Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 41 (2000), No. 3, 449--457

Persistent URL: http://dml.cz/dmlcz/119180

Comment.Math.Univ.Carolin. 41,3 (2000)449–457

Equivalence of the properties (β) and (NUC) in Orlicz spaces

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Abstract. We obtain the equivalence of the properties (β) and (NUC) in Orlicz function spaces. This answers a question raised by Y. Cui, R. Pluciennik and T. Wang.

Keywords: Orlicz spaces, property (β), property (NUC)
Classification: 46E30, 46E40, 46B20

Introduction

Let $(X, \|.\|)$ be a real Banach space, and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. the unit sphere) of X. For any subset A of X, we denote by $\mathrm{conv}(A)$, the convex hull of A. Clarkson [2] introduced the concept of uniform convexity. The norm $\|.\|$ is called *uniformly convex* (write (UC)) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ inequality $\|x - y\| > \varepsilon$ implies

$$\left\|\frac{1}{2}(x+y)\right\| < 1-\delta.$$

For any $x \notin B(X)$, the *drop* determined by x is the set

$$D(x, B(X)) = \mathrm{conv}(\{x\} \cup B(X)).$$

Rolewicz [12] introduced the notion of drop property for Banach spaces. A Banach space X has the *drop property* (write(D)) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A sequence $\{x_n\} \subset X$ is said to be *ε-separated* for some $\varepsilon > 0$ if

$$\mathrm{sep}(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon.$$

A Banach space X is said to be *nearly uniformly convex* (write (NUC)) if for every $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that for every sequence $\{x_n\} \subset B(X)$ with $\mathrm{sep}(\{x_n\}) > \varepsilon$, we have

$$\mathrm{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset.$$

A Banach space X is said to have *property* (β) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\alpha(D(x, B(X))\backslash B(X)) < \varepsilon$$

whenever $1 < \|x\| < 1 + \delta$. Here α is the Kuratowski measure of noncompactness on bounded subsets of X. Rolewicz [12] showed that property (β) follows from the uniform convexity and that property (β) implies (NUC). All of these concepts are related as follows:

$$(UC) \Rightarrow (\beta) \Rightarrow (NUC) \Rightarrow (D) \Rightarrow (Rfx), \tag{1}$$

where (Rfx) denotes reflexivity. The implications cannot be reversed in general (see [5], [7], [8], [9], [11], and [12]).

Denote by $\mathbb{R}$ the set of real numbers.

A map $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is vanishing at 0, even, convex and not identically equal to 0. We say that the Orlicz function Φ satisfies $\triangle_2$-*condition* if there exist a constant $k > 2$ and $u_0 > 0$ such that

$$\Phi(2u) \leq k\Phi(u),$$

for every $|u| \geq u_0$.

Let (G, Σ, μ) be a nonatomic measure space with a finite measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G. Let l^0 stand for the space of all real sequences. By the *Orlicz function space* L_Φ, we mean

$$L_\Phi = \{x \in L^0 : I_\Phi(cx) = \int_G \Phi(cx(t))d\mu < \infty \text{ for some } c > 0\}.$$

Analogously, we define the *Orlicz sequence space* l_Φ by the formula

$$l_\Phi = \{x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx_i) < \infty \text{ for some } c > 0\}.$$

L_Φ and l_Φ are equipped with the so called *Luxemburg norm*

$$\|x\| = \inf\{\varepsilon > 0 : I_\Phi(\frac{x}{\varepsilon}) \leq 1\}$$

or with the equivalent norm

$$\|x\|_0 = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

called the *Orlicz norm.* It is well known that for any $x \neq 0$ if, for some k,

$$I_\Psi(p(|kx|)) = 1,$$

where Ψ is the complementary function of Φ and p is the right hand derivative of Φ, then

$$\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)).$$

Write L_Φ, l_Φ, L^0_Φ and l^0_Φ for the spaces $(L_\Phi, \|.\|)$, $(l_\Phi, \|.\|)$, $(L_\Phi, \|.\|_0)$, and $(l^0_\Phi, \|.\|_0)$ respectively.

The Orlicz function Φ is *strictly convex* if

$$\Phi(\frac{u+v}{2}) < \frac{\Phi(u) + \Phi(v)}{2}$$

for all $u, v \in \mathbb{R}$, $u \neq v$.
The Orlicz function Φ is said to be *uniformly convex on* $[u_0, \infty)$, where $u_0 > 0$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi(\frac{u+v}{2}) \leq (1-\delta)\frac{\Phi(u) + \Phi(v)}{2}$$

holds true for all $u, v \in [u_0, \infty)$ satisfying

$$|u - v| \geq \varepsilon. \max\{u, v\}.$$

For more details we refer to [1] or [9].

In the course of the proof, we use the fact from Theorem 3 and 4 in [3] which states for L_Φ and L^0_Φ that

$$(\beta) \Leftrightarrow \Phi \text{ is uniformly convex on } [u_0, \infty) \text{ for every } u_0 > 0 \text{ and } \Phi \text{ satisfies } \triangle_2\text{-condition.}$$

Results. In [3], it was shown that properties (β), (NUC) and (D) are equivalent for Orlicz sequence space l_Φ, that is the second and the third implication in (1) can be reversed. The authors gave an example showing that the implication $(\beta) \Rightarrow$ (UC) is not true for spaces l_Φ and l^0_Φ. But they continue to show in contrast to the sequence case that the properties (UC) and (β) are equivalent for Orlicz function spaces L_Φ and L^0_Φ. The only problem left open in the paper concerning the implication in (1) is whether or not (NUC) $\Rightarrow$ (β) in Orlicz spaces L_Φ and L^0_Φ. We show here the answer is affirmative. The proof of the result is mostly based on ingredients in the proofs appearing in [3].

Theorem. *The properties* (β) *and* (*NUC*) *are equivalent for* L_Φ *and* L^0_Φ.

Before we give the proof of the Theorem, we prove a simple but useful result. It is a characterization of uniform convexity of Φ. The author has been informed by the referee that the following lemma is related to some results of S. Chen and H. Hudzik, *On some convexities of Orlicz and Orlicz-Bochner spaces*, Comment. Math. Univ. Carolinae, **29.1** (1988).

Lemma. *For an Orlicz function Φ, Φ is uniformly convex if and only if for any $\varepsilon > 0$ and any $u_0 > 0$, there exists $\delta > 0$ such that for all couples $(u, v) \subset (u_0, \infty)$ satisfying $v - u \geq \varepsilon v$ we have*

$$\Phi(ru + sv) \leq (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for some $r, s \in (0, 1)$ with $r + s = 1$.

Proof: We only prove the "sufficiency". Suppose Φ is not uniformly convex. Thus there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exist u, v with

$$(u, v) \subset (0, \infty), v - u \geq \varepsilon v, \text{ and } p(v) < (1 + \delta)p(u).$$

We now show that

$$\Phi(ru + sv) > (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for all $r, s \in (0, 1)$ with $r + s = 1$.
Write $w = ru + sv$ for such a pair (r, s). Then put

$$\mathrm{I} = \frac{r\Phi(u) + s\Phi(v) - \Phi(u)}{w - u}, \quad \mathrm{II} = \frac{\Phi(w)}{w - u},$$

and

$$\mathrm{III} = \frac{\Phi(w) - \Phi(u)}{w - u}.$$

We estimate

$$\mathrm{I} = \frac{s(\Phi(v) - \Phi(u))}{s(v - u)} = \frac{\Phi(v) - \Phi(u)}{v - u} < (1 + \delta)p(u),$$

$$\mathrm{II} > \frac{\Phi(w) - \Phi(u)}{w - u} > p(u),$$

and

$$\mathrm{III} > p(u).$$

Thus,

$$\frac{r\Phi(u) + s\Phi(v) - \Phi(ru + sv)}{\Phi(ru + sv)} = \frac{\frac{r\Phi(u)+s\Phi(v)-\Phi(u)}{w-u} - \frac{\Phi(ru+sv)-\Phi(u)}{w-u}}{\frac{\Phi(ru+sv)}{w-u}}$$

$$= \frac{\text{I-III}}{\mathrm{II}} < \frac{(1 + \delta)p(u) - p(u)}{p(u)} = \delta.$$

