Surjective factorization of holomorphic mappings

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Abstract. We characterize the holomorphic mappings $f$ between complex Banach spaces that may be written in the form $f = T \circ g$, where $g$ is another holomorphic mapping and $T$ belongs to a closed surjective operator ideal.

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1. Introduction and preliminary results

In recent years many authors [1], [2], [7], [9], [10], [15], [19], [20] have studied conditions on a holomorphic mