Lev Vasil′evich Sabinin

Smooth quasigroups and loops: forty-five years of incredible growth

Commentationes Mathematicae Universitatis Carolinae, Vol. 41 (2000), No. 2, 377--400

Persistent URL: http://dml.cz/dmlcz/119171

Smooth quasigroups and loops: forty-five years of incredible growth

Lev V. Sabinin

Abstract. The remarkable development of the theory of smooth quasigroups is surveyed.

Keywords: smooth quasigroups and loops, odules, loopuscular and odular algebras, Bol loops, Moufang loops, Bol algebras, Mal'cev algebras, nonlinear geometric algebra, nonassociative geometry, F-quasigroups, transsymmetric spaces, hyperalgebra, hyporeductivity, pseudoreductivity

Classification: 20N05, 53A60, 53C22

0. There might be different opinions on the date of birth of the theory of smooth quasigroups. But nobody would deny that the pioneering work of Mal'cev [A.I. Mal'cev 55] was a milestone of exceptional importance in the area. Thus we regard this very work as a starting point of smooth quasigroups and loops theory. In this survey we concentrate our efforts only on main ideological achievements; it would be not productive and, moreover, impossible to survey the whole scope of published papers in the field.

Instead we present a large (but not comprehensive) bibliography on the subject.

Near 1955, after remarkable results of the algebraic theory of quasigroups, the spiritual state of World Mathematical Mind reached a proper strength to create something new. A very natural idea to generalize the Lie groups theory in a non-associative way obtained the shape and force and appeared in the paper [A.I. Mal'cev 55]. This initiated attempts to construct the foundation of smooth quasigroups and loops in different parts of the world (Russia, Japan, Germany, Central Europe).

Some attempts of geometric nature were related to web geometry. We will not be concerned with this matter, which is treated in the article of M. Akivis and V. Goldberg. Here we note only that web geometry gives a useful instrument to study isotopically invariant properties of quasigroups. The study of properties which are not isotopically invariant needs different approaches.

From [L.V. Sabinin 98a] one may see how, by means of the construction of *antiproduct*, web theory is reduced to standard considerations of loop theory.

Among those who influenced the development of the theory of smooth quasigroups, I would especially mention the following personalities (in alphabetical order): A.S. Fedenko, A.N. Grishkov, M. Kikkawa, O. Kowalski, E.N. Kuz'min, A.I. Ledger, O. Loos, A.I. Mal'cev, P. Nagy, L.V. Sabinin, A.A. Sagle, K. Strambach.

1. In the paper [A.I. Mal'cev 55] a *local diassociative analytic loop* $\mathcal{Q} = \langle Q, \cdot, \varepsilon\rangle$ was studied. Diassociativity means that any two elements generate a subgroup. Since the multiplication in a loop is a binary operation and the loop is diassociative, one may write the analogue of the *Campbell-Hausdorff series*, which depends only on one skew-symmetric bilinear operation (multiplication $\times$) in the tangent space $V = T_\varepsilon Q$, at $\varepsilon \in Q$. Thus, at least locally, a diassociative loop $\mathcal{Q} = \langle Q, \cdot, \varepsilon\rangle$ can be restored by means of its *tangent algebra* $\langle V, +, 0_V, \times\rangle$. This algebra is a *binary-Lie algebra*, that is, any two of its elements generate a subalgebra which is a Lie algebra. Any binary-Lie algebra generates, by means of the Campbell-Hausdorff formula, a diassociative local loop. Thus we get a diassociative smooth loops — binary-Lie algebras theory generalizing the Lie groups — Lie algebras theory. Later, Gainov [A.T. Gainov 57] described a binary Lie algebra V by the identities:

$$(1) \qquad \begin{cases} \mathrm{J}(\xi, \eta, \xi \times \eta) = 0, \quad \xi \times \xi = 0, \quad \xi, \eta \in V, \\ \mathrm{J}(\xi, \eta, \zeta) = \xi \times (\eta \times \zeta) + \zeta \times (\xi \times \eta) + \eta \times (\zeta \times \xi). \end{cases}$$

In the same paper of A.I. Mal'cev a smooth *Moufang loop* was considered, as well. A Moufang loop can be defined as a loop $\langle Q, \cdot, \varepsilon\rangle$ with the identities

$$(2) \qquad L_x \circ L_y \circ L_x = L_{x\cdot(y\cdot x)}, \quad L_x z \stackrel{\text{def}}{=} x \cdot z$$
(*left Bol identity*),

$$(3) \qquad R_x \circ R_y \circ R_x = R_{(x\cdot y)\cdot x}, \quad R_x z \stackrel{\text{def}}{=} z \cdot x$$
(*right Bol identity*).

Being diassociative, a smooth Moufang loop can be infinitesimally described by a skew-symmetric algebra with identities. Mal'cev called it a Moufang-Lie algebra. Now it is called a *Mal'cev algebra*. Its defining identities are:

$$(4) \qquad \xi \times \xi = 0\,, \quad \mathrm{J}(\xi, \tau, , \xi \times \eta) = \mathrm{J}(\xi, \tau, \eta) \times \xi\,, \quad \xi, \eta, \tau \in V.$$

At that time the question whether any Mal'cev algebra uniquely defines a smooth Moufang loop or not was open. This question was answered in positive by Kuz'min [E.N. Kuz'min 70, 71]. Thus the infinitesimal theory for smooth Moufang loops — Mal'cev algebras has been constructed in the full analogy with the Lie groups — Lie algebras theory. Kuz'min classified, also, all simple Mal'cev algebras over $\mathbb{R}$. With a few rather interesting exceptions, these turned out to be Lie algebras [E.N. Kuz'min 68]. Earlier the same, in fact, was done by Sagle [A.A. Sagle 62a], who intensively studied Mal'cev algebras [A.A. Sagle 61, 62a, b]. Later some results on global smooth Moufang loops were obtained [F.S. Kerdman 79].

In the light of the above, we may speak of a Mal'cev school in the smooth loops theory (Mal'cev-Gainov-Kuz'min and others), the approach of which was purely algebraic, without any concern for the geometry. Much later Grishkov [A.N. Grishkov 86a, b] finalized, to a certain extent, the above theory. He constructed the structure theory of binary-Lie algebras. In particular, he showed that any simple binary-Lie algebra (over $\mathbb{R}$) is a Mal'cev algebra.

There are still some open problems in the field: is the group generated by left (right) translations L_x, $L_x y = x \cdot y$, (R_x, $R_x y = y \cdot x$) of a diassociative C^3-smooth loop a Lie group? Note that this is true for Moufang loops.

2. The other sources of the smooth quasigroup theory had their backgrounds in differential geometry. O. Loos [O. Loos 69] retold the theory of *symmetric spaces* (see, for example, [S. Helgason 62, 78]) in the language of smooth quasigroups, although he did not use the term *'quasigroup'* [O. Loos 69]. It was a remarkable breakthrough, since earlier no good geometric examples of smooth quasigroups had been discovered. It was followed by the notion of *generalized symmetric spaces* [A.J. Ledger 67], [A.J. Ledger, M. Obata 68], [A.S. Fedenko 73, 77], [O. Kowalski 74, 80] which has been well studied. Again, despite the geometric approach, these results have a smooth quasigroups nature. In the language of smooth quasigroups, a generalized symmetric space is simply a smooth (partial or global) quasigroup $\langle Q, \cdot \rangle$ with the identities

$$x \cdot (x \cdot y) = y \quad (\textit{key identity}), \tag{5}$$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \quad (\textit{left distributive identity}). \tag{6}$$

3. Meanwhile, a very important fact was discovered by Kikkawa [M. Kikkawa 64]. By means of parallel translations of geodesic arcs along geodesic arcs in an affinely connected manifold (M, ∇), he constructed so-called *geodesic loops* of an affinely connected manifold. More precisely, M may be considered to have the binary operations (local or global)

$$x \underset{a}{\cdot} y \overset{\text{def}}{=} \operatorname{Exp}_x \tau_x^a \operatorname{Exp}_a^{-1} y, \tag{7}$$

where τ_x^a means the parallel translation from a to x along the geodesic arc $\breve{ax}$ and Exp_a is the exponential map at $a \in M$. In this way we get a loop $\langle M, \underset{a}{\cdot}, a \rangle$ with neutral element $a \in M$. This construction justified the role of smooth quasigroups and loops in differential geometry. But at that time the fundamental question, when the system of smooth loops on a manifold is a system of geodesic loops for some affinely connected space, was still open.

4. Later, under the influence of [O. Loos 69], Kikkawa introduced the notion of *homogeneous Lie loops* and developed their theory. Actually, such loops are *left A-loops*, that is, loops with the identity

$$\begin{cases} \ell(a\,, b) \circ L_x \circ [\ell(a\,, b)]^{-1} = L_{\ell\,(a\,, b)\,x}\,, \\ L_c\, z \overset{\text{def}}{=} c \cdot z\,, \quad \ell(a, b) \overset{\text{def}}{=} (L_{a \cdot b})^{-1} \circ L_a \circ L_b\,. \end{cases} \tag{8}$$

This means that $\ell(a,b)$ is an automorphism of the considered loop. He introduced in this case the canonical reductive affine connection and the proper infinitesimal object, a *triple Lie algebra* [K. Yamaguti 58a, b] (a binary-ternary linear algebra with identities). See [M. Kikkawa 72, 73, 74, 75a, b, c, 84 a, b, 85, 91].

5. In 1972 Sabinin [L.V. Sabinin 72a, b, c] established the fundamental relations between homogeneous spaces and left loops. As it is known, any left homogeneous space G/H with a fixed cross section Q defines a left loop on itself by projecting on Q the group product of two elements from Q along a left coset:

$$\forall\, q_1, q_2 \in Q, \quad q_1 * q_2 \overset{\text{def}}{=} \pi_Q(q_1 \cdot q_2) \tag{9}$$

(this is due to [R. Baer 39, 40]). But there is a converse construction discovered by Sabinin [L.V. Sabinin 72a, b, c]: by means of a left loop and its *transassociant*, one may uniquely construct a left homogeneous space (*semidirect product of a loop by its transassociant*) and its cross-section such that the left loop induced on it by projecting along the left cosets is isomorphic to the original left loop.

This result was generalized to left quasigroups, as well.

Note that despite the fact that the above results are purely algebraic, they are most important in the smooth setting (after obvious localization). Thus the *Baer-Sabinin* construction was established.

6. In [L.V. Sabinin, L.B. Sharma 76] the notion of Bol loop was generalized to so-called left *half Bol loops* and corresponding examples were given. This concept has played an important role in current research. See [L.V. Sabinin, L.V. Sbitneva 94], [L.V. Sabinin, Yu.A. Selivanov 94]. For the smooth case, see [L.V. Sabinin, C. Castillo 99].

7. In [L.V. Sabinin 77], the fundamental concepts of *odule* and *odular structure* were introduced and applied to affinely connected manifolds. The notion of geodesic loop (Kikkawa) was enriched by the introduction of the *canonical unary operations* (which resulted in the notion of *geodesic odules* and *diodules*). In this way a purely algebraic description of an affinely connected space as an *odular* (*diodular*) *universal algebra* with the *geoodular identities* was obtained. Such an algebraic description is impossible in the language of loops only. In this scheme an affinely connected manifold (M, ∇) is a smooth universal algebra $\langle M, L, N, (\omega_t)_{t\in\mathbb{R}}\rangle$ with ternary operations

$$L(x,a,y) = L^a_x y = x \underset{a}{\cdot} y, \quad N(x,a,y) = N^a_x y = x \underset{a}{+} y \tag{10}$$

and binary operations

$$\omega_t(a,x) = t_a x \tag{11}$$

such that, for any fixed $a \in M$, $\langle M, \underset{a}{\cdot}, a, (t_a)_{t\in\mathbb{R}}\rangle$ is a smooth local left odule, which means that $\langle M, \underset{a}{\cdot}, a\rangle$ is a loop with two-sided neutral $a \in M$ and

$$(12) \qquad (t+u)_a x = t_a x \underset{a}{\cdot} u_a x \quad (\textit{monoassociativity}),$$

$$(13) \qquad t_a(u_a x) = (tu)_a x \quad (\textit{pseudoassociativity}),$$

$$(14) \qquad 1_a x = x \quad (\textit{unitarity}).$$

2) $\langle M, \underset{a}{+}, a, (t_a)_{t\in\mathbb{R}}\rangle$ is a local vector space.

3) The identities

$$(15) \qquad L^{u_z x}_{t_z x} \circ L^{z}_{t_z x} = L^{z}_{u_z x} \quad (\textit{the first geoodular identity}),$$

$$(16) \qquad L^{z}_{x} \circ t_z = t_x \circ L^{z}_{x} \quad (\textit{the second geoodular identity}),$$

$$(17) \qquad L^{z}_{x}(y \underset{z}{+} w) = L^{z}_{x} y \underset{x}{+} L^{z}_{x} w, \quad (\textit{the third geoodular identity}).$$

are valid.

Such an algebra is called a *geodiodular algebra.* A geodiodular algebra has a *natural affine connection* defined by

$$(18) \qquad \nabla_{X_a} Y = \left\{ \frac{d}{dt} \left([(L^{a}_{g(t)})_{*,a}]^{-1} Y_{g(t)} \right) \right\}_{t=0}, \qquad g(0) = a, \quad \dot{g}(0) = X_a,$$

Y being a vector field, $L^a_c q = L(c, a, q)$ (see (10)).

The initial geodiodular structure may be restored by means of its natural connection ∇ as

$$(19) \qquad L(x, a, y) = x \underset{a}{\cdot} y = \mathrm{Exp}_x\, \tau^a_x\, \mathrm{Exp}^{-1}_a y,$$

$$(20) \qquad \omega_t(a, z) = t_a z = \mathrm{Exp}_a\, t\, \mathrm{Exp}^{-1}_a z,$$

$$(21) \qquad N(x, a, y) = x \underset{a}{+} y = \mathrm{Exp}_a\, (\mathrm{Exp}^{-1}_a x + \mathrm{Exp}^{-1}_a y).$$

Later, an algebraic study of affine connections was presented in the Ph.D. Dissertation of our student Afanasiev [A.J. Afanasiev 84].

In addition, some important classes of affinely connected spaces (*reductive, symmetric*, etc.) were described by algebraic identities. Thus the identities

$$(22) \qquad \left\{ \begin{aligned} L^b_a\, L(x,y,z) &= L\,(L^b_a x, L^b_a y, L^b_a z), \\ L^b_a\, \omega_t(x,y) &= \omega_t\,(L^b_a x, L^b_a y) \end{aligned} \right.$$

describe *reductive spaces*, and the identities

$$\begin{cases} (-1)_a\, L(x,y,z) = L((-1)_a x, (-1)_a y, (-1)_a z), \\ (-1)_a\, \omega_t(x,y) = \omega_t((-1)_a x, (-1)_a y). \end{cases} \tag{23}$$

describe *symmetric spaces*. On this matter, see [L.V. Sabinin 81, 91b, 99a, b].

In the Dr.Ph. Dissertation [H.M. Karanda 72], the fundamental fact that a geodesic loop of symmetric space is a left Bol loop was disclosed. Later, it was shown [L.V. Sabinin 81] that a locally symmetric space is nothing but a C^3-smooth *left Bol-Bruck loop*, that is a left Bol loop with the *left Bruck identity*

$$(x\cdot y)\cdot(x\cdot y) = x\cdot(y\cdot(y\cdot x)). \tag{24}$$

Equivalently, one can replace (24) by the *automorphic inverse identity*

$$(x\cdot y)^{-1} = x^{-1}\cdot y^{-1}.$$

It is interesting that in 1977 we did not know the results of Kikkawa [M. Kikkawa 64].

The above results have introduced a new ideology into geometry which allows one to treat an affinely connected space as a smooth algebraic system (*nonlinear geometric algebra and nonassociative geometry*).

8. Despite the work of A. Mal'cev and Kikkawa, it still was valuable to explore certain classes of smooth loops close to Lie groups. Since a geodesic odule of any symmetric space turned out to be a Bol loop, it was significant to study smooth Bol loops.

The infinitesimal theory of C^3-smooth left (right) Bol loops was developed by L. Sabinin and his student P. Miheev [L.V. Sabinin, P.O. Miheev 82b, 84, 85a, b], [P.O. Miheev 86a]. For a contemporary treatment, see [L.V. Sabinin 99b].

In the above works the proper infinitesimal object, a binary-ternary linear algebra with the identities

$$\begin{cases} (\eta;\xi,\xi) = 0, \quad (\xi;\eta,\zeta) + (\eta;\zeta,\xi) + (\zeta;\xi,\eta) = 0, \\ ((\xi;\nu,\tau);\eta,\zeta) + (\xi;(\eta;\nu,\tau),\zeta) + (\xi;\eta,(\zeta;\nu,\tau)) = ((\xi;\eta,\zeta);\nu,\tau) \end{cases} \tag{25}$$

(*triple Lie algebra*) and

$$\begin{aligned} &\xi\cdot\xi = 0\,, \\ &((\tau\cdot\zeta);\xi,\eta) = (\tau;\xi,\eta)\cdot\zeta + \tau\cdot(\zeta;\xi,\eta) + ((\xi\cdot\eta);\tau,\zeta) + (\tau\cdot\zeta)\cdot(\xi\cdot\eta)\,, \end{aligned} \tag{26}$$

was introduced. Such an object is called a *Bol algebra*.

If $\langle Q,\cdot,\varepsilon\rangle$ is a C^3-smooth left Bol loop and

$$A_\lambda(x) = \left[\frac{\partial(x\cdot y)^\alpha}{\partial y^\lambda}\right]_{y=\varepsilon} \frac{\partial}{\partial x^\alpha} \tag{27}$$

are so-called left fundamental vector fields then

$$(28)\qquad \begin{aligned}(\xi \cdot \eta) &= [A_\lambda, A_\mu](\varepsilon)\xi^\lambda \eta^\mu,\\ (\xi;\eta,\zeta) &= [A_\lambda,[A_\mu,A_\nu]](\varepsilon)\xi^\lambda\eta^\mu\zeta^\nu\end{aligned}$$

define the *tangent Bol algebra* on $V = T_\varepsilon(Q)$.

A Bol algebra uniquely defines a local left Bol loop and vice versa, in full analogy with the Lie algebras–Lie groups theory. It is interesting that initially this theory was obtained by means of differential-geometric machinery, but later (see [L.V. Sabinin 91b, 99b]), it was elaborated in a more algebraic way.

9. The first comprehensive account of the geometric smooth quasigroups and loops theory was given in [L.V. Sabinin 81]. For a contemporary treatment of the whole theory, see [L.V. Sabinin 99b].

10. Along with the above, geometric studies by means of the smooth quasigroup and loop approach took place. Generalized symmetric spaces were treated in the quasigroup language [L.V. Sbitneva 79, 82a, 82c, 84a, b]. This resulted in the concept of a *perfect s-space*, which was well studied in algebraic and geometric ways.

11. In 1988 the idea to create a general infinitesimal theory of smooth loops was completely realized in [L.V. Sabinin 88b]. The infinitesimal object in this theory is much more complicated (a so-called *hyperalgebra*).

Thus, let $\langle Q,\ \cdot\ ,\ \varepsilon\rangle$ be a local C^k-smooth ($k \geq 3$) loop, let

$$A^i_j(x) = \left[\frac{\partial\,(x\cdot y)^i}{\partial y^j}\right]_{y=\varepsilon}$$

be its left fundamental vector fields (see (27)), and let Exp be the *exponential map*.