Hence

$$\Phi(ru + sv) > \frac{1}{\delta}(r\Phi(u) + s\Phi(v) - \Phi(ru + sv))$$

and so

$$(1+\frac{1}{\delta})\Phi(ru+sv) > \frac{1}{\delta}(r\Phi(u)+s\Phi(v)).$$

This implies

$$\Phi(ru+sv) > (1-\frac{\delta}{1+\delta})(r\Phi(u)+s\Phi(v)) > (1-\delta)(r\Phi(u)+s\Phi(v)).$$

□

Proof of Theorem: We first consider the space L_Φ. From Theorem 3 in [3] we only need to show that (NUC) implies uniform convexity of Φ on $[u,\infty)$ for all $u>0$. For this, it is enough to show that Φ is strictly convex on $[0,\infty)$ and that there exists $v>0$ such that Φ is uniformly convex on $[v,\infty)$.

If Φ is not strictly convex, we obtain an interval $[a,b]$ in $[0,\infty)$, $G^0 \subset G$, $G' \subset G \backslash G^0$ and $c>0$ as in [3] such that Φ is affine on $[a,b]$ and

$$\Phi(\frac{a+b}{2})\mu(G^0)+\Phi(c)\mu(G')=1.$$

Then, for each n, we obtain a partition $\{G_1^n, G_2^n, \ldots, G_{2^n}^n\}$ of G^0 such that

$$\mu(G_i^n)=2^{-n}\mu(G^0) \quad (i=1,2,\ldots,2^n).$$

Define

$$x_n = a\chi_{E_{1,n}} + b\chi_{E_{2,n}} + c\chi_{G'},$$

where

$$E_{1,n}=\bigcup_{k=1}^{2^{n-1}} G_{2k-1}^n,\ E_{2,n}=\bigcup_{k=1}^{2^{n-1}} G_{2k}^n, \quad (n=1,2,\ldots).$$

We show that $\{x_n\}$ violates the property (NUC) by showing that

$$x_n \in B(L_\Phi) \ \text{ for each } \ n\geq 1,\ \operatorname{sep}(\{x_n\}) > \frac{b-a}{\Phi^{-1}(\frac{2}{\mu(G^0)})}$$

and

$$\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(L_\Phi)=\emptyset \ \text{ for all } \ \delta>0.$$

Since

$$\begin{aligned} I_\Phi(x_n) &= \frac{\Phi(a)+\Phi(b)}{2}\mu(G^0)+\Phi(c)\mu(G') \\ &= \frac{\Phi(a+b)}{2}\mu(G^0)+\Phi(c)\mu(G')=1, \end{aligned}$$

we first have $\|x_n\| = 1$.
Secondly, we have $\|x_n - y_n\| = \frac{b-a}{\Phi^{-1}(\frac{2}{\mu(G^0)})} > 0$ whenever $n \neq m$.
Finally let $r_1, \ldots, r_n \geq 0$ and $r_1 + \ldots + r_n = 1$. Put $x = r_1 x_1 + \ldots + r_n x_n$. Since the values of x on G^0 are convex combinations of a and b with coefficients in $[0,1]$, an easy calculation shows that

$$I_\Phi(x) = \frac{\Phi(a) + \Phi(b)}{2}\mu(G^0) + \Phi(c)\mu(G') = 1.$$

Thus $\|x\| = 1 > 1 - \delta$ for all $\delta > 0$. Therefore Φ is strictly convex.

We now show that Φ is uniformly convex on $[u, \infty)$ for "large" u. Again we suppose for the contrary that there exists $\varepsilon > 0$ such that for any $\delta > 0$ we can find u, v by the Lemma and $G^0 \subset G$ so that $0 < u < v$,

$$\Phi(u)\mu(G) \geq 1, \;\; v - u \geq \varepsilon v, \;\; \frac{\Phi(u) + \Phi(v)}{2}\mu(G^0) = 1,$$

and

$$\Phi(ru + sv) > (1 - \delta)(r\Phi(u) + s\Phi(v))$$

for all pairs (r, s) in $(0, 1)$ with $r + s = 1$.
Define

$$x_n = u\chi_{E_{1,n}} + v\chi_{E_{2,n}}$$

where $E_{1,n}$ and $E_{2,n}$ are constructed as above. Again we show that $\{x_n\}$ violates the property (NUC) by showing that

