}$  exists in  $Set$ ; hence, every element  $\alpha \in K$  is given by a compatible family of elements  $\alpha_D \in SHD$ ,  $D \in \mathcal{D}$ . Every  $\alpha \in K$  defines a diagram

$$H_{\alpha} : \mathcal{D} \rightarrow \mathcal{A} \quad \text{with} \quad H_{\alpha}D = (HD)_{\alpha_D}, \quad H_{\alpha}d = (Hd)_{\alpha_{D'}},$$

for all  $d : D \rightarrow D'$  in  $\mathcal{D}$ . (Note that  $H_{\alpha}d$  is well-defined since  $(SHd)(\alpha_{D'}) = \alpha_D$ .)

**Lemma** ([7]). *A diagram  $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$  has a colimit in  $\Pi\mathcal{A}$  if  $SH^{op}$  has a limit  $K$  in  $Set$  and  $H_{\alpha}$  has a multicolimit in  $\mathcal{A}$ , for every  $\alpha \in K$ .*

**PROOF:** For every  $\alpha \in K$ , a multicolimit of  $H_{\alpha}$  is given by a small family of cocones  $\lambda_{\alpha,i} : H_{\alpha} \rightarrow \Delta L_{\alpha,i}$ ,  $i \in I_{\alpha}$ . With  $L = (L_{\alpha,i})_{\alpha \in K, i \in I_{\alpha}}$ , this defines a cocone  $\lambda : H \rightarrow \Delta L$  when we put

$$(\lambda_D)_{\alpha,i} = (\lambda_{\alpha,i})_D : (HD)_{\alpha_D} = H_{\alpha}D \rightarrow L_{\alpha,i}$$

for every  $D \in \mathcal{D}$ . In order to see that every cocone  $\beta : H \rightarrow \Delta B$  factors uniquely through  $\lambda_{\alpha,i}$ , for a unique pair  $(\alpha, i)$ , we may without loss of generality assume  $B \in \mathcal{A}$ . The naturality of the family  $\beta_D$  ( $D \in \mathcal{D}$ ) defines a uniquely determined element  $\alpha \in K$ , and the morphisms  $\beta_D : (HD)_{\alpha_D} \rightarrow B$  define in fact a cocone  $\overline{\beta} : H_{\alpha} \rightarrow \Delta B$ . Hence, there are uniquely defined  $i \in I_{\alpha}$  and  $f : L_{\alpha,i} \rightarrow B$  in  $\mathcal{A}$  with  $\overline{\beta} = \Delta f \cdot \lambda_{\alpha,i}$ , which gives the desired factorization  $\beta = \Delta f \cdot \lambda$ .  $\square$

**2.6.** A functor  $U : \mathcal{A} \rightarrow \mathcal{X}$  is *right multiadjoint* if its extension

$$\Pi U : \Pi\mathcal{A} \longrightarrow \Pi\mathcal{X}$$

with  $\Pi U \cdot J_{\mathcal{A}} = J_{\mathcal{X}} \cdot U$  is right adjoint ([7]). Analogously one defines  $U$  to be *multisolid* (“strongly localizing semitopological” in [18]) if  $\Pi U$  is solid; this means that  $U$  is faithful, and that for every  $U$ -sink  $\sigma$  as in 2.3 the category  $(X \downarrow U)_{\sigma}$  has a multi-initial object (i.e., a multicolimit of the empty diagram).

The characterization 2.3 of solid functors should lead to a characterization of multisolid functors when we exploit it for  $\Pi U$  in lieu of  $U$ . In fact, let us agree that a *counit* of a right multiadjoint functor  $U : \mathcal{A} \rightarrow \mathcal{X}$  at  $A \in \mathcal{A}$  is simply the counit

of  $\Pi U$  at  $A \in \Pi \mathcal{A}$ . For a class  $\mathcal{E}$  of morphisms in  $\mathcal{A}$  we call  $\mathcal{A}$  *multi- $\mathcal{E}$ -cocomplete* if the multipushout of a morphism in  $\mathcal{E}$  along any morphism exists in  $\mathcal{A}$ , with every component of it belonging to  $\mathcal{E}$ , and if the multicointersection of any family of morphisms in  $\mathcal{E}$  with common domain exists in  $\mathcal{A}$ , with every component of it belonging to  $\mathcal{E}$ . Hence, if  $\mathcal{A}$  is multi- $\mathcal{E}$ -cocomplete, then  $\Pi \mathcal{A}$  is  $\mathcal{E}^\Pi$ -cocomplete, with  $\mathcal{E}^\Pi$  those morphisms of  $\Pi \mathcal{A}$  whose components lie in  $\mathcal{E}$ ; conversely if  $\Pi \mathcal{A}$  is  $\mathcal{F}$ -cocomplete for a class of morphisms in  $\Pi \mathcal{A}$ , then  $\mathcal{A}$  is multi- $\mathcal{F}^1$ -cocomplete, with  $\mathcal{F}^1$  the class of those morphisms of  $\mathcal{A}$  which appear as a component of some morphism in  $\mathcal{F}$ . These observations prove the following results which essentially appeared in [13]:

**Proposition.** *Equivalent are for a functor  $U : \mathcal{A} \rightarrow \mathcal{X}$ :*

- (i)  $U : \mathcal{A} \rightarrow \mathcal{X}$  is *multisolid*;
- (ii)  $U : \mathcal{A} \rightarrow \mathcal{X}$  is *right multiadjoint*, and there is a class  $\mathcal{E}$  of morphisms in  $\mathcal{A}$  containing all isomorphisms and being closed under composition with them, such that
  1. the counits of  $U$  lie in  $\mathcal{E}$ ,
  2.  $\mathcal{A}$  is multi- $\mathcal{E}$ -cocomplete.

**Corollary** ([13]). *If  $\mathcal{X}$  is multicocomplete and  $\mathcal{A}$  cowellpowered, a faithful functor  $U : \mathcal{A} \rightarrow \mathcal{X}$  is multisolid if and only if  $\mathcal{A}$  is multicocomplete.*

### 3. Multitotal categories

**3.1 Definition.** A category  $\mathcal{A}$  is called *multitotal* if the Yoneda embedding  $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, Set]$  is right multiadjoint.