We recall that, by definition,

$$(29)\qquad \frac{d\,(\operatorname{Exp} tq)^i}{dt} = A^i_j\,(\operatorname{Exp} tq)\,q^j\,,\quad \operatorname{Exp}(0) = \varepsilon\,,$$

and adjoin to the loop its *canonical operations*

$$(30)\qquad t\,x = \operatorname{Exp}\, t \operatorname{Exp}^{-1} x\,,\quad x + y = \operatorname{Exp}(\operatorname{Exp}^{-1} x + \operatorname{Exp}^{-1} y).$$

Let us also consider $\tilde{l}\,(a,b) = [\,\ell\,(a,b)\,]_{*,\varepsilon}$ and $C^{\,i}_{j\,k}(x)$ defined by the unique representation

$$(31)\qquad [\,A_j\,,A_k\,]\,(x) = C^{\,i}_{j\,k}\,(x)\,A_i\,(x)\,.$$

We introduce the functions

$$
(32)\qquad \begin{aligned} c^{i}_{jk}(q) &= C^{i}_{j\,k}(\operatorname{Exp} q)\,, \\ l^{\,p}_{m}(v\,,w) &= \tilde{l}^{\,p}_{m}(\operatorname{Exp} v\,,\ \operatorname{Exp} w)\,; \quad q,v,w \in T_{\varepsilon}(Q)\,. \end{aligned}
$$

(Note that in the normal coordinates $\operatorname{Exp} q = q\,$.)

Let us define on $T_{\varepsilon}(Q) = V$ the operations

$$
(33)\qquad \begin{aligned} \nu(v,w) &= l^{\,p}_{m}(v,w)\, w^{m}\,(\partial_p)_{\varepsilon}\,, \\ d(q,w) &= c^{i}_{jk}(q)\, q^{j} w^{\,k}\,(\partial_i)_{\varepsilon}\,. \end{aligned}
$$

The tangent vector space $T_{\varepsilon}(Q) = V$ equipped with these operations is called the *ν-hyperalgebra tangent to the loop $\langle Q, \cdot, \varepsilon\rangle$*. Such an algebra possesses properties which we collect in the following definition.

Definition. Let V be a vector space (over an arbitrary field $\mathcal{K}$), $\dim V = n$, $d(q,w)$ be a binary operation on V admitting a representation in an arbitrary basis $e_1,\dots,e_n$ of the form

$$
d(q,v) = d^{k}_{ij}(q)\, q^{i}\, v^{j}\, e_k
$$

and such that $d(q,q) = 0$. Then we say that V is a *hyperalgebra*.

If, additionally, a binary operation $\nu(v,w)$ is given on V, and $\nu(v,w)$ admits the representation in an arbitrary base $e_1,\dots,e_n$ of the form

$$
\nu(q,w) = \nu^{i}_{j}(q,w)\, w^{j}\, e_i\,,
$$

where $\nu(0,w) = w$, $\nu^{i}_{j}(v,0)\, w^{j} = w^{i}$, then we say that V is a *ν-hyperalgebra* (with multioperator ν).

Proposition. *Any C^r-smooth ($r \geq 1$) ν-hyperalgebra determines uniquely a local C^r-smooth loop $\langle Q, \cdot, \varepsilon\rangle$ such that its tangent ν-hyperalgebra is isomorphic to the initial ν-hyperalgebra.*

Morphisms of smooth loops induce morphisms of the corresponding ν-hyperalgebras and vice versa.

Although the operations ν and d are uniquely defined, the above indicated representations for them are not unique. If ν and d are analytic then we can obtain a countable system of multilinear operations (with identities) which is equivalent to the original ν-hyperalgebra (under some conditions of convergence). We call such a system of multilinear operations also a ν-hyperalgebra.

A C^k-smooth ($k \geq 2$) loop is right monoalternative, that is, $(x \cdot ty) \cdot uy = x \cdot (t+u)y$ ($t,u \in \mathbb{R}$), if and only if for its tangent ν-hyperalgebra $\nu(v,w) = w$.

In different applications one needs derivatives of ν and d. For example, it is the case for geometric odules, where one needs first derivatives of ν. This leads us to a modification of the concept of ν-hyperalgebra (equivalent to the initial one). In such a way an *F-hyperalgebra* can be constructed.

Let us note, finally, where, in the context of ν-hyperalgebra, Lie groups appear. A C^3-smooth loop is a Lie group if $d(v,w)$ is bilinear and satisfies the identities of a Lie algebra, and $\nu(v,w) = w$ (i.e., the loop is right monoalternative). Additional identities in a loop influence very much the structure of its tangent ν-hyperalgebra (the identity of associativity demonstrates such a case).

Comment. Roughly speaking, in the analytic case one needs a countable set of multilinear operations instead of one binary operation as in the case of Lie groups — Lie algebras, or binary and ternary operations as in the case of Bol loops — Bol algebras. Of course, some (rather weak) identities should be added.

The complete treatment of the infinitesimal theory of smooth loops may be found in [L.V. Sabinin 91b, 99b]. In this connection we note that the earlier treatment of such a theory [L.V. Sabinin, P.O. Miheev 86, 87, 90] (presented in a differential-geometric way) is now only of historical interest due to the unsatisfactory definition of hyperalgebra there.

12. The concept of odular structure (L.V. Sabinin) has given a rise, to a generalization of *G-spaces* [H. Buseman 55]. Thus in [O.A. Matveev 86, 87] the notion of *geodetic space* was introduced. This is a smooth manifold with a system of smooth unary operations (local or global) and characteristic identities

$$u_x(t_x y) = (ut)_x y, \quad 1_x y = y, \quad t_x y = (1-t)_y x \qquad (t,u \in \mathbb{R},\ x,y \in M). \tag{34}$$

We note here that any affinely connected space is a geodetic space. In the smooth case any geodetic space can be equipped with a unique affine connection of zero torsion with the same geodesics $\{t_x y\}_{t\in\mathbb{R}}$. For that it is enough to take the *tangent affine connection* to the loopuscular structure $L(x,z,y) = L^z_x\, y = (2)_z(1/2)_x\, y$,

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt} \left([\, (L^a_{g(t)})_{*,a}]^{-1}\, Y_{g(t)} \right) \right\}_{t=0}, \qquad g(0) = a, \quad \dot g(0) = X_a, \tag{35}$$

Y being a vector field.

O.A. Matveev developed the algebraic theory of geodesic maps for affinely connected spaces (or, equivalently, for smooth geoodular spaces).

13. The problem of when a smooth odule is a geodesic odule for some affine connection was solved in [L.V. Sabinin 87]. It was proved that a smooth odule $\langle Q, \cdot, \varepsilon, (t)_{t\in\mathbb{R}} \rangle$ is geodesic for some affine connection if and only if it is *geometric*, that is if the *identity of geometricity*

$$\ell(x,y)\, ty = t\, \ell(x,y)\, y, \qquad \ell(x,y) = (L_{x\cdot y})^{-1} \circ L_x \circ L_y \quad (t \in \mathbb{R},\ x,y \in Q), \tag{36}$$

is valid. The above affine connection is not unique and the arbitrariness of its choice was described as well. On this matter, see also [L.V. Sabinin 95b].

In this setting, the concept of *holonomial odule* was introduced [L.V. Sabinin 87] as an odule $\langle Q, \cdot, \varepsilon\,(t)_{t\in\mathbb{R}}\rangle$ with an additional ternary operation $h(a, b; x)$ and the system of identities

(37) $h\,(a\,, b\,; t\,x) = t\,h\,(a\,, b\,; x)$ (homogeneity identity),

(38) $h\,(a\,, a\cdot b\,; t\,b) = \ell\,(a\,, b)\,t\,b$ (joint identity),

(39) $h\,(c\cdot ta\,, c\cdot ua\,; h\,(c\,, c\cdot ta\,; x)) = h(c\,, c\cdot ua\,; x)$ (h-identity),

(40) $h\,(\varepsilon\,, q\,; x) = x$ (ε-identity).

This structure uniquely describes the geoodular space by

$$x \underset{a}{\cdot} y = x\,\cdot\,h\,(a\,, x\,; a\backslash y)\,, \quad t_{\,a}\,y = a\,\cdot\,t\,(a\backslash y) \quad (a\backslash y = (L_a)^{-1}y). \tag{41}$$

This notion is equivalent to the notion of geoodular structure in the sense that it allows one to concentrate all information about a geoodular structure (locally) into one fixed geodesic odule [L.V. Sabinin 87, 91b, 99b]. One may also say that the above construction is, in particular, an algebraic version of the equations of E. Cartan in polar coordinates for an affine connection.

14. A delicate analysis of Bol loops and left A-loops has given rise to the notion of *hyporeductive* and *pseudoreductive loops*, as well as to the notion of *hyporeductive homogeneous space* (a generalization of the *reductive space* [P.K. Rashevski 50, 51, 52], [S. Kobayashi, K. Nomizu 63, 69]). The proper infinitesimal theory has been constructed, see [L.V. Sabinin 90b, 90c, 90d, 91a, 99b]. Later a differential-geometric treatment of the above was given [I.A. Nourou 92]. On some generalizations, see [N.A. Gluhova 91].

15. The very interesting problem of describing *spaces of constant curvature* (and, more generally, *projectively flat spaces*) in a purely algebraic way was solved in [L.V. Sabinin, O.A. Matveev, S.S. Yantranova 86], [L.V. Sabinin, S.S. Yantranova 84], [S.S. Yantranova 89]. It turns out that a geodesic diodule Q of such a space is a *left Bol-Bruck loop* with the *pseudolinear identity*

$$\begin{gathered} x\cdot y = \alpha(x,y)x + \beta(x,y)y, \\ x, y \in Q; \quad \alpha(x,y), \beta(x,y) \in \mathbb{R}. \end{gathered} \tag{42}$$

This result was also generalized to *symmetric spaces of index* 1.

