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Induced near-homeomorphisms

WŁODZIMIERZ J. CHARATONIK

Abstract. We construct examples of mappings $f$ and $g$ between locally connected continua such that $2^f$ and $C(f)$ are near-homeomorphisms while $f$ is not, and $2^g$ is a near-homeomorphism, while $g$ and $C(g)$ are not. Similar examples for refinable mappings are constructed.

Keywords: cell-like, continuum, dendrite, hyperspace, induced mapping, monotone, near-homeomorphism, refinable

Classification: 54B20, 54E40

For a metric continuum $X$ we denote by $2^X$ and $C(X)$ the hyperspaces of all nonempty closed and of all nonempty closed connected subsets of $X$, respectively. Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we let $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ denote the corresponding induced mappings. The following theorem is known ([7, Lemma 2.1, p.750]).

1. Theorem. For any continua $X$ and $Y$ and a mapping $f : X \to Y$ the following three statements are equivalent:

(a) $f : X \to Y$ is monotone;
(b) $2^f : 2^X \to 2^Y$ is cell-like;
(c) $C(f) : C(X) \to C(Y)$ is cell-like.

As applications of these results we show that if $f$ is a monotone mapping between locally connected continua, then $2^f$ is a near-homeomorphism between Hilbert cubes. Moreover, if the continua $X$ and $Y$ contain no free arcs, then $C(f)$ is a near-homeomorphism, too. We show appropriate examples of mappings $f$ and $g$ such that $2^f$ and $C(f)$ are near-homeomorphisms while $f$ is not, and $2^g$ is a near-homeomorphism, while $g$ and $C(g)$ are not. Finally, we present examples of non-refinable mappings whose induced mappings are near-homeomorphisms, in particular are refinable. Several questions are asked.

All spaces considered in this paper are assumed to be metric. A mapping means a continuous function. A continuum means a compact connected space. Given a continuum $X$ with a metric $d$, we denote by $2^X$ the hyperspace of all nonempty closed subsets of $X$ equipped with the Hausdorff metric $H$ defined by

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$
(equivalently: with the Vietoris topology: see e.g. [6, (0.1), p. 1 and (0.12), p. 10]. Furthermore, we denote by $C(X)$ the hyperspace of all subcontinua of $X$, i.e., of all connected elements of $2^X$. The reader is referred to Nadler’s book [6] for needed information on the structure of hyperspaces.

Given a mapping $f : X \to Y$ between continua $X$ and $Y$, we consider mappings (called the induced ones)

\[ 2^f : 2^X \to 2^Y \quad \text{and} \quad C(f) : C(X) \to C(Y) \]

defined by

\[ 2^f(A) = f(A) \quad \text{for every} \quad A \in 2^X \quad \text{and} \quad C(f)(A) = f(A) \quad \text{for every} \quad A \in C(X). \]

A continuous mapping $\omega : 2^X \to \mathbb{R}$ is called a Whitney map provided that $\omega(\{x\}) = 0$ for each point $x \in X$, and that if $A$ and $B$ are nonempty closed subsets of $X$ with $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$.

A continuum is said to have trivial shape if it is the intersection of a decreasing sequence of compact absolute retracts. A mapping $f : X \to Y$ between continua $X$ and $Y$ is called cell-like if, for each point $y \in Y$ the preimage $f^{-1}(y)$ is a continuum of trivial shape. In particular, cell-like mappings are monotone, i.e. the preimages of points are connected.

A mapping $f : X \to Y$ between continua $X$ and $Y$ is called a near-homeomorphism if $f$ is the uniform limit of homeomorphisms from $X$ onto $Y$. A proof of the following proposition is straightforward.

2. Proposition. If a surjective mapping $f : X \to Y$ between continua $X$ and $Y$ is a near-homeomorphism, then the two induced mappings $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ are near-homeomorphisms, too.

We will show that the converse implications do not hold.

An arc $ab$ in a space $X$ is said to be free provided that $ab \setminus \{a, b\}$ is an open subset of $X$.

3. Theorem. Let continua $X$ and $Y$ be locally connected, and let a mapping $f : X \to Y$ be monotone. Then $2^X$ and $2^Y$ are homeomorphic to the Hilbert cube, and the induced mapping $2^f$ is a near-homeomorphism. If, moreover, $X$ and $Y$ do not contain free arcs, then $C(X)$ and $C(Y)$ are homeomorphic to the Hilbert cube, and $C(f)$ is a near-homeomorphism.

Proof: The hyperspaces $2^X$ and $2^Y$ are homeomorphic to the Hilbert cubes by [6, (1.97), p. 137]. Similarly, if $X$ and $Y$ do not contain free arcs, then $C(X)$ and $C(Y)$ are homeomorphic to the Hilbert cube by [6, (1.98), p. 138]. Then by Theorem 1 the two induced mappings $2^f$ and $C(f)$ are cell-like mappings between Hilbert cubes, so they are near-homeomorphisms by [5, Theorem 7.5.7, p. 357 and Corollary 7.8.4, p. 372].

The next example shows that even if continua $X$ and $Y$ are homeomorphic, the conditions that both induced mappings are near-homeomorphisms do not imply that $f$ is a near-homeomorphism.
4. Example. There are a locally connected continuum $X$ and a mapping $f : X \to X$ such that the induced mappings $2f$ and $C(f)$ are near-homeomorphisms, while $f$ is not.

Proof: To describe the example recall that a Gehman dendrite is a dendrite $G$ having the Cantor ternary set in $[0, 1]$ as the set $E(G)$ of its end points, such that all ramification points of $G$ (the set of which is denoted by $R(G)$) are of order 3 and are situated in $G$ in such a way that $E(G) = \text{cl} \ R(G) \setminus R(G)$ (see the figure).

Let $e_0$ and $e_1$ denote two end points of $G$ being of the maximal distance apart, i.e., these end point of $G$ correspond to points 0 and 1 of the Cantor set when it is embedded into $[0, 1]$ in the natural way. Let $r$ be a ramification point of $G$ lying in the left half of $G$ and having the maximal distance from $e_0$. Let $K$ be the component of $G \setminus \{r\}$ containing the end point $e_1$, and let $D$ be the closure of the union of two other components of $G \setminus \{r\}$. Note that $D$ is a copy of $G$ diminished thrice with respect to the size of $G$. Thus there is a homothety $h : D \to G$ with the center $e_0$ and the ratio 3, which maps homeomorphically $D$ onto $G$. Therefore, if $g : G \to D$ is a monotone retraction of $G$ onto $D$ which shrinks $K$ to the singleton $\{r\}$ and which is the identity on $D$, then the composition $h \circ g : G \to G$ is a monotone mapping which is not a near-homeomorphism. The above construction is due to Dr. K. Omiljanowski, see [1, Example 5.3, p.177].

Let $f = (h \circ g) \times \text{id} : G \times [0, 1] \to G \times [0, 1],$ and observe that the induced mappings $2f$ and $C(f)$ are near-homeomorphisms, again by Theorem 3.

To see that $f$ is not a near-homeomorphism note that $f((e_1, 0)) = (v, 0),$ where $v$ is the highest point of $G,$ and that each neighborhood of the point $(e_1, 0)$ contains
the Cartesian product of a triod by an interval, while small neighborhoods of \((v, 0)\) do not contain such products.

5. Example. There are a locally connected continuum \(X\) and a mapping \(f : X \to X\) such that the induced mapping \(2^f\) is a near-homeomorphism, while \(f\) and \(C(f)\) are not.