It is easy to prove a “multiversion” of the characterization 2.1 of total categories:

**3.2 Proposition.** *The following conditions are equivalent for a category  $\mathcal{A}$ :*

- (i)  $\mathcal{A}$  is *multitotal*;
- (ii) every *small-partitioned diagram*  $H : \mathcal{D} \rightarrow \mathcal{A}$  has a *multicolimit* in  $\mathcal{A}$ ;
- (iii) every diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  for which  $\text{colim } \mathcal{A}(A, H-)$  exists in *Set* for all  $A \in \mathcal{A}$ , has a *multicolimit* in  $\mathcal{A}$ .

**PROOF:** (i) $\Rightarrow$ (ii). A small-partitioned diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  defines a functor  $E : \mathcal{A}^{op} \rightarrow Set$  which assigns to an object  $A$  the set of connected components of  $(A \downarrow H)$ . A multicolimit  $\lambda_i : H \rightarrow \Delta L_i$  ( $i \in I$ ) is obtained from a  $\Pi Y_{\mathcal{A}}$ -universal arrow  $(\eta_i : E \rightarrow Y_{\mathcal{A}} L_i)_{i \in I}$  for  $E \in [\mathcal{A}^{op}, Set]$ : one just evaluates  $(\eta_i)_{HD} : EHD \rightarrow \mathcal{A}(HD, L_i)$  at the component  $(D, 1_{HD})$  in  $(HD \downarrow H)$  to define  $(\lambda_i)_D$ , for every  $i \in I$  and  $D \in \mathcal{D}$ .

(ii) $\Rightarrow$ (i). For any  $E \in [\mathcal{A}^{op}, Set]$  one considers a multicolimit  $(\lambda_i)_{i \in I}$  of the small-partitioned forgetful functor  $H : \text{el } E \rightarrow \mathcal{A}$ , where  $\text{el } E$  is the “element category” with objects  $(A, x)$ ,  $A \in \mathcal{A}$ ,  $x \in EA$ . Then the  $\Pi Y_{\mathcal{A}}$ -universal arrow

$(\eta_i)_{i \in I}$  for  $E$  is obtained as  $(\eta_i)_A(x) = (\lambda_i)_{(A,x)}$ , for every  $i \in I$ ,  $A \in \mathcal{A}$  and  $x \in EA$ .

(ii)  $\Leftrightarrow$  (iii) follows from the fact that, given a diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  and  $A \in \mathcal{A}$ ,  $\text{colim } \mathcal{A}(A, H-)$  exists in  $\text{Set}$  iff  $(A \downarrow H)$  has just a set of connected components.  $\square$

**3.3 Corollary.** *If  $\Pi\mathcal{A}$  is total, then  $\mathcal{A}$  is multitotal.*

PROOF: We check condition (ii) of 3.2. Given a small-partitioned diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$ , it is easy to see that then also  $J_{\mathcal{A}}H : \mathcal{D} \rightarrow \Pi\mathcal{A}$  is small-partitioned, so that  $\text{colim } J_{\mathcal{A}}H$  exists in  $\Pi\mathcal{A}$ , by hypothesis. But this is, by definition, a multicolimit of  $H$  in  $\mathcal{A}$ .  $\square$

**3.4.** The question which remains is whether multitotality of  $\mathcal{A}$  is also a sufficient condition for  $\Pi\mathcal{A}$  to be total. In other words: does right adjointness of  $\Pi Y_{\mathcal{A}}$  imply right adjointness of  $Y_{\Pi\mathcal{A}}$ ? For every category  $\mathcal{A}$  denote by  $[(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi}$  the full subcategory of  $[(\Pi\mathcal{A})^{op}, \text{Set}]$  of all coproduct-preserving functors; then one has a functor  $\Sigma$  which makes the upper triangle of the diagram (1) below commutative (up to natural isomorphism):

$$(1) \quad \begin{array}{ccc} & \Pi\mathcal{A} & \\ \Pi Y_{\mathcal{A}} \swarrow & & \searrow Y_{\Pi\mathcal{A}} \\ \Pi[\mathcal{A}]^{op}, \text{Set} & \xrightarrow{\Sigma} & [(\Pi\mathcal{A})^{op}, \text{Set}] \\ \uparrow & & \uparrow \\ [\mathcal{A}]^{op}, \text{Set} & \xrightarrow{\sim} & [(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi} \end{array}$$

$\Sigma$  is the product-preserving extension of the functor that assigns to  $E \in [\mathcal{A}^{op}, \text{Set}]$  the functor  $\Sigma E$  with  $(\Sigma E)(C_k)_K = \prod_{k \in K} EC_k$ ; of course,  $\Sigma E \in [(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi}$ . It is clear that the restriction of  $\Sigma$  creates an equivalence of categories, as indicated in (1). Now it is a straightforward exercise to show that the existence of a left adjoint to  $\Pi Y_{\mathcal{A}}$  gives a  $(Y_{\Pi\mathcal{A}})$ -universal arrow for every  $F \in [\mathcal{A}^{op}, \text{Set}]_{\Pi}$ , hence a “partial left adjoint” to  $Y_{\Pi\mathcal{A}}$ . We now establish the existence of “a totally defined” left adjoint.

**3.5 Example.** The category  $\Pi\text{Set}$  is total. To see this, we check the conditions of Corollary 2.3. First observe that  $\Pi\text{Set}$  is certainly cocomplete (see 2.5). Now, a morphism  $f : A \rightarrow B$  in  $\Pi\text{Set}$  is an epimorphism if and only if  $\varphi = Sf : SB \rightarrow SA$  is injective and every map  $f_j : A_{\varphi(j)} \rightarrow B_j$  ( $j \in J = SB$ ) is epic in  $\text{Set}$  (see 6.3 in [18]). This characterization shows that wellpoweredness and cowellpoweredness of  $\text{Set}$  give cowellpoweredness of  $\Pi\text{Set}$ . Finally,  $\Pi\text{Set}$  has a (single-object) generator, namely the triple  $(1, \emptyset, \emptyset)$ : the singleton set 1 and (one copy of) the empty set are needed to distinguish distinct morphisms  $f, g : A \rightarrow B$  in  $\Pi\text{Set}$  with  $Sf = Sg$ , and two copies of  $\emptyset$  are needed to distinguish  $f$  and  $g$  in case  $Sf \neq Sg$ .

**3.6 Proposition.** *The functor  $S_{\mathcal{A}} : (\Pi\mathcal{A})^{op} \rightarrow \mathcal{S}et$*

- *has a left adjoint  $L_{\mathcal{A}}$  if  $\mathcal{A}$  has a terminal object;*
- *has a right adjoint  $R_{\mathcal{A}}$  if  $\mathcal{A}$  has a multi-initial object.*

PROOF: For 1 terminal in  $\mathcal{A}$  and every set  $I$ , let  $L_{\mathcal{A}}I = (1)_I$  be the constant  $I$ -indexed family with value 1. Then  $\text{id}_I : I \rightarrow S_{\mathcal{A}}L_{\mathcal{A}}I$  serves as an  $S_{\mathcal{A}}$ -universal arrow. For an initial object  $O = (O_t)_{t \in T}$  in  $\Pi\mathcal{A}$  and every set  $I$ , let  $R_{\mathcal{A}}I = (O_i)_{(t,i) \in T \times I}$ . Then the projection  $S_{\mathcal{A}}R_{\mathcal{A}}I \rightarrow I$  is an  $S_{\mathcal{A}}$ -couniversal arrow.  $\square$

**3.7 Theorem.** *A category  $\mathcal{A}$  is multitotal if and only if  $\Pi\mathcal{A}$  is total.*

PROOF: We need to prove necessity (see 3.3). To this end we assume  $\mathcal{A}$  to be multitotal and prove totality of  $\Pi\mathcal{A}$  by checking condition 2.1(ii). Hence, let  $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$  be a small-partitioned diagram in  $\Pi\mathcal{A}$ . From 3.6 one has adjoint situations

$$L_{\mathcal{S}et} \dashv S_{\mathcal{S}et} \quad \text{and} \quad S_{\mathcal{A}} \dashv R_{\mathcal{A}},$$

and we can form

$$\overline{H} = L_{\mathcal{S}et}^{op} S_{\mathcal{A}}^{op} H : \mathcal{D} \rightarrow \Pi\mathcal{S}et.$$

By adjointness, for all  $X \in \Pi\mathcal{S}et$  there are canonical isomorphisms

$$(X \downarrow \overline{H}) \cong (S_{\mathcal{S}et}X \downarrow S_{\mathcal{A}}^{op}H) \cong (R_{\mathcal{A}}S_{\mathcal{S}et}X \downarrow H),$$

so that  $\overline{H}$  must be small-partitioned since  $H$  is. Consequently,  $\text{colim} \overline{H}$  exists in  $\Pi\mathcal{S}et$ , by 3.5. This is a limit of  $\overline{H}^{op}$  in  $(\Pi\mathcal{S}et)^{op}$ , which is preserved by the right adjoint functor  $S_{\mathcal{S}et}$ . Since

$$S_{\mathcal{S}et}\overline{H}^{op} = S_{\mathcal{A}}H^{op},$$

we see that a limit of  $S_{\mathcal{A}}H^{op}$  exists in  $\mathcal{S}et$ .

In order to see that  $\text{colim} H$  exists in  $\Pi\mathcal{A}$ , according to Lemma 2.5 it is sufficient to show that for every  $\alpha \in K = \lim S_{\mathcal{A}}H^{op}$  in  $\mathcal{S}et$ , the diagram  $H_{\alpha} : \mathcal{D} \rightarrow \mathcal{A}$  has a multicolimit in  $\mathcal{A}$ . In fact, since  $\mathcal{A}$  is multitotal, thanks to 3.2 it suffices to prove that  $H_{\alpha}$  is small-partitioned. Hence, given  $A \in \mathcal{A}$ , we must show that  $A \downarrow H_{\alpha}$  has only a small set of connected components.

Let  $O = (O_t)_{t \in T}$  be initial in  $\Pi\mathcal{A}$  and form  $\overline{A} = A \times O$  in  $\Pi\mathcal{A}$ . Every object  $(D, f)$ , with  $f : A \rightarrow H_{\alpha}D$ , of  $(A \downarrow H_{\alpha})$  defines an object  $(D, \overline{f})$  of  $(\overline{A} \downarrow H)$ , as follows: for each  $i \in S_{\mathcal{A}}HD - \{\alpha_D\}$ , let  $\overline{f}_i : O_t \rightarrow (HD)_i$  be the morphism determined by initiality of  $O$ , and for  $i = \alpha_D$  let  $\overline{f}_i = f$ . Since  $H$  is small-partitioned,  $(\overline{A} \downarrow H)$  has only a set of connected components. Hence, it now suffices to show that, for any pair of objects  $(D, f)$ ,  $(D', f')$  in  $(A \downarrow H_{\alpha})$ , any zig-zag of  $(\overline{A} \downarrow H)$  between  $(D, \overline{f})$ ,  $(D', \overline{f}')$  gives a zig-zag of  $(A \downarrow H_{\alpha})$  between  $(D, f)$ ,  $(D', f')$ .



Consider the first step of the zig-zag between  $\bar{f}$ ,  $\bar{f}'$ , which is given by one of the following commutative triangles:

$$(2) \quad \begin{array}{ccc} & \bar{A} & \\ \bar{f} \swarrow & & \searrow g_1 \\ HD & \xrightarrow{Hd_1} & HD_1 \end{array} \qquad \begin{array}{ccc} & \bar{A} & \\ \bar{f} \swarrow & & \searrow g_1 \\ HD & \xleftarrow{Hd_1} & HD_1 \end{array}$$

(a) Case  $g_1 = Hd_1 \cdot \bar{f}$ . Since  $(S_{\mathcal{A}}Hd_1)(\alpha_{D_1}) = \alpha_D$ , we know that  $S_{\mathcal{A}}g_1 = S_{\mathcal{A}}\bar{f} \cdot S_{\mathcal{A}}Hd_1$  takes  $\alpha_{D_1}$  to the index of  $A$ . Hence, the morphism  $(g_1)_{\alpha_{D_1}} : A \rightarrow (HD_1)_{\alpha_{D_1}} = H_{\alpha}D_1$  gives us the first step of the zig-zag between  $(D, f)$ ,  $(D', f')$ .

(b) Case  $\bar{f} = Hd_1 \cdot g_1$ . Since  $(S_{\mathcal{A}}Hd_1)(\alpha_D) = \alpha_{D_1}$ , we know that, again,  $S_{\mathcal{A}}g_1$  takes  $\alpha_{D_1}$  to the index of  $A$ . Hence, also in this case  $(g_1)_{\alpha_{D_1}}$  gives the first step of the zig-zag between  $(D, f)$ ,  $(D', f')$ .