16. Meanwhile, analyzing the quasigroup structure of generalized symmetric spaces, L.V. Sabinin and L.L. Sabinina discovered the remarkable *transsymmetric spaces* [L.V. Sabinin, L.L. Sabinina 90, 91]. In the quasigroup language, these are simply smooth left quasigroups satisfying the *left F-identity*

$$x\cdot(y\cdot z) = (x\cdot y)\cdot(Fx\cdot z) \quad (F\colon Q\to Q) \tag{43}$$

with the property of *correctness:* there exist $(\rho_{Fx})^{-1}$, where $\rho_z\, x \stackrel{\text{def}}{=} x \cdot z$ (not everywhere defined right division).

The comprehensive theory of transsymmetric spaces in the language of smooth left F-quasigroups was developed in [L.L. Sabinina 92]. See, also, [L.L. Sabinina 93, 94, 95], [L.V. Sabinin, L.L. Sabinina 95]. *Perfect transsymmetric spaces* were introduced and studied in [L.V. Sabinin, L.L. Sabinina, L.V. Sbitneva 99].

17. Some first steps to solve the *generalized fifth problem of Hilbert for loops* were undertaken in [L.V. Sabinin, L.L. Sabinina, R. Jimenez 97]. In particular, it would be interesting to know, when a topological left Bol loop is analytic?

18. At this time the problem of constructing a theory of *smooth loop actions* (which would be helpful in applications to physics) is very significant. Instead of a group $\langle Q, \cdot, \varepsilon\rangle$ action,

$$a \in Q \mapsto (f_a : M \to M), \quad f_a \circ f_b = f_{a \cdot b},\ f_\varepsilon = \text{id},$$

one may try, for example,

$$a \in Q \mapsto (f_a : M \to M), \quad f_a f_b x = f_{a \overset{x}{\cdot} b}\, x,\ f_\varepsilon = \text{id}, \quad a, b, \in Q, \tag{44}$$

where $\{\langle Q, \overset{x}{\cdot}, \varepsilon\rangle\}_{x \in Q}$ is a family of loops. There is some preliminary non-rigorous treatment of the matter in [I.A. Batalin 81]. As to *Bol loops actions*, the work [T. Nono 61] may give some hint.

19. Finally we would like to indicate some results on applications of smooth quasigroups and loops. See [M.V. Karasev, V.P. Maslov 91] (Nonlinear Poisson brackets.), [P. Kuusk, J. Ord, E. Paal 94, 95] (General relativity), [J. Lohmus, E. Paal, L. Sorgsepp 94] (Survey on non-associative methods in physics), [A.I. Nesterov 89, 90, 97] (Non-associative methods in physics), [E.P. Osipov 89] (Anomalies in field theory), [E. Paal 88, 89] (Moufang symmetries in physics), [L.V. Sabinin, P.O. Miheev 93] (Special relativity), [L.V. Sabinin, A.I. Nesterov 97a] (Thomas precession), [L.V. Sabinin, A.I. Nesterov 97b] (Generalized coherent states), [A. Ungar 90, 94, 97] (Special relativity), [D.V. Yuriev 87, 92] (Chiral anomalies; Noncommutative geometry), [M. Bangoura 94] (Poisson brackets in mechanics).

References

I.L. Afanasiev

[**84**] *Algebraic structures of an affine connection*, (Russian), Ph.D. Dissertation, Friendship of Nations University, Moscow, 1984.

M.A. Akivis

[**78**] *Geodesic loops and local triple systems in a space with an affine connection*, (Russian), Siberian Math. J. **19** (1978), no. 2, 243–253; English transl.: Siberian Math. J. **19** (1978), no. 2, 171–178, **MR** 58 (1979) #7438.

M.Yu. Al-Houjeiri
[**87**] *Bol algebras for involutive pair of rank* 1, (Russian), Ph.D. Dissertation, Inst. of Math. of Moldova Acad. Sci., Kishinev, 1987.

E. Artin
[**57**] *Geometric Algebra*, Interscience Publishers, New York, 1957.

R. Baer
[**39**] *Nets and groups*, Trans. Amer. Math. Soc. **46** (1939), 110–141, **MR** 1 (1940), p. 6.
[**40**] *Nets and groups, II*, Trans. Amer. Math. Soc. **47** (1940), 435–439, **MR** 2 (1941), p. 4.

M. Bangoura
[**94**] *Quasi-groupes de Lie-Poisson*, (French), C.R. Acad. Sci. Paris Série I **319** (1994), no. 9, 975–978.

I.A. Batalin
[**81**] *Quasigroups construction and first class constraints*, J. Math. Phys. **22** (9) (1981), 1837–1850.

V.D. Belousov
[**67**] *Foundations of the Theory of Quasigroups and Loops*, (Russian), "Nauka", Moscow, 1967, 223 pp., **MR** 36 (1968) #1569.
[**81**] *Elements of Quasigroups Theory*, (Russian), Kishinev University Press, 1981, 115 pp.

R.H. Bruck
[**71**] *A Survey of Binary Systems*, (third printing, corrected), Springer-Verlag, Berlin, 1971, **MR** 20 (1959) #76.

H. Buseman
[**55**] *The Geometry of Geodesics*, Academic Press, New-York, 1955.

O. Chein, H. Pflugfelder, J.D.H. Smith (eds.)
[**90**] *Loops and Quasigroups: Theory and Applications*, Heldermann Verlag, Berlin, 1990.

S.-S. Chern
[**82**] *Web geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 1–8, **MR** 84g:53024.

T.V. Daneeva
[**90**] *On geometry of special loops*, (Russian), Ph.D. Dissertation, Friendship of Nations University, Moscow, 1990.

A.I. Dolgarev
[**91**] *EM-spaces*, (Russian), Ph.D. Dissertation, Friendship of Nations University, Moscow, 1991.

L.P. Eisenhart
[**26**] *Riemannian Geometry*, Princeton University Press, 1926.
[**27**] *Non-Riemannian Geometry*, Colloquium Publications of the Amer. Math. Soc. 8, New York, 1927.

A.S. Fedenko
[**73**] *Regular spaces with symmetries*, (Russian), Mathematical Notes **14** (1973), no. 1, 113–120, **MR** 49 (1975) #7956.
[**77**] *Spaces with Symmetries*, (Russian), Belarus Univ. Press, Minsk, 1977, 168 pp.

A.T. Gainov
[**57**] *Identical relations for binary-Lie rings*, (Russian), Advances in Math. Sciences **12** (1957), 141–146, **MR** 20 (1959) #890.

[**63**] *Binary-Lie algebras of lower ranks*, (Russian), Algebra and Logic **2** (1963), 21–40, **MR** 28 (1964) #3067.

V.M. Galkin

[**88**] *Quasigroups*, (Russian), Results in Science: Algebra. Topology. Geometry, vol. 26, VINITI, Moscow, 1988.

N.A. Gluhova

[**91**] *On the theory of generalized hyporeductive loops*, Webs and Quasigroups, Tver Univ. Press, Tver, 1991, pp. 153–163.

A.N. Grishkov

[**86a**] *Binary-Lie algebras and alternative loops*, (Russian), Dr.Sci. Dissertation, Minsk, 1986, 264 pp.

[**86b**] *Binary-Lie algebras and alternative loops*, (Russian), Synopsis of Dr.Sci. Dissertation, Minsk, 1986, 22 pp.

M. Hall, Jr.

[**43**] *Projective planes*, Trans. Amer. Math. Soc. **54** (1943), 229–277, **MR** 5 (1944), p. 72.

[**59**] *The Theory of Groups*, MacMillan Co., New York, 1959, **MR** 21 (1960) #1966.

S. Helgason

[**62**] *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962, **MR** 26 (1963) #2986.

[**78**] *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, 1978.

K.H. Hofmann

[**58**] *Topologische Loops*, Math. Z. **70** (1958), 13–37, **MR** 21 (1960) #1362.

[**61**] *Non-associative Topological Algebra*, Lecture Notes, Tulane University, 1961.

K.H. Hofmann, K. Strambach

[**86**] *The Akivis algebra of homogeneous loop*, Mathematika **33** (1986), no. 1, 87–95, **MR** 88d:17003.

[**91**] *Torsion and curvature in smooth loops*, (German), Publ. Math. Debrecen **38** (1991), no. 3–4, 189–214, **MR** 92e:53023.

J.P. Holmes, A.A. Sagle

[**78**] *Problems in H-spaces and non-associative algebras*, Kumamoto J. Sci. **13** (1978/79), 1–5, **MR** 58 (1979) #13010.

[**80**] *Analytic H-spaces, Campbell-Hausdorff formula and alternative algebras*, Pacific J. Math. **91** (1980), no. 1, 105–134, **MR** 82d:17005.

N. Jacobson

[**62**] *Lie Algebras*, Wiley Interscience Publishers, New York, 1962, **MR** 26 (1963) #1345.

H.M. Karanda

[**72**] *On geometry of symmetric loops*, (Russian), Ph.D. Dissertation, Friendship of Nations University, Moscow, 1972.

M.V. Karasev, V.P. Maslov

[**91**] *Nonlinear Poisson Brackets. Geometry and Quantization*, (Russian), "Nauka" Press, Moscow, 1991. There is an English translation.

H. Karzel, H. Wefelscheid

[**93**] *Groups with involutory automorphisms and K-loops; Applications to space-time-world and hyperbolic geometry I.*, Results in Math. **29** (1993), 338–354.

[**95**] *A geometric construction of the K-loop of a hyperbolic space*, Geometria Dedicata **58** (1995), 227–236.

F.S. Kerdman

[**79**] *Analytic Moufang loops in the large*, (Russian), Algebra and Logic **18** (1979), 523–555, **MR** 82c:22006.