$$x_n \in B(L_\Phi) \;\text{ for each }\; n \geq 1, \; \mathrm{sep}(\{x_n\}) > \frac{\varepsilon}{2}$$

and

$$\mathrm{conv}(\{x_n\}) \cap (1 - \delta)B(L_\Phi) = \emptyset.$$

We estimate for $n, m \geq 1$ and $n \neq m$,

$$I_\Phi(x_n) = (\Phi(u) + \Phi(v))\frac{\mu(G^0)}{2} = 1,$$

and

$$I_\Phi(2\frac{x_n - x_m}{\varepsilon}) = \Phi(2\frac{v-u}{\varepsilon})\frac{\mu(G^0)}{2} \geq \Phi(v)\mu(G^0) > \frac{\Phi(u) + \Phi(v)}{2}\mu(G^0) = 1.$$

Thus $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{\varepsilon}{2}$.
Next let $x = r_1 x_1 + \ldots + r_n x_n$ be a convex combination of $x_1, \ldots, x_n$ and estimate

$$I_\Phi(x) > (1 - \delta)(\Phi(u) + \Phi(v))\frac{\mu(G^0)}{2} = 1 - \delta,$$

whence $\|x\| > 1 - \delta$.

We consider now the space L^0_Φ. If Φ is not strictly convex, we obtain as in [3], positive numbers a, b, c, and subsets G^0 of G and G' of $G \backslash G^0$ so that p is constant on $[a, b]$, $\mu(G \backslash G^0) > 0$, and

$$\Psi(p(a))\mu(G^0) + \Psi(p(c))\mu(G') = 1.$$

Denote

$$k = 1 + \Phi(\frac{a+b}{2})\mu(G^0) + \Phi(c)\mu(G').$$

Put

$$x_n = \frac{1}{k}(a\chi_{E_{1,n}} + b\chi_{E_{2,n}} + c\chi_{G'}).$$

Since $I_\Psi(p(kx_n)) = 1$, we have

$$\|x_n\|_0 = \frac{1}{k}(1 + I_\Phi(kx_n)) = 1.$$

Also it is seen that for some A with $\mu(A) = \frac{\mu(G^0)}{2}$,

$$\|x_n - x_m\|_0 = \|\frac{a-b}{k}\chi_A\|_0 = \frac{b-a}{k}\|\chi_A\|_0 = \frac{b-a}{k}\mu(A)\Psi^{-1}(\frac{1}{\mu(A)}) > 0$$

whenever $n \neq m$. Now if $x = r_1x_1 + \cdots + r_nx_n$ is a convex combination of $x_1, \ldots, x_n$, we obtain

$$I_\Phi(kx) = \frac{\Phi(a) + \Phi(b)}{2}\mu(G^0) + \Phi(c)\mu(G') = k - 1,$$

and

$$I_\Psi(p(kx)) = \Psi(p(a))\mu(G^0) + \Psi(p(c))\mu(G') = 1.$$

Thus $\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)) = 1$. This contradicts the property (NUC).

If Φ is not uniformly convex outside a neighborhood of zero, we can find an $\varepsilon > 0$ such that for each $\delta > 0$ there exist numbers u, v with $v - u \geq \varepsilon v > \varepsilon u > 0$,

$$\frac{\Psi(p(v)) + \Psi(p(u))}{2}\mu(G) \geq 1,$$

and

$$p(v) < (1 + \delta)p(u).$$

We choose $G^0 \subset G$ so that

$$\frac{\Psi(p(v))+\Psi(p(u))}{2}\mu(G^0)=1$$

and put

$$k=\frac{u(p(u))+v(p(v))}{2}\mu(G^0).$$

Define

$$x_n=\frac{1}{k}(u\chi_{E_{1,n}}+v\chi_{E_{2,n}}),$$

where $E_{1,n},\ E_{2,n}$ are defined as before.
Since

$$I_\Psi(p(kx_n))=\frac{\Psi(p(v))+\Psi(p(u))}{2}\mu(G^0)=1$$

and

$$I_\Phi(kx_n)=\frac{\Phi(u)+\Phi(v)}{2}\mu(G^0),$$

we see that

$$\|x_n\|_0=\frac{2+(\Phi(u)+\Phi(v))\mu(G^0)}{2k}=1.$$

We also see that for $n\neq m$ we have for some A with $\mu(A)=\frac{\mu(G^0)}{2}$,