Inductively one finishes the proof of the last claim, so that the proof of the Theorem is complete.  $\square$

**3.8 Corollary** ([18]). *Every multitotal category  $\mathcal{A}$  is connectively hypercomplete, that is, every connected diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  for which  $\lim \mathcal{A}(A, H-)$  exists in  $\mathbf{Set}$  for all  $A \in \mathcal{A}$ , has a limit in  $\mathcal{A}$ .*

PROOF: With 3.7 and 2.2,  $\Pi\mathcal{A}$  is total and therefore hypercomplete. Consider a connected diagram  $H : \mathcal{D} \rightarrow \mathcal{A}$  with the indicated property; equivalently, with the property that there is only a small set of cones  $\Delta A \rightarrow H$  for every  $A \in \mathcal{A}$ . Since  $\mathcal{D}$  is connected, it is easy to see that there is only a small set of cones  $\Delta A \rightarrow J_{\mathcal{A}}H$  for every  $A \in \Pi\mathcal{A}$ , so that  $(L_i)_{i \in I} = \lim J_{\mathcal{A}}H$  exists in  $\Pi\mathcal{A}$ . Again, connectedness of  $\mathcal{D}$  determines a unique index  $i_0$  such that  $L_{i_0} = \lim H$ .  $\square$

**Corollary.** *Every multitotal category  $\mathcal{A}$  has equalizers, pullbacks and intersections of (arbitrarily large) families of monomorphisms.*

**3.9.** Theorem 3.7 makes it easy to establish the interrelationship between multitotal categories and multisolid functors:

**Theorem.** *Let  $U : \mathcal{A} \rightarrow \mathcal{X}$  be a functor. Then:*

- (1) *if  $\mathcal{X}$  is multitotal and  $U$  multisolid, then also  $\mathcal{A}$  is multitotal;*
- (2) *if  $\mathcal{A}$  is multitotal and  $U$  faithful and right multiadjoint, then  $U$  is multisolid.*

PROOF: (1) The hypotheses together with Theorem 3.7 imply that  $\Pi\mathcal{X}$  is total and  $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\mathcal{X}$  is solid, so that  $\Pi\mathcal{A}$  is total, see Theorem 2.5. Hence,  $\mathcal{A}$  is multitotal.

(2) Using again Theorem 3.7 we see that  $\Pi\mathcal{A}$  is total and  $\Pi U$  is faithful and right adjoint, by hypothesis, so that  $\Pi U$  is solid by Theorem 2.5. Hence,  $U$  is multisolid.  $\square$

**3.10.** If  $\mathcal{A}$  has a generator  $\mathcal{G}$ , one may apply Proposition 2.6 and then Theorem 3.9(1) to the canonical functor  $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$  to obtain:

**Corollary.** *Every cowellpowered category with a generator and multicolimits is multitotal.*

In the next section, we study multitotal categories with various types of generators.

#### 4. Multitotal categories with generators

**4.1.** Recall that a generator  $\mathcal{G}$  of  $\mathcal{A}$  is *strong* if a morphism  $f : A \rightarrow B$  in  $\mathcal{A}$  is an isomorphism whenever all maps  $\mathcal{A}(G, f) : \mathcal{A}(G, A) \rightarrow \mathcal{A}(G, B)$  are bijective,  $G \in \mathcal{G}$ ; in other words, if the canonical functor  $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$  is conservative. In order to relate a (strong) generator of a category  $\mathcal{A}$  to the category  $\Pi\mathcal{A}$ , denote by  $O$  an initial object of  $\Pi\mathcal{A}$  (i.e., a multi-initial object of  $\mathcal{A}$ ), provided that it exists. For every  $A \in \mathcal{A}$ , we put  $\overline{A} = A \times O$  (product in  $\Pi\mathcal{A}$ ). For  $\mathcal{G}$  a set of objects in  $\mathcal{A}$ , let

$$\overline{\mathcal{G}} = \{\overline{G} \mid G \in \mathcal{G}\} \cup \{O \times O\};$$

and for  $\mathcal{H}$  a set of objects in  $\Pi\mathcal{A}$ , let  $\mathcal{H}^1$  be the set of objects of  $\mathcal{A}$  which are components of objects in  $\mathcal{H}$ . Then we have the following:

**Lemma.** (1) *If  $\mathcal{G}$  is a (strong) generator of  $\mathcal{A}$  and if  $\mathcal{A}$  has a multi-initial object, then  $\overline{\mathcal{G}}$  is a (strong) generator of  $\Pi\mathcal{A}$ .*

(2) *If  $\mathcal{H}$  is a (strong) generator of  $\Pi\mathcal{A}$ , then  $\mathcal{H}^1$  is a (strong) generator of  $\mathcal{A}$ .*

PROOF: (1) If  $\mathcal{G}$  is a generator of  $\mathcal{A}$  and if  $p, q : A \rightarrow B$  are distinct morphisms of  $\Pi\mathcal{A}$ , then there is  $k \in SB$  such that either  $Sp(k) \neq Sq(k)$ , or  $Sp(k) = l = Sq(k)$  and  $p_k \neq q_k : A_l \rightarrow B_k$ . In the former case, consider  $h : O \times O \rightarrow A$  with  $Sh(Sp(k)) \neq Sh(Sq(k))$ , then  $ph \neq qh$ ; in the latter case, choose  $h_0 : G \rightarrow A_l$  with  $p_k h_0 \neq q_k h_0$  and  $G \in \mathcal{G}$ , and let  $h : \overline{G} \rightarrow A$  be a morphism with  $l$ -component  $h_0$ , then  $ph \neq qh$ .

If  $\mathcal{G}$  is a strong generator of  $\mathcal{A}$  and if  $f : A \rightarrow B$  is a morphism in  $\Pi\mathcal{A}$  such that  $\Pi\mathcal{A}(G, f)$  is bijective for all  $G \in \overline{\mathcal{G}}$ , then (a)  $Sf$  is bijective since, on the one hand, the bijectivity of  $\Pi\mathcal{A}(O \times O, f)$  is clearly equivalent to the bijectivity of  $\text{Set}^{op}(2, Sf)$ , and, on the other hand,  $\text{Set}^{op}(2, Sf)$  is surjective (injective) iff  $Sf$  is injective (surjective, respectively); and (b) each component  $f_k$  of  $f$  is an isomorphism since, thanks to the morphisms from  $G \times O$ ,  $G \in \mathcal{G}$ , to  $B$ ,  $\mathcal{A}(G, f_k)$  is bijective. Thus  $f$  is an isomorphism, hence,  $\overline{\mathcal{G}}$  is a strong generator.

(2) If  $\mathcal{H}$  is a generator of  $\Pi\mathcal{A}$  and  $p, q : A \rightarrow B$  are distinct morphisms of  $\mathcal{A}$ , choose  $h : H \rightarrow A$  with  $ph \neq qh$  in  $\Pi\mathcal{A}$ . The unique component  $h_1 : H_1 \rightarrow A$  of  $h$  then fulfills  $ph_1 \neq qh_1$  and  $H_1 \in \mathcal{H}^1$ . If  $\mathcal{H}$  is a strong generator of  $\Pi\mathcal{A}$ , it follows easily that also  $\mathcal{H}^1$  is a strong generator of  $\mathcal{A}$ , from the observation that, given  $H \in \mathcal{H}$  and a morphism  $f : A \rightarrow B$  in  $\mathcal{A}$ ,  $\Pi\mathcal{A}(H, f)$  is essentially the map  $\prod_{i \in SH} \mathcal{A}(H_i, f)$ .  $\square$

**4.2.** We use the notation of [GU] and denote by  $\widetilde{\text{Set}}$  any small-indexed discrete power of  $\text{Set}$ .

**Theorem.** *The following conditions on a category  $\mathcal{A}$  are equivalent:*

- (i)  $\mathcal{A}$  is multitotal and has a generator,
- (ii)  $\Pi\mathcal{A}$  is total and has a generator,
- (iii)  $\Pi\mathcal{A}$  admits a solid functor into  $\widetilde{\text{Set}}$ ,
- (iv)  $\mathcal{A}$  admits a multisolid functor into  $\widetilde{\text{Set}}$ .

PROOF: (i) $\Rightarrow$ (ii) by 3.7 and 4.1(1). (ii) $\Rightarrow$ (iii) by (2) of Theorem 2.3. (iii) $\Rightarrow$ (iv):  $\mathcal{A}$  is always multireflective in  $\Pi\mathcal{A}$ ; hence, composition of the solid functor  $\Pi\mathcal{A} \rightarrow \widetilde{\text{Set}}$  with  $J_{\mathcal{A}}$  gives a multisolid functor  $\mathcal{A} \rightarrow \widetilde{\text{Set}}$ . (iv) $\Rightarrow$ (i):  $\mathcal{A}$  is multitotal by 3.9(1); furthermore, since  $\Pi\mathcal{A}$  is solid over  $\Pi\widetilde{\text{Set}}$  and since  $\Pi\widetilde{\text{Set}}$  has a generator by 4.1(1), also  $\Pi\mathcal{A}$  and then  $\mathcal{A}$  has a generator, by 4.1(2).  $\square$

**4.3 Corollary.** *The following conditions on a category  $\mathcal{A}$  are equivalent:*

- (i)  $\mathcal{A}$  is multitotal and has a strong generator,
- (ii)  $\Pi\mathcal{A}$  is total and has a strong generator,
- (iii)  $\Pi\mathcal{A}$  admits a solid, conservative functor into  $\widetilde{\text{Set}}$ ,
- (iv)  $\mathcal{A}$  admits a multisolid, conservative functor into  $\widetilde{\text{Set}}$ .

PROOF: Thanks to the strong generator part of Lemma 4.1, one can mimic the proof of 4.2.  $\square$

**4.4.** A generator  $\mathcal{G}$  of  $\mathcal{A}$  is *regular* if the canonical functor  $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$  reflects regular epimorphisms; equivalently,  $f : A \rightarrow B$  in  $\mathcal{A}$  is a regular epimorphism whenever every map  $\mathcal{A}(G, f)$ ,  $G \in \mathcal{G}$ , is surjective.

**Remark.** Every cocomplete category with a regular generator is total, see [4]. These are, by [3], precisely the reflective subcategories of monadic categories over  $\widetilde{\text{Set}}$ .

**Lemma.**

- (1) *Let  $\mathcal{A}$  have a multi-initial object. Then:*
  - a. *A morphism  $f : A \rightarrow B$  in  $\Pi\mathcal{A}$  is a regular epimorphism if its components are regular epimorphisms in  $\mathcal{A}$  and if  $Sf$  is injective; these conditions are also necessary whenever  $\mathcal{A}$  has a terminal object.*
  - b. *If  $\mathcal{G}$  is a regular generator of  $\mathcal{A}$ , then  $\overline{\mathcal{G}}$  is a regular generator of  $\Pi\mathcal{A}$ .*
- (2) *If  $\mathcal{H}$  is a regular generator of  $\Pi\mathcal{A}$ , then  $\mathcal{H}^1$  is a regular generator of  $\mathcal{A}$ .*

PROOF: (1)a. Put  $\varphi = Sf$ . By hypothesis, each  $f_j$  is a coequalizer of a pair  $p_j, q_j : K_j \rightarrow A_{\varphi(j)}$  in  $\mathcal{A}$ . Put  $I = SA$ ,  $J = SB$ , and  $K = (K_j)_{j \in J}$  and define  $p, q : K \times O \times O \rightarrow A$  by letting  $Sp$  map  $I - \varphi(J)$  into the first copy of  $SO$  and  $Sq$  into the second one. Then  $f$  is a coequalizer of  $p, q$  in  $\Pi\mathcal{A}$ . Conversely, assuming

$f$  to be the coequalizer of some pair  $g, h : L \rightarrow A$  in  $\Pi\mathcal{A}$ , we first consider  $j_1, j_2 \in J$  with  $\varphi(j_1) = \varphi(j_2) = i$ . With any commutative square

$$\begin{array}{ccc}
 & & B_{j_1} \\
 & \nearrow^{f_{j_1}} & \\
 A_i & & \\
 & \searrow_{f_{j_2}} & \\
 & & B_{j_2} \\
 & & \nearrow \\
 & & C
 \end{array}$$

in  $\mathcal{A}$  (which certainly exists when  $\mathcal{A}$  has a terminal object) one obtains morphisms  $s, t : B \rightarrow C$  in  $\Pi\mathcal{A}$  with  $sf = tf$ , hence  $s = t$  and then  $j_1 = j_2$ . Injectivity of  $\varphi$  now gives that each  $f_j$  is a coequalizer of  $g_{\varphi(j)}, h_{\varphi(j)}$  in  $\mathcal{A}$ .

(1)b. and (2) follow from an easy analysis of the proof of 4.1.  $\square$

**Remark.** For categories  $\mathcal{A}$  which do not have a terminal object the converse implication of (1)a. above is false, in general: suppose that a pair  $p, q : A \rightarrow B$  in  $\mathcal{A}$  has a multicoequalizer  $c_k : B \rightarrow C_k (k \in K)$  in  $\mathcal{A}$ . Then the corresponding coequalizer  $c : B \rightarrow C = (C_k)_{k \in K}$  in  $\Pi\mathcal{A}$  is a regular epimorphism and  $Sc$  is a constant function.

**4.5 Theorem.** *The following conditions on a category  $\mathcal{A}$  are equivalent:*

- (i)  $\mathcal{A}$  is multitotal and has a regular generator;
- (ii)  $\mathcal{A}$  is multicocomplete and has a regular generator;
- (iii)  $\Pi\mathcal{A}$  is cocomplete and has a regular generator;
- (iv)  $\Pi\mathcal{A}$  admits a solid functor into  $\widetilde{\mathcal{S}et}$  which reflects regular epimorphisms;
- (v)  $\mathcal{A}$  is equivalent to a multireflective subcategory of a monadic category over  $\widetilde{\mathcal{S}et}$ ;
- (vi)  $\mathcal{A}$  admits a multisolid functor into  $\widetilde{\mathcal{S}et}$  which reflects regular epimorphisms.

PROOF: (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) follows from 4.4(2). (iii) $\Rightarrow$ (iv): In the presence of a regular generator, the cocomplete category  $\Pi\mathcal{A}$  is actually total (see Remark 4.4), so that 2.3(2) becomes applicable. (iv) $\Rightarrow$ (v): Since  $\Pi\mathcal{A}$  has coequalizers, it is equivalent to a full reflective subcategory of a monadic category over  $\widetilde{\mathcal{S}et}$  (see also [3]); statement (v) follows since  $\mathcal{A}$  is multireflective in  $\Pi\mathcal{A}$ . (v) $\Rightarrow$ (vi): The forgetful functor of a monadic category over  $\widetilde{\mathcal{S}et}$  is solid and reflects regular epimorphisms. One easily shows that the latter statement remains true if we restrict the functor to a multireflective subcategory. (vi) $\Rightarrow$ (i): In order to see that  $\mathcal{A}$  has a regular generator, one uses 4.4(1) to show that when the multisolid functor  $U : \mathcal{A} \rightarrow \widetilde{\mathcal{S}et}$  reflects regular epimorphisms, the same is true for the solid functor  $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\widetilde{\mathcal{S}et}$ , so that the existence of a regular generator in  $\Pi\widetilde{\mathcal{S}et}$  implies the same for  $\Pi\mathcal{A}$ ; now one applies 4.4(2) again.  $\square$

## 5. Product completions and dense generators

**5.1.** Recall that a generator  $\mathcal{G}$  of  $\mathcal{A}$  is said to be *dense* provided that the full subcategory of  $\mathcal{A}$  generated by  $\mathcal{G}$  (which we denote also by  $\mathcal{G}$ ) is dense in  $\mathcal{A}$ , that is, every object  $A$  of  $\mathcal{A}$  is a canonical colimit of the forgetful functor  $D_A : (\mathcal{G} \downarrow A) \rightarrow \mathcal{A}$ .

In the previous section, we related several types of generators of a category  $\mathcal{A}$  to the category  $\Pi\mathcal{A}$ , and this enabled us to obtain results on multitotal categories analogous to the ones of [3] for total categories. Similarly to (2) of Lemma 4.1, in the case of dense generators, we have the following:

**Lemma.** *If  $\mathcal{H}$  is a dense generator of  $\Pi\mathcal{A}$ , then  $\mathcal{H}^1$  is a dense generator of  $\mathcal{A}$ .*

PROOF: Let  $A$  be an object in  $\mathcal{A}$  and let  $D^1 : (\mathcal{H}^1 \downarrow A) \rightarrow \mathcal{A}$  and  $D : (\mathcal{H} \downarrow A) \rightarrow \Pi\mathcal{A}$  be the corresponding canonical diagrams into  $\mathcal{A}$  and  $\Pi\mathcal{A}$ , respectively. If  $\gamma : D^1 \rightarrow \Delta B$  is a cocone for  $D^1$ , define  $\bar{\gamma} : D \rightarrow \Delta B$  by considering, for each  $\Pi\mathcal{A}$ -morphism  $h : H \rightarrow A$  with  $H \in \mathcal{H}$ , the morphism  $\bar{\gamma}_h : H \rightarrow B$  such that  $\left( (\bar{\gamma}_h)_* : H_{(S\bar{\gamma}_h)(*)} \rightarrow B \right) = \left( \gamma_{h_*} : H_{(Sh)(*)} \rightarrow B \right)$ . It is easy to show that  $\bar{\gamma}$  is a cocone for  $D$ . Thus, there exists a unique morphism  $w : A \rightarrow B$  such that  $w \cdot h = \bar{\gamma}_h$  for every  $h : H \rightarrow A$  with  $H \in \mathcal{H}$ . It is now easily verified that  $w : A \rightarrow B$  is also the unique morphism which fulfills the equality  $w \cdot g = \gamma_g$  for every  $g : G \rightarrow A$  with  $G \in \mathcal{H}^1$ .  $\square$

**5.2.** However, for dense generators there is no analogous statement to (1) of Lemma 4.1. More precisely, we shall show next that the existence of a dense generator of  $\Pi\mathcal{A}$  in the presence of a dense generator of  $\mathcal{A}$  depends on the following large-cardinal axiom:

(M) There do not exist arbitrarily large measurable cardinals.

The statement (M) means that we can find a cardinal  $\rho$  such that no cardinal larger or equal to  $\rho$  is measurable, or, equivalently, every ultrafilter closed under intersections of less than  $\rho$  members contains a singleton set (see A.5 in [2]).

**Remark.** Recall that a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called *coinitial* (the dual of *cofinal*) provided that for every object  $B$  of  $\mathcal{B}$  the comma-category  $(F \downarrow B)$  is connected; this implies that for every diagram  $G : \mathcal{B} \rightarrow \mathcal{X}$  we have  $\text{colim } G = \text{colim } G \cdot F$  (more precisely, if  $(c_B : GB \rightarrow C)_{B \in \text{Ob}(\mathcal{B})}$  is a colimit of  $G$ , then  $(c_{FA} : GFA \rightarrow C)_{A \in \text{Ob}(\mathcal{A})}$  is a colimit of  $G \cdot F$ ).