M. Kikkawa

[**64**] *On local loops in affine manifolds*, J. Sci. Hiroshima Univ. Ser. A-I Math. **28** (1964), 199–207, **MR** 32 (1966) #4627.

[**72**] *On locally reductive spaces and tangent algebras*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci. **5** (1972), 1–13, **MR** 47 (1974) #990.

[**73**] *On some quasigroups of algebraic models of symmetric spaces*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci. **6** (1973), 9–13, **MR** 48 (1974) #6304.

[**74**] *On some quasigroups of algebraic models of symmetric spaces. II*, Mem. Fac. Lit. Sci. Shimane Univ. Natur. Sci. **7** (1974), 29–35, **MR** 49 (1975) #7384.

[**75a**] *Geometry of homogeneous Lie loops*, Hiroshima Math. J. **5** (1975), no. 2, 141–179, **MR** 52 (1976) #4182.

[**75b**] *A note on subloops of a homogeneous Lie loop and subsystems of its triple algebra*, Hiroshima Math. J. **5** (1975), no. 3, 439–446, **MR** 52 (1976) #9117.

[**75c**] *On some quasigroups of algebraic models of symmetric spaces. III*, Mem. Fac. Lit. Shimane Univ. Natur. Sci. **9** (1975), 7–12, **MR** 54 (1977) #442.

[**84a**] *Naturally reductive metrics on homogeneous systems*, Indag. Math. **46** (1984), 203–208, **MR** 85k:53048.

[**84b**] *Totally geodesic embeddings of homogeneous systems into their enveloping Lie groups*, Mem. Fac. Sci. Shimane Univ. **18** (1984), 1–8, **MR** 86f:53056.

[**85**] *Canonical connections of homogeneous Lie loops and* 3*-webs*, Mem. Fac. Sci. Shimane Univ. **19** (1985), 37–55, **MR** 87j:53077.

[**91**] *Projectivity of Homogeneous Left Loops*, Nonassociative Algebras and Related Topics, World Scientific, 1991, pp. 77–99.

S. Kobayashi, K. Nomizu

[**63**] *Foundations of Differential Geometry*, vol. 1, Interscience-Wiley, New York-London, 1963, xi + 329 pp., **MR** 27 (1964) #2945.

[**69**] *Foundations of Differential Geometry*, vol. 2, Interscience-Willey, New York-London, 1969, xiii + 470 pp., **MR** 38 (1969) #6501.

O. Kowalski

[**74**] *Riemannian manifolds with general symmetries*, Math. Z. **136** (1974), no. 2, 137–150, **MR** 49 (1975) #6097.

[**80**] *Generalized Symmetric Spaces*, Lecture Notes in Mathematics 805, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 187 pp., **MR** 83d:53036.

A.G. Kurosh

[**65**] *Lectures on General Algebra*, (Russian), 2$^{\text{d}}$ ed., Nauka, Moscow, 1973; English translation of 1$^{\text{st}}$ ed.: International Series of Monographs in Pure and Appl. Math. 70, Pergamon Press, 1965, 364 pp., **MR** 31 (1966) #3483.

[**74**] *General Algebra*, (Russian), Nauka, Moscow, 1974, 159 pp., **MR** 52 (1977) #13569.

P. Kuusk, J. Örd, E. Paal

[**94**] *Geodesic multiplication and the theory of gravity*, J. Math. Phys. **31 (1)** (1994), 321–334.

[**95**] *Geodesic multiplication and geometrical BRIST-like operators*, Proc. Estonian Acad. Sci. (Phys.-Math.) **44** (1995), no. 4, 437–449.

P. Kuusk, E. Paal

[**92**] *Geodesic multiplication as a tool for classical and quantum gravity*, Transactions of Tallinn Technical University **733** (1992), 33–41; Tallinn.

[**96a**] *Geodesic loops and BRIST-like cohomology*, Proc. Estonian Acad. Sci. (Phys.-Math) **45** (1996), no. 2/3, 128–133.

[**96b**] *Geodesic multiplication and BRIST-like operators*, General Relativity and Gravitation **28** (1996), no. 8, 991–998.

E.N. Kuz'min

[**68**] *Simple Mal'cev algebras over a field of characteristic zero*, (Russian), Dokl. Akad. Nauk SSSR **181** (1968), no. 6, 1324–1326; English transl.: Soviet Math. Dokl. **9** (1968), no. 4, 1034–1036, **MR** 39 (1970) #1508.

[**70**] *La relation entre les algèbres de Mal'cev et les boucles de Moufang analytiques*, C.R. Acad. Sci. Paris Ser. A-B **271** (1970), no. 23, A1152–A1155, **MR** 42 (1971) #4661.

[**71**] *The connection between Mal'cev algebra and analytic Moufang loops*, (Russian), Algebra and Logic **10** (1971), no. 1, 3–22; English transl.: Algebra and Logic **10** (1971), no. 1, 1–14, **MR** 45 (1973) #6968.

[**77**] *Levi's theorem for Mal'cev algebras*, (Russian), Algebra and Logic **16** (1977), no. 4, 424–431, **MR** 58 (1979) #28113.

A.J. Ledger

[**67**] *Espaces de Riemann symétriques généralisés*, C.R. Acad. Sci. Paris Sér. A **264** (1967), no. 22, 947–948, **MR** 36 (1968) #4487.

A.J. Ledger, M. Obata

[**68**] *Affine and Riemannian s-manifolds*, J. Differential Geom. **2** (1968), no. 4, 451–459, **MR** 39 (1970) #6206.

S. Lie, F. Engel

[**1888**] *Theorie de Transformationsgruppen*, Vol. 1 and 2, Teubingen, Leipzig, 1988, reprinted in 1930.

[**1893**] *Theorie de Transformationsgruppen*, Vol. 3, Teubingen, Leipzig, 1893, reprinted in 1930.

N.I. Lobachevski

[**56**] *On the Principles of Geometry*, (Russian), Foundations of Geometry, GITTL Press, Moscow, 1956, pp. 27–49.

E.K. Loginov

[**96**] *Linear representations of analytic Moufang loops*, Algebras, Groups and Geometries **13** (1996), no. 4, 471–478.

J. Lohmus, E. Paal, L. Sorgsepp

[**89**] *On currents and symmetries associated with Mal'cev algebras*, Preprint F-50 (1989), Institute of Physics. Estonian Acad. Sci., Tartu.

[**94**] *Nonassociative Algebras in Physics*, Hadronic Press, 1994, pp. 260.

O. Loos

[**69**] *Symmetric Spaces*, Vol. 1, 2, Benjamin, New York, 1969, **MR** 39 (1970) #365a,b.

A.I. Mal'cev

[**55**] *Analytic loops*, (Russian), Mat. Sb. (N.S.) **36 (78)** (1955), no. 3, 569–576, **MR** 16 (1955), p. 997.

[**70**] *Algebraic Systems*, (Russian), Nauka, Moscow, 1970, 392 pp., **MR** 44 (1972) #142.

O.A. Matveev

[**86**] *On manifolds with geodesics*, (Russian), Webs and Quasigroups (1986), Kalinin University Press, 44–49.

[**87**] *The odular theory of geodesic mappings*, Ph.D. Dissertation (Russian), Friendship of Nations University, Moscow, 1987.

[**91**] *On locally invariant Affine connections*, (Russian), Webs and Quasigroups, Tver Univ. Press, 1991, pp. 78–97.

P.O. Miheev

[**86a**] *The geometry of smooth Bol loops*, (Russian), Ph.D. Dissertation, Friendship of Nations University, Moscow, 1986.

[**86b**] *On G-property of local analytic Bol loops*, (Russian), Webs and Quasigroups (1986), 54–59, **MR** 88i:22005.

[**88**] *Idempotent quasigroups and manifolds with geodesics*, (Russian), Webs and Quasigroups (1988), 41–46, **MR** 89f:53066.

[**90a**] *Commutator algebras of right monoalternative algebras*, (Russian), Mathematical Explorations **113** (1990), 62–65, **MR** 91i:17051.

[**90b**] *Quasigroups of transformations*, Transactions of Inst. of Physics **66** (1990), Tartu, Estonia, 54–66.

[**90c**] *On loops with left pseudospecial property*, (Russian), Webs and Quasigroups (1990), 39–45, **MR** 91f:53014.

[**91**] *Pseudoautomorphisms of smooth loops*, Algebras, Groups and Geometries **8** (1991), no. 2, 131–144.

[**92**] *Nuclei of loops with right monoalternative property*, (Russian), Webs and Quasigroups (1992), 64–68.

[**93a**] *On groups enveloping Moufang loops*, (Russian), Advances in Mathematical Sciences **48** (1993), no. 2, 191–192; English transl.: Russian Mathematical Survey **48** (1993), no. 2, London Math. Soc.

[**93b**] *On diassociative analytic laws of composition*, (Russian), Mathematical Notes **54** (1993), no. 5, 72–77.

[**94**] *Analytic loops with identities of the hypospecial type*, (Russian), Advances in Mathematical Sciences **49** (1994), no. 4, 298; English transl.: Russian Mathematical Survey **49** (1994), no. 4, London Math. Soc..

D. Montgomery, L. Zippin

[**55**] *Topological Transformation Groups*, Wiley Interscience Publishers, New York, 1955, **MR** 17 (1956), p. 383.

G.D. Mostow

[**55**] *Some new decomposition theorems for semi-simple groups*, Mem. Amer. Math. Soc. **14** (1955), 31–54, **MR** 16 (1955), p. 1087.

R. Moufang

[**35**] *Zur Struktur von Alternativkörpern*, Math. Ann. **110** (1934), 416–430, **Zbl.** 10 (1935), p. 4.

P. Nagy, M. Funk

[**93**] *On collineation group generated by Bol reflections*, J. Geom. **48** (1993), 63–78.

P. Nagy, K. Strambach

[**94**] *Loops as invariant sections in groups and their geometries*, Canad. J. Math. **16** (1994), no. 5, 1027–1056.