Proof: Let \(X = G\) be the Gehman dendrite, and let the mappings \(g\) and \(h\) have the same meaning as in the previous example. Put \(f = h \circ g\), and observe that \(2^f\) is a near-homeomorphism, again by Theorem 3. So, we need only to verify that \(C(f)\) is not a near-homeomorphism. Denote, as previously, by \(v\) the top of \(G\). Thus \(f(e_1) = v\). Note that if \(N\) is a closed connected neighborhood of \(e_1\), then \(\dim C(N) = \infty\) by [6, (1.103), p.142], while dimension of the hyperspace of sub-continua of a small closed connected neighborhood of \(v\) is two. Therefore there is no homeomorphism from \(C(X)\) to \(C(X)\) sending \(\{e_1\}\) into a neighborhood of \(\{v\}\). This shows that \(C(f)\) is not a near-homeomorphism. The proof is complete.

6. Questions. Let a mapping \(f : X \to Y\) between continua \(X\) and \(Y\) be such that the induced mapping \(C(f)\) is a near-homeomorphism (in particular, \(C(X)\) and \(C(Y)\) are homeomorphic). Does it imply that \(2^f\) is a near-homeomorphism? The same question, if \(X = Y\).

Now we are going to discuss relations between refinable induced mappings. Let us start with a definition. A surjective mapping \(f : X \to Y\) is called refinable (see [2, p.263]; see also a survey article [4] for more information) if for each \(\varepsilon > 0\) there is a surjective \(\varepsilon\)-mapping \(g : X \to Y\) (called \(\varepsilon\)-refinement of \(f\)) which is \(\varepsilon\)-close to \(f\), that is, \(\rho(f, g) < \varepsilon\) (where \(\rho\) means the supremum metric on the functional space \(Y^X\)) and \(\diam g^{-1}(y) < \varepsilon\) for each \(y \in Y\). In particular, every near-homeomorphism is refinable, while in general the continua \(X\) and \(Y\) do not have to be homeomorphic. However, if there exists a refinable mapping from \(X\) onto \(Y\), then \(X\) has to be \(Y\)-like, in particular

\[
\dim X \leq \dim Y.
\]

It is known that if \(f\) is refinable, then \(2^f\) is refinable, [3, Theorem 2.4 (i), p.3].

Now we will investigate other possible relations between the three conditions:

(A) \(f\) is refinable;
(B) \(2^f\) is refinable;
(C) \(C(f)\) is refinable.

8. Example. Let \(f : [0, 1]^2 \to [0, 1]\) be the natural projection. Then \(2^f\) is a near-homeomorphism (in particular it is refinable), while \(C(f)\) and \(f\) are not refinable.

Proof: \(2^f\) is a near-homeomorphism by Theorem 3. \(C(f)\) and \(f\) are not refinable because inequality (7) is not satisfied.

□
9. Example. Let $f : [0, 1]^3 \to [0, 1]^2$ be the natural projection. Then $2^f$ and $C(f)$ are near-homeomorphisms (in particular they are refinable), while $f$ is not refinable.

**Proof:** The argument is exactly the same as for the previous example. □

The following two questions remain open.

10. **Question** (Hosokawa, [3, p. 2]). Does $f$ refinable imply $C(f)$ refinable?

11. **Question.** Does $C(f)$ refinable imply $2^f$ refinable?

A surjective mapping $f : X \to Y$ is called *monotonely refinable* if it is refinable, and each $\varepsilon$-refinement of $f$ can be chosen to be a monotone mapping. In particular each near-homeomorphism is monotonely refinable. It is known that if the mapping $f$ is monotonely refinable, then the two induced mappings, $2^f$ and $C(f)$ also are monotonely refinable, [3, Theorem 2.4 (ii), p. 3]. Example 9 shows that none of the two opposite implications is true. Furthermore, by Example 8, $2^f$ is monotonely refinable does not imply that $C(f)$ is monotonely refinable.

12. **Question.** Does $C(f)$ monotonely refinable imply $2^f$ monotonely refinable?

**References**


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