**Theorem.** *The following assertions are equivalent:*

- (i)  $\Pi\mathcal{A}$  has a dense generator, for every category  $\mathcal{A}$  with a dense generator and a multi-initial object;
- (ii)  $\text{III}$  has a dense generator (with  $\mathbb{I}$  the terminal category);
- (iii) the set-theoretic axiom (M) holds.

PROOF: Since  $\text{III} = \text{Set}^{op}$ , we have (ii) $\Leftrightarrow$ (iii); this was proved by J. Isbell in [10], by showing that, for any cardinal  $\rho$ , the sets of cardinality less than  $\rho$  form

a codense cogenerator of  $\mathcal{Set}$  if and only if no cardinal  $\geq \rho$  is measurable. As (i) $\Rightarrow$ (ii) is trivial, (iii) $\Rightarrow$ (i) remains to be shown. Let  $\mathcal{G}$  be a dense generator of  $\mathcal{A}$ , and let  $O = (O_t)_{t \in T}$  be a multi-initial object of  $\mathcal{A}$ . By hypothesis, there is a cardinal number  $\rho$  such that  $\hat{\rho} = \{I \mid \text{card } I < \rho\}$  is a codense in  $\mathcal{Set}$ . Without loss of generality, we may assume  $O_t \in \mathcal{G}$  for all  $t \in T$  and  $\rho > \max\{\text{card } T, \aleph_0\}$ . We claim that

$$\mathcal{G}^\rho = \{G \in \Pi\mathcal{A} \mid \text{card}(SG) < \rho \text{ and } G_i \in \mathcal{G} \text{ for all } i \in SG\}$$

is dense in  $\Pi\mathcal{A}$ . Hence, for every  $A \in \Pi\mathcal{A}$ , we must show that  $A$  is a colimit of the canonical diagram  $D_A : (\mathcal{G}^\rho \downarrow A) \rightarrow \Pi\mathcal{A}$ . For that we use Lemma 2.5 and first show that  $SD_A^{op}$  has limit  $SA$  in  $\mathcal{Set}$ .

In fact, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{G}^\rho \downarrow A)^{op} & \xrightarrow{D_A^{op}} & (\Pi\mathcal{A})^{op} \\ S_A \downarrow & & \downarrow S \\ (SA \downarrow \hat{\rho}) & \xrightarrow{D_{SA}} & \mathcal{Set} \end{array}$$

with  $S_A$  induced by  $S$ . Since  $\lim D_{SA} = SA$  (canonically), it suffices to show that  $S_A$  is coinitial. Hence, for every object  $(J, \varphi : SA \rightarrow J)$  in  $(SA \downarrow \hat{\rho})$  we must show that the comma category  $(S_A \downarrow (J, \varphi))$  is connected. In fact, we can define a morphism  $c_\varphi : O^J \rightarrow A$  by

$$(c_\varphi)_k : O^J \xrightarrow{p_{\varphi(k)}} O \xrightarrow{!_{A_k}} A_k$$

for every  $k \in SA$  (where  $p_{\varphi(k)}$  is a projection and  $!_{A_k}$  is determined by initiality of  $O$  in  $\Pi\mathcal{A}$ ). Hence, we have an object  $(O^J, c_\varphi) \in (\mathcal{G}^\rho \downarrow A)$ , and the diagram

$$\begin{array}{ccc} SO^J = T \times J & \xrightarrow{\pi_2} & J \\ & \searrow S_{c_\varphi} & \nearrow \varphi \\ & SA & \end{array}$$

commutes. This means that  $((O^J, c_\varphi), \pi_2)$  is an object of  $(S_A \downarrow (J, \varphi))$ ; in fact, it is weakly initial in that category, as we shall show next. Given any object  $((G, f), \psi)$  in  $(S_A \downarrow (J, \varphi))$ , so that  $\psi : S_A(G, f) \rightarrow (J, \varphi)$  is a morphism in  $(SA \downarrow \hat{\rho})$ , we claim that the morphism  $d_\psi : O^J \rightarrow G$  with

$$(d_\psi)_i : O^J \xrightarrow{p_{\psi(i)}} O \xrightarrow{!_{G_i}} G_i$$

for all  $i \in SG$  makes the following diagrams commute:

$$\begin{array}{ccc}
 SG & \xrightarrow{Sd_\psi} & SO^J \\
 \psi \searrow & & \nearrow \pi_2 \\
 Sf \searrow & J & \nearrow Sc_\varphi \\
 \varphi \uparrow & & \\
 SA & & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xleftarrow{d_\psi} & O^J \\
 f \searrow & & \nearrow c_\varphi \\
 & A & 
 \end{array}$$

This is obvious for the diagram on the left, while initiality of  $O$  in  $\Pi\mathcal{A}$  implies commutativity of the right-hand diagram. Consequently we have a morphism  $d_\psi : ((O^J, c_\varphi), \pi_2) \rightarrow ((G, f), \psi)$  in  $(SA \downarrow (J, \varphi))$ . This concludes the proof of  $\lim SD_A^{op} = SA$ .

According to Lemma 2.5, it now suffices to show that for every  $k \in SA$ ,  $A_k$  is the canonical colimit of the diagram  $D_k : (\mathcal{G}^\rho \downarrow A) \rightarrow \mathcal{A}$ ,  $(G, f) \mapsto G_{(Sf)(k)}$ . But  $D_k$  factors as

$$\begin{array}{ccc}
 (\mathcal{G}^\rho \downarrow A) & \xrightarrow{D_k} & \mathcal{A} \\
 F_k \searrow & & \nearrow D_{A_k} \\
 & (\mathcal{G} \downarrow A_k) & 
 \end{array}$$

with  $F_k : (G, f) \mapsto (G_{(Sf)(k)}, f_k)$ . Since, by hypothesis,  $A_k$  is the canonical colimit of  $D_{A_k}$ , it suffices to show that  $F_k$  is cofinal in order to finish the proof. We show that, given any object  $(H, g) \in (\mathcal{G} \downarrow A_k)$ , the comma category  $((H, g) \downarrow F_k)$  is connected. Let  $g^{(k)} : H \times O \rightarrow A$  be the  $\Pi\mathcal{A}$ -morphism with  $(g^{(k)})_k = g : H \rightarrow A_k$  and  $(g^{(k)})_l = !_{A_l} : O \rightarrow A_l$  for all  $l \neq k$ . Then  $1_H : (H, g) \rightarrow F_k(H \times O, g^{(k)})$  is a morphism in  $(\mathcal{G} \downarrow A_k)$ , so  $((H \times O, g^{(k)}), 1_H)$  is an object in  $((H, g) \downarrow F_k)$ . Consider another object  $((G, f), h)$  in that category; denote  $i = (Sf)(k)$ . We analyse two cases:

(a)  $(Sf)^{-1}(\{i\}) = \{k\}$ . Thus one can define a  $\Pi\mathcal{A}$ -morphism  $h^{(k)} : H \times O \rightarrow G$  as above, and since  $f_k \cdot h = g$  one has  $f \cdot h^{(k)} = g^{(k)}$ . Consequently,  $h^{(k)} : (H \times O, g^{(k)}) \rightarrow (G, f)$  is a morphism in  $(\mathcal{G}^\rho \downarrow A)$  with  $f \cdot F_k h^{(k)} = g$ ; hence, we have  $h^{(k)} : ((H \times O, g^{(k)}), 1_H) \rightarrow ((G, f), h)$  in  $((H, g) \downarrow F_k)$ .

(b)  $(Sf)^{-1}(\{i\}) \neq \{k\}$ . We show that  $((G, f), h)$  belongs to the same connected component of the category  $((H, g) \downarrow F_k)$  as an object  $((\overline{G}, \overline{f}), \overline{h})$  which is of type (a), so the proof will be complete. Put  $\overline{G} = G \times G_i$  and let  $\overline{f} : G \times G_i \rightarrow A$  be the obvious morphism such that  $\overline{f}_l = (f \cdot \pi_1)_l$  for all  $l \neq k$ , where  $\pi_1$  is the first projection of  $G \times G_i$  and  $(S\overline{f})^{-1}(i) = \{k\}$ , and put  $\overline{h} = h$ . Since  $\overline{f}_k \cdot \overline{h} = f_k \cdot h = g$ ,  $((\overline{G}, \overline{f}), \overline{h})$  is an object in  $((H, g) \downarrow F_k)$ . Let  $t : G \rightarrow \overline{G}$  be the  $\Pi\mathcal{A}$ -morphism such that  $St$  identifies the two copies of  $i$  of  $S\overline{G}$ , and all components of  $t$  are identities.

Then we have that  $\bar{f} \cdot t = f$ , so  $t : (G, f) \rightarrow (\bar{G}, \bar{f})$  is a morphism in  $(\mathcal{G}^p \downarrow A)$ ; furthermore,  $F_k t \cdot h = \bar{h}$ , thus we have a morphism  $t : ((G, f), h) \rightarrow ((\bar{G}, \bar{f}), \bar{h})$  in  $((H, g) \downarrow F_k)$ .  $\square$

**5.3 Remark.** The dense generator of  $\Pi A$  obtained in 5.2 from a dense generator of a category  $\mathcal{A}$  with a multi-initial object (in the presence of the axiom (M)) is distinct from the set  $\bar{\mathcal{G}} = \{G \times O \mid G \in \mathcal{G}\} \cup \{O \times O\}$  used in Section 4. (We have seen there that  $\bar{\mathcal{G}}$  is a (strong or regular) generator of  $\Pi A$  whenever  $\mathcal{A}$  has a multi-initial object  $O$  and a (strong or regular, respectively) generator  $\mathcal{G}$ .) Actually, we can prove that the category  $\Pi \text{Set}$  does not have a dense generator of the form  $1 \times O$  plus  $O^J$ ,  $J \in \mathcal{J}$  (assuming that  $\mathcal{J}$  is dense in  $\text{Set}^{op}$ ). Consider  $A = (X_1, X_2)$  where  $X_1 = X_2 = 1$ . The canonical diagram  $D$  of  $A$  consists of

- (a) some copies of  $O^J$ ;
- (b) four copies of  $1 \times O$  indexed by the obvious four morphisms  $f : 1 \times O \rightarrow A$  determined by  $Sf : \{1, 2\} \rightarrow \{1, 2\}$  uniquely.

Each of the four copies of  $1 \times O$  is connected with the objects of type (a) by morphisms  $O^J \rightarrow 1 \times O$ , but there is no morphism in the opposite direction. The colimit of the canonical diagram is, by Lemma 2.5,

$$\text{colim } D = (\text{colim } D_\varphi)_{\varphi \in \text{lim } SD}.$$

If  $\mathcal{J}$  were dense in  $\text{Set}^{op}$ , we would have, for the subdiagram  $D'$  of  $D$  of all objects  $O^J$ ,  $\text{lim } SD' = \{1, 2\}$  (canonically). Thanks to the morphisms  $O^J \rightarrow 1 \times O$ , we have  $\text{lim } SD = \text{lim } SD'$ . It is easy to see that the choice of index 1 gives a diagram with two distinct copies of 1 without any morphism between them; thus  $\text{colim } D_1$  will have two elements. Analogously,  $\text{colim } D_2$  has two elements. Hence  $(A_1, A_2) \neq (\text{colim } D_1, \text{colim } D_2)$ .

**5.4.** Although the existence of a dense generator of  $\mathcal{A}$  does not guarantee the same for  $\Pi A$ , the relationship between dense generators and multitotal categories is similar to the one for totality:

**Theorem.** *The following statements are equivalent for a category  $\mathcal{A}$ :*

- (i)  $\mathcal{A}$  is multitotal with a dense generator;
- (ii)  $\mathcal{A}$  is multisolid over  $\widetilde{\text{Set}}$  and has a dense generator;
- (iii)  $\mathcal{A}$  is equivalent to a full multireflective subcategory of a ranked monadic category over  $\widetilde{\text{Set}}$ ;
- (iv)  $\mathcal{A}$  is a full multireflective subcategory of a locally presentable category;
- (v)  $\mathcal{A}$  is a full multireflective subcategory of a Grothendieck topos.

PROOF: One proceeds similarly as in the total case, see [3].  $\square$

**Remark.** From the Theorem above and from 6.16 of [2], it follows that multitotal categories with dense generators are precisely the locally multipresentable categories, provided that the set-theoretic Vopěnka Principle holds.



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