[**98**] *Loops, their cores and symmetric spaces*, Israel J. Math. **105** (1998), 285–322, **MR** 99h:20105.

A.I. Nesterov

[**89**] *The methods of nonassociative algebra in physics*, Dr. Sci. Dissertation (Russian), Institute of Physics of Estonian Acad. Sci., Tartu, Estonia, 1989, 198 pp.

[**90**] *Quasigroup ideas in physics*, Transactions of Inst. of Physics, Estonian Acad. Sci. **66** (1990), 67–78.

[**97**] *Quasigroups, Asymptotic Symmetries and Conservation Laws in General Relativity*, Physical Review (Rapid Communications) **D56** (1997), no. 12, 1–5.

A.I. Nesterov, V.A. Stepanenko

[**86**] *On Methods of Nonassociative Algebra in Geometry and Physics*, (Russian); Preprint no. 400 Φ, Institute of Physics, Acad. Sci. USSR, Krasnoyarsk, 1986, 48 pp.

I.A. Nourou

[**92**] *Geometry of smooth hyporeductive loops*, (Russian), Dr.Ph. Dissertation, Friendship of Nations University, Moscow, 1992, 68 pp.

[**94**] *Examples of hypospecial loops*, Algebras, Groups and Geometries **13** (1994), no. 4, 499–514.

T. Nono

[**58**] *On geodesic subspaces of group spaces*, J. Sci. Hiroshima Univ. Ser. A **21** (1958), no. 3, 167–176, **MR** 20 (1959) #3934.

[**61**] *Sur les familles triples locales de transformations locales de Lie*, J. Sci. Hiroshima Univ., Ser. A-I Math. **25** (1961), 357–366, **MR** 28 (1964) #2174b.

E.P. Osipov

[**89**] *The central extension of Kac–Moody–Mal'cev algebras*, Letters in Mathematical Physics **18** (1989), 35–42.

E. Paal

[**88**] *Analytic Moufang transformations*, Preprint F-46 (1988), Institute of Physics, Estonian Acad. Sci., Tartu.

[**89**] *On nonassociative enlargement of group theory methods based on Moufang loops and Mal'cev algebras*, (Russian), Dr.Ph. Dissertation, Institute of Physics of Estonian Acad. Sci., Tartu, Estonia, 1989, 136 pp.

H.O. Pflugfelder

[**90**] *Quasigroups and Loops: An Introduction*, Heldermann Verlag, Berlin, 1990.

L.S. Pontryagin

[**73**] *Continuous Groups*, 3^{d} ed., (Russian), Nauka, Moscow, 1973, 519 pp., **MR** 50 (1975) #10141; English translation of the 2^{nd} Russian edition: Gordon and Breach Science Publishers, Inc., New York-London-Paris, 1966, xv + 543 pp., **MR** 34 (1967) #1439.

P.K. Rashevski

[**50**] *Symmetric spaces of affine connection with torsion*, (Russian), Proceedings of Sem. on Vector and Tensor Analysis **8** (1950), Moscow University, 82–92, **MR** 12 (1951), p. 534.

[**51**] *On the geometry of homogeneous spaces*, (Russian), Dokl. Akad. Nauk SSSR **80** (1951), 169–171, **MR** 13 (1952), p. 383.

[**52**] *On the geometry of homogeneous spaces*, (Russian), Proceedings of Sem. on Vector and Tensor Analysis **9** (1952), Moscow University, 49–74, **MR** 14 (1953), p. 795.

D.A. Robinson

[**66**] *Bol loops*, Trans. Amer. Math. Soc. **123** (1966), 341–354, **MR** 33 (1967) #2755.

[**68**] *A Bol loop isomorphic to all loop isotopes*, Proc. Amer. Math. Soc. **19** (1968), 671–672, **MR** 36 (1968) #6530; **MR** 37 (1969) #1470.

[**72**] *Bol quasigroups*, Publ. Math. Debrecen **19** (1972), 151–153, **MR** 48 (1974) #4175.

[**77**] *A special embedding of Bol loops in groups*, Acta Math. Acad. Sci. Hungar. **30** (1977), 95–103, **MR** 56 (1978) #15813.

[**79**] *A loop-theoretic study of right-sided quasigroups*, Ann. Soc. Sci. Bruxelles Ser. I **93** (1979), no. 1, 7–16, **MR** 80k:20072.

[**80**] *The Bryant-Schneider group of a loop*, Ann. Soc. Sci. Bruxelles Ser. I **94** (1980), no. 2–3, 69–81, **MR** 82c:20122.

L.V. Sabinin

[**58a**] *On the geometry of subsymmetric spaces*, (Russian), Scientific Reports of Higher School, Ser. Phys.-Math. Sci. **3** (1958), 46–49.

[**58b**] *On the structure of the groups of motions of homogeneous Riemannian spaces with axial symmetry*, (Russian), Scientific Reports of Higher School, Ser. Phys.-Math. Sci. **6** (1958), 127–138.

[**61**] *On the geometry of tri-symmetric Riemannian spaces*, (Russian), Siberian Math. J. **2** (1961), no. 2, 266–278, **MR** 24 #A2350.

[**70**] *On the classification of tri-symmetrical spaces*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **194** (1970), no. 3, 518–520; English transl.: Soviet Math. Dokl. **11** (1970), no. 5, 1245–1247, Amer. Math. Soc.

[**72a**] *On the equivalence of categories of loops and homogeneous spaces*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **205** (1972), no. 3, 533–536; English transl.: Soviet Math. Dokl. **13** (1972), no. 4, 970–974, Amer. Math. Soc. **MR** 46 (1973) #9220.

[**72b**] *The geometry of loops*, (Russian), Mathematical Notes **12** (1972), no. 5, Nauka Press, 605–616; English transl.: Mathematical Notes **12** (1973), no. 5, 799–805, **MR** 49 (1975) #5216.

[**72c**] *On the geometry of loops*, (Russian), Abstracts of the 5-th Conference on contemporary problems in Differential Geometry (Samarkand, 20–24 October 1972), 1972, p. 192.

[**72d**] *Tri-symmetric spaces with simple compact groups of motions*, (Russian), Proceedings of Sem. on Vector and Tensor Analysis **16** (1972), 202-226, Moscow University, **MR** 48 #1110.

[**77**] *Odules as a new approach to a geometry with a connection*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **233** (1977), no. 5, 800–803; English transl.: Soviet Math. Dokl. **18** (1977), no. 2, 515–518, Amer. Math. Soc. **MR** 57 (1979) #1340.

[**81**] *Methods of Nonassociative Algebra in Differential Geometry*, (Russian), Supplement to Russian translation of S. Kobayashi and K. Nomizu "Foundations of Differential Geometry" Vol. 1, (1981), Nauka Press, Moscow, 293–339, **MR** 84b:53002.

[**82**] *On the geometry of almost symmetric spaces*, (Russian), Applied problems of Differential Geometry, Moscow District Pedagogical Institute, Dep. No. 1648–82, VINITI Press, Moscow, 1982, pp. 14–15.

[**86a**] *On the theory of special smooth loops*, (Russian), Principle of inclusion and invariant tensors, Moscow District Pedagogical Institute, Dep. No. 426–B86, VINITI Press, Moscow, 1986, pp. 113–117.

[**86b**] *Tangent affine connections of loopuscular structures*, (Russian), Webs and Quasigroups (1986), 86–89, Kalinin University Press, **MR** 88m:53043.

[**87**] *Geometric odules*, (Russian), Webs and Quasigroups (1987), Kalinin University Press, 88–98, **Zbl.** 637.53014, **MR** 88i:53041.

[**88a**] *On nonlinear geometric algebra*, (Russian), Webs and Quasigroups (1988), 32–37, Kalinin University Press, **MR** 89e:20121.

[**88b**] *Differential equations of smooth loops*, (Russian), Proceedings of Sem. on Vector and Tensor Analysis **23** (1988), 133–146, Moscow University, **MR** 91h:53002.

[**89**] *Differential geometry and quasigroups*, (Russian), Proceedings of Institute of Mathematics, Siberian branch of Acad. Sci. of the USSR **14** (1989), Nauka Press, Novosibirsk, 208–221, **MR** 91f:53015.

[**90a**] *Algebraic structures of nonlinear geometric algebra*, (Russian), Quasigroups and systems of quasigroups, Mathematical Explorations **113** (1990), Stiinca Press, Kishinev, 83–88, **MR** 91f:53016.

[**90b**] *On the infinitesimal theory of smooth hyporeductive loops*, (Russian), Webs and Quasigroups (1990), 33–39, Kalinin University Press, **MR** 91e:53003.

[**90c**] *Smooth hyporeductive loops*, (Russian), Variational methods in contemporary geometry. Collection of papers (1990), 50–69, Friendship of Nations University Press, Moscow, **MR** 92g:22008.

[**90d**] *On smooth hyporeductive loops*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **314** (1990), no. 3, 565–568; English transl.: Soviet Math. Dokl. **42** (1991), no. 2, 524–526, American Math. Soc. **MR** 92d:22002.

[**90e**] *Quasigroups, geometry and physics*, Proceedings of International meeting "Quasigroups and nonassociative algebras in Physics", Transactions of Institute of Physics, Estonian Acad. Sci. **66** (1990), 24–53, Tartu, Estonia, **MR** 93d:20125.

[**91a**] *Smooth hyporeductive loops*, Proceedings of the International Conference "Nonlinear Geometric Algebra–89", (Friendship of Nations University, Moscow, January 24–31, 1989), Webs and Quasigroups (1991), 129–137, Tver University Press, **MR** 92f:53003.

[**91b**] *Analytic Quasigroups and Geometry*, Monograph (Russian), Friendship of Nations University Press, Moscow, 1991, 112 pp., **MR** 95d:53013.