$$\begin{aligned}\|x_n-x_m\|_0=\frac{v-u}{k}\|\chi_A\|=\frac{v-u}{k}\mu(A)\Psi^{-1}(\frac{1}{\mu(A)})\\ =\frac{v-u}{2k}\mu(G^0)\Psi^{-1}(\frac{2}{\mu(G^0)})>\frac{\varepsilon}{2k}vp(v)\mu(G^0)\geq\frac{\varepsilon}{2}.\end{aligned}$$

This follows from the fact that

$$k=\frac{up(u)+vp(v)}{2}\mu(G^0)\leq vp(v)\mu(G^0),$$

and

$$2=(\Psi(p(u))+\Psi(p(v)))\mu(G^0)>\Psi(p(v))\mu(G^0).$$

Now if $x=r_1x_1+\ldots+r_nx_n$ is a linear convex combination of $x_1,\ldots,x_n$, put

$$y=p(u)\chi_{E_{1,n}}+p(v)\chi_{E_{2,n}}.$$

Thus $I_\Psi(y) = 1$. It is straightforward to see that

$$\begin{aligned}
\|kx\|_0 \geq \int_G kxy &= [p(u)((1+r_n)u + (1-r_n)v) + p(v)((1+r_n)v + (1-r_n)u)]\frac{\mu(G^0)}{4} \\
&\geq [up(u) + vp(v) + (u+v)p(u)]\frac{\mu(G^0)}{4} \\
&= \frac{up(u)+vp(v)}{4}\mu(G^0) + \frac{vp(u)+up(u)}{4}\mu(G^0) \\
&> \frac{k}{2} + (\frac{vp(v)}{1+\delta} + up(u))\frac{\mu(G^0)}{4} \\
&= \frac{k}{2} + \frac{up(u)+vp(v)}{4}\mu(G^0) - \frac{\delta}{1+\delta}vp(v)\frac{\mu(G^0)}{4} \\
&> k - \delta vp(v)\frac{\mu(G^0)}{4} > k - \frac{\delta k}{2}.
\end{aligned}$$

Thus $\|x\|_0 > 1 - \frac{\delta}{2} > 1 - \delta$.
This shows that $\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(L^0_\Phi) = \emptyset$. □

Acknowledgment. The author would like to thank Professor R. Pluciennik for his helpful conversations during his short visit to Chiang Mai University, Thailand. The author also wish to thank an anonymous referee for his suggestions which led to substantial improvements of this paper.

References

[1] Chen S., *Geometry of Orlicz spaces*, Dissertationes Mathematicae 356, Warszawa, 1996.

[2] Clarkson J.A., *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[3] Cui Y., Pluciennik R., Wang T., *On property (β) in Orlicz spaces*, Arch. Math. **68** (1997), 1–13.

[4] Huff R., *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 473–549.

[5] Kutzarova D.N., *A nearly uniformly convex space which is not a (β) space*, Acta Univ. Carolinae Math. Phys. **30** (1989), 95–98.

[6] Kutzarova D.N., *An isomorphic characterization of property (β) of Rolewicz*, Note Mat. **10.2** (1990), 347–354.

[7] Kutzarova D.N., *On condition (β) and Δ-uniform convexity*, C.R. Acad. Bulgar Sci. **42.1** (1989), 15–18.

[8] Montesinos V., *Drop property equals reflexivity*, Studia Math. **87** (1987), 93–100.

[9] Montesinos V., Torregrosa J.R., *A uniform geometric property of Banach spaces*, Rocky Mountain J. Math. **22.2** (1992), 683–690.

[10] Musielak J., *Orlicz spaces and modular spaces*, LNM 1034, pp. 1–222, Berlin-Heidelberg-New York, 1983.

[11] Rolewicz S., *On drop property*, Studia Math. **85** (1987), 27–35.

[12] Rolewicz S., *On Δ-uniform convexity and drop property*, Studia Math. **87** (1987), 181–191.

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(*Received* February 5, 1999)