[**91c**] *On the theory of φ-spaces*, (Russian), Proceedings of Sem. on Vector and Tensor Analysis **24** (1991), 180-185, Moscow University.

[**92a**] L. Bokut, Yu. Ershov, O. Kegel, A. Kostrikin (eds.), *Smooth quasigroups and loops. New results*, Contemporary Mathematics, Proceedings of the International Conference on Algebra (dedicated to the Memory of A.I. Mal'cev), vol. **131** (Part 1), American Math. Soc., 1992, pp. 707–712, **MR** 93i:22002.

[**92b**] *On flat geoodular spaces*, Webs and Quasigroups (1992), 4–9, Tver University Press, **MR** 94f:53023.

[**94a**] *On differential equations of smooth loops*, (Russian), Advances in Mathematical Sciences **49** (1994), no. 2, 165–166, Moscow Math. Soc.; English transl.: Russian Mathematical Survey **49** (1994), no. 2, 172–173, London Math. Soc., **MR** 95f:22009.

[**94b**] *Geoodular axiomatics of affine spaces*, Note di Matematica **14** (1994), no. 1, 109–113, **MR** 98c:20121.

[**95a**] *On gyrogroups of A. Ungar*, (Russian), Advances in Math. Sciences **50** (1995), no. 5, 251–252, Moscow Math. Soc.; English transl.: Russian Mathematical Survey **50** (1995), no. 5, London Math. Soc.

[**95b**] *Differential equations of geometric odule*, (Russian), Dokl. Akad. Nauk **344** (1995), no. 6, 745–748; English transl.: Reports of Russian Acad. Sci. (Math.) **52** (1995), no. 2, 268–270, **MR** 96m:53033.

[**96a**] *The theory of smooth hyporeductive and pseudoreductive loops*, Algebras, Groups and Geometries **13** (1996), 1–24, **MR** 97c:22003.

[**96b**] *Homogeneous spaces and quasigroups*, (Russian), Communications of Higher School (Math.) (1996), no. 7, 77–84, Kazan University Press; English transl.: Russian Mathematics (Iz. VUZ) **40** (1996), no. 7, 74–81, **MR** 97k:53053.

[**97**] *On the infinitesimal theory of reductive loops*, Reports of Russian Acad. Sci. (Math.) **353** (1997), no. 1, 26–28 (Russian); English transl.: Doklady Mathematics **55** (1997), no. 2, 185–187.

[**98a**] *Quasigroups, geometry and nonlinear geometric algebra*, Acta Applicandae Mathematicae **50** (1998), 46–66, **MR** 99h:20106.
[**98b**] *Smooth Loops I*, Algebras, Groups and Geometries **15** (1998), 127–153.
[**99a**] *Methods of non-associative algebra in differential geometry*, Proceedings of the 7-th International Conference on Differential Geometry and Applications, (Brno, Czech Republic, August 10-14, 1998), pp. 419–427.
[**99b**] *Smooth Quasigroups and Loops*, Monograph. Mathematics and Its Applications, Vol. 492, XVI+250, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.
[**2000**] *Smooth quasigroups and loops. Recent achievements and open problems*, Proceedings of IV International Conference on Non-associative Algebra and its Applications, (San Paulo, Brazil, July 19–24, 1998), Marcel Dekker, 2000, pp. 337–344.

L.V. Sabinin, C. Castillo
[**99**] *Generalized Bol loops*, Herald of Friendship of Nations University (Math.) **6** (1999), no. 1, 182–190.

L.V. Sabinin, O.A. Matveev
[**95**] *Geodesic loops and some classes of affinely connected manifolds* (*Survey on Odular Geometry*), Herald of Friendship of Nations University (Math.) **2** (1995), no. 1, 135–143.

L.V. Sabinin, O.A. Matveev, S.S. Yantranova
[**86**] *On the identity of quasi-linearity in differentiable linear geodiodular manifolds*, (Russian), Invariant tensors, Moscow District Pedagogical Institute, Dep. No. 6553-B86, VINITI Press, Moscow, 1986, pp. 9–18.

L.V. Sabinin, P.O. Miheev
[**82a**] *On a symmetric connection in the space of an analytic Moufang loop*, (Russian), Reports of Acad. Sci. of the USSR **262** (1982), no. 4, 807–809; English transl.: Soviet Math. Dokl. **25** (1982), no. 1, 136–138, Amer. Math. Soc., **MR** 84e:53027.
[**82b**] *On analytic Bol loops*, (Russian), Webs and Quasigroups (1982), Kalinin University Press, 102–109, **MR** 84c:22007.
[**84**] *On the geometry of smooth Bol loops*, (Russian), Webs and Quasigroups (1984), Kalinin University Press, 144–154, **Zbl.**568.53009, **MR** 88c:53001.
[**85a**] *On the differential geometry of Bol loops*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **281** (1985), no. 5, 1055-1057; English transl.: Soviet Math. Dokl. **31** (1985), no. 2, 389–391, Amer. Math. Soc., **MR** 86k: 53022.
[**85b**] *The Theory of Smooth Bol loops*, Lecture Notes, Friendship of Nations University Press, Moscow, 1985, 81 pp., (Bilingual: in English and in Russian), **MR** 87j:22030.
[**85c**] *On local analytic loops with the right alternative identity*, (Russian), Problems of the theory of Webs and Quasigroups (1985), Kalinin University Press, pp. 72–75, **Zbl.** 572.20056, **MR** 88d:22006.
[**87**] *On the infinitesimal theory of local analytic loops*, (Russian), Reports of Acad. Sci. of the USSR (Math.) **297** (1987), no. 4, 801–804; English transl.: Soviet Math. Dokl. **36** (1988), no. 3, 545–548, Amer. Math. Soc., **MR** 89g:22003.
[**88**] *Smooth quasigroups and geometry*, (Russian), Problems in Geometry **20** (1988), VINITI Press, Moscow, 75–110, **MR** 90b:53063.
[**90**] *Quasigroups and differential geometry*, in Quasigroups and Loops: Theory and Applications, Collective monograph, (O. Chein, H. Pflugfelder and J.D.H Smith, eds.), Heldermann Verlag, Berlin, 1990, Chapter XII, pp. 357–430, **MR** 93g:20133.
[**93**] *On the law of addition of velocities in special relativity*, (Russian), Advances in Mathematical Sciences **48** (1993), no. 5, 183–184, Moscow Math. Soc.; English transl.: Russian Mathematical Survey **48** (1993), no. 5, London Math. Soc., **MR** 95a:83011.

[**94**] *Almost symmetric and antisymmetric spaces of affine connection*, (Russian), Reports of Russian Acad. Sci. (Math.) **337** (1994), no. 4, 454–455, **MR** 95g:53016.

L.V. Sabinin, A.I. Nesterov

[**97a**] *Smooth loops and Thomas precession*, Hadronic Journal **20** (1997), 219–237.

[**97b**] *Smooth loops, generalized coherent states and geometric phases*, International Journal of Theoretical Physics **36** (1997), no. 9, 1981–1989, **MR** 98i:81095.

L.V. Sabinin, L.L. Sabinina

[**90**] *On the geometry of transsymmetric spaces*, (Russian), Webs and Quasigroups (1990), 64–68, Kalinin University Press, **MR** 91f:53046.

[**91**] *On geometry of transsymmetric spaces*, Proceedings of the International Conference "Nonlinear Geometric Algebra–89" (Friendship of Nations University, Moscow, January 24–31, 1989), Webs and Quasigroups (1991), 117–122, Tver University Press, **MR** 92f:53003.

[**95**] *On the theory of left F-quasigroups*, Algebras, Groups and Geometries **12** (1995), 127–137, **MR** 96d:20066.

[**99**] *On Bol-Bruck loops*, Webs and Quasigroups (1998–99), 106–108.

L.V. Sabinin, L.L. Sabinina, R. Jimenez

[**97**] *On C^1-smooth commutative Moufang loops and distributive quasigroups*, Webs and Quasigroups (1996/97), 86–88, Tver University Press.

L.V. Sabinin, L.L. Sabinina, L.V. Sbitneva

[**98**] *On the notion of gyrogroup*, Aequationes Mathematicae **56** (1998), no. 1, 11–17, **MR** 99i:83004.

[**99**] *Perfect transsymmetric spaces*, Publicaciones Mathematicae **54** (1999).

L.V. Sabinin, L.V. Sbitneva

[**90**] *Reductive spaces and F-quasigroups*, (Russian), Webs and Quasigroups (1990), Kalinin University Press, 89–94, **MR** 91h:20093.

[**91**] *Reductive spaces and left F-quasigroups*, Proceedings of the International Conference "Nonlinear Geometric Algebra–89" (Friendship of Nations University, Moscow, January 24–31, 1989), Webs and Quasigroups (1991), 123–128, Tver University Press.

[**94**] *Half Bol loops*, Webs and Quasigroups (1994), 50–54, Tver University Press.

[**96**] *Left quasigroups and reductive spaces*, Algebras, Groups and Geometries **13** (1996), 479–487, **MR** 57k:20119.

L.V. Sabinin, Yu. A. Selivanov

[**94**] *On left loops with the second half Bol property*, Webs and Quasigroups (1994), 55–56, Tver University Press, **MR** 97e:53022.

L.V. Sabinin, B.L. Sharma

[**76**] *On the existence of half-Bol loops*, Ann. Stiint. Univ. Al. I. Cuza. Iasi. Sect. Ia Mat. (N.S.) **22** (1976), no. 2, 147–148, Romania, **MR** 55 (1977) #3132.

L.V. Sabinin, A.M. Shelekhov

[**88**] *On the problem of universality of the identity of geometricity*, (Russian), Webs and Quasigroups (1988), 84–87, Kalinin University Press, **MR** 89f:22004.

L.V. Sabinin, S.S. Yantranova

[**84**] *On canonical reductants of spaces with constant curvature*, (Russian), Webs and Quasigroups (1984), 76–83, Kalinin University Press, **MR** 88c:53001.

L.L. Sabinina

[**86a**] *On left distributive quasigroups*, (Russian), Webs and Quasigroups (1986), Kalinin University Press, 90–91, **MR** #88i:20106.

[**86b**] *On the structure of left distributive quasigroups*, (Russian), Inclusion principle and Invariant Tensors, Moscow District Pedagogical Institute (1986), VINITI Press, Dep. No. #426 B 86.

[**88**] *On the theory of F-quasigroups*, (Russian), Webs and Quasigroups (1988), Kalinin University Press, 127–130, **MR** #89i:20112.

[**89**] *On the geometry of F-quasigroups*, (Russian), Differential Geometry and Multiplicative Integral, Moscow District Pedagogical Institute (1989), VINITI Press, Dep. No. #3299 B 89.

[**92**] *On the geometry of smooth F-quasigroups*, (Russian), Dr.Ph. Dissertation, Friendship of Nations University, Moscow, 1992.

[**93**] *On geodesic loops of transsymmetric spaces*, Proceedings of International Conference on Nonassociative Algebra and Applications (7–11 July 1993, Oviedo, Spain).

[**94**] *Transsymmetric spaces*, Proceedings of XXII International Conference on Differential Geometric Methods in Theoretical Physics, (20–25 September 1993, Ixtapa–Zihuatanejo, Mexico), Advances in Applied Clifford Algebras (Proc. Suppl.) **4(S1)** (1994), Universidad Nacional Autonoma de México, 509–514.

[**95**] *On geodesic loops of transsymmetric spaces*, Algebras, Groups and Geometries **12** (1995), 119-125.

A.A. Sagle

[**61**] *Mal'cev algebras*, Trans. Amer. Math. Soc. **101** (1961), no. 3, 426–458, **MR** 26 (1963) #1343.

[**62a**] *Simple Mal'cev algebras over fields of characteristic zero*, Pacific J. Math. **12** (1962), no. 3, 1057–1078, **MR** 27 (1964) #184.

[**62b**] *On derivations of semi-simple Mal'cev algebras*, Portug. Math. **21** (1962), 107–109, **MR** 25 (1963) #3967.

[**64**] *On anti-commutative algebras with an invariant form*, Canad. J. Math. **16** (1964), 370–378, **MR** 29 (1965) #3510.

[**65a**] *On simple algebras obtained from homogeneous general Lie triple systems*, Pacific J. Math. **15** (1965), 1397–1400, **MR** 32 (1966) #5798.

[**65b**] *On simple extended Lie algebras over fields of characteristic zero*, Pacific J. Math. **15** (1965), 621–648, **MR** 32 (1966) #7612.

[**65c**] *Remarks on simple extended Lie algebras*, Pacific J. Math. **15** (1965), 613–620, **MR** 32 (1966) #7613.

[**65d**] *On anticommutative algebras and general Lie triple systems*, Pacific J. Math. **15** (1965), 281–291, **MR** 31 (1966) #1283.

[**65e**] *On anti-commutative algebras and analytic loops*, Canad. J. Math. **17** (1965), 550–558, **MR** 31 (1966) #214.

[**74**] *Jordan algebras and connections on homogeneous spaces*, Trans. Amer. Math. Soc. **187** (1974), 405–427, **MR** 49 (1975) #3776.

[**76**] *Power-associative algebras and Riemannian connections*, Pacific J. Math. **65** (1976), 493–498, **MR** 54 (1977) #11224.

A.A. Sagle, J.R. Schumi

[**73**] *Multiplications on homogeneous spaces, non-associative algebras and connections*, Pacific J. Math. **48** (1973), 247–266, **MR** 51 (1976) #11361.

[**77**] *Anti-commutative algebras and homogeneous spaces with multiplications*, Pacific J. Math. **68** (1977), 255–269, **MR** 57 (1979) #10647.

A.A. Sagle, R.E. Walde

[**73**] *Introduction to Lie Groups and Lie Algebras*, Academic Press, New York and London, 1973, **MR** 50 (1975) #13374.

L.V. Sbitneva

[**79**] *Perfect s-structures*, (Russian), Differential Geometry of Figures' Manifolds **10** (1979), Kaliningrad Univ. Press, 97–103, **MR** 82j:53059.

[**82a**] *On Lie algebras of perfect s-spaces*, (Russian), Webs and Quasigroups (1982), Kalinin University Press, 128–133, **MR** 84h:53068.

[**82b**] *On the existence of nontrivial perfect s-structures*, (Russian), Problems in Homological Algebra. Collection of papers (1982), Yaroslavl Univ. Press, 137–138, **MR** 86m:53064.

[**82c**] *On the structure of perfect s-spaces*, (Russian), Applied problems of Differential Geometry (1982), no. 1, Moscow District Pedagogical Institute, VINITI Press, Dep. No. 1648–82.

[**84a**] *Perfect s-spaces*, (Russian), Synopsis of Dr.Ph. Dissertation, Friendship Univ. Press, Moscow, 1984, pp. 10.

[**84b**] *Perfect s-spaces*, (Russian), Dr.Ph. Dissertation, Friendship of Nations University, Moscow, 1984, pp. 87.

[**86a**] *On the infinitesimal theory of smooth M-loops*, (Russian), Webs and Quasigroups (1986), Kalinin University Press, pp. 92–95, **MR** 88k:20098.

[**86b**] *On the theory of smooth M-loops*, (Russian), Inclusion Principle and Invariant Tensors (1986), Moscow District Pedagogical Institute, 121–125, VINITI Press, Dep. No. 426 - V - 86.

[**88**] *On the conditions of right monoalternativity for geometric odules*, (Russian), Webs and Quasigroups (1988), Kalinin University Press, pp. 46–48, **MR** 89f:22005.

[**99a**] *Remarks on Lie triple spaces*, (Russian), Webs and Quasigroups (1999), Tver University Press, 104–105.

[**99b**] *Transsymmetric spaces. New results*, Proceedings of the 7-th International Conference 'Differential Geometry and Applications' (Brno, Czech Republic, August 10–14, 1998), 1999, pp. 437–442.

[**99c**] *Algebraic structures of perfect transsymmetric spaces*, Aportaciones Matemáticas, Serie Communicaciones **25** (1999), 161–165.

[**2000**] *Algebraic Structure of Transsymmetric Spaces*, in Nonassociative Algebra and Its Applications, Costa, Grishkov, Guzzo, Peresi (eds.), Marcel Dekker, New York, 2000, pp. 349–355.

R.D. Schafer

[**66**] *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966, **MR** 35 (1968) #1643.

B.L. Sharma

[**76**] *Left loops which satisfy the left Bol identity*, Proc. Amer. Math. Soc. **61** (1976), no. 2, 189–195, **MR** 54 (1977) #10467.

J.D.H. Smith

[**76a**] *Mal'cev Varieties*, Lecture Notes in Mathematics, vol. 554, Springer-Verlag, Berlin-New York, 1976, **MR** 55 (1978) #5499.

[**76b**] *Centralizer rings of multiplication groups on quasigroups*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 427–431, **MR** 53 (1977) #3178.

[**88**] *Multilinear algebras and Lie's theorem for formal n-loops*, Arch. Math. **51** (1988), Basel, 169–177, **Zbl.** 627.22003.

[**92**] *Quasigroup representation theory*, Universal Algebra and Quasigroups Theory, (A. Romanovska, J.D.H. Smith (eds.)), Heldermann Verlag, Berlin, 1992, pp. 195–207.

K. Strambach

[**75a**] *Reguläre idempotente Multiplikationen*, Math. Z. **145** (1975), 43–62, **MR** 54 (1977) #1215.

[**75b**] *Rechtsdistributive Quasigruppen auf* 1*-Manningfaligkeiten*, Math. Z. **145** (1975), 63–68, **MR** 54 (1976) #3425.
[**76**] *Distributive quasigroups*, Foundation of Geometry (Proc. Conf. Univ. Toronto, 1974), Univ. Toronto Press, Toronto, 1976, pp. 251–276, **MR** 54 (1977) #2861.

A. Ungar
[**90**] *Weakly associative groups*, Results in Math. **17** (1990), 149–168.
[**94**] *The holomorphic automorphism group of the complex disk*, Aequationes Mathematicae **17** (1994), no. 2, 240–254.
[**97**] *Thomas precession: its underlying gyrogroup axioms and their use in hyperbolic geometry and relativistic physics*, Foundations of Physics **27** (1997), 881–951.

A. Vanžurová
[**99**] *Sabinin's method for classification of local Bol loops*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II **59** (1999), 209–220.

K. Yamaguti
[**58a**] *On algebras of totally geodesic subspaces* (*Lie triple systems*), J. Sci. Hiroshima Univ. Ser. **A 21** (1957/1958), no. 2, 107–113, **MR** 20 (1959) #6482.
[**58b**] *On the Lie triple systems and its generalization*, J. Sci. Hiroshima Univ. Ser. **A 21** (1958), no. 3, 155–160, **MR** 20 (1959) #6483.

S.S. Yantranova
[**89**] *Canonical reductants of symmetric spaces of rank* 1, (Russian), Dr.Ph. Dissertation, Friendship of Nations University, Moscow, 1989.

D.V. Yuriev
[**87**] *Octavion and superoctavion symmetries in exceptional gauge theories*, Theoretical and Mathematical Physics **73** (1987), no. 1, 74–78.
[**92**] *Noncommutative geometry, chiral anomalies in the quantum projective* $\mathfrak{sl}(2,\mathbb{C})$*-invariant field theory and* $j\ell(2,\mathbb{C})$*-invariance*, J. Math. Phys. **33** (1992), 2819–2822.

Department of Mathematics, Quintana Roo University, 77019 Chetumal, Quintana Roo, Mexico

E-mail: lsabinin@balam.cuc.uqroo.mx

(*Received* October 7, 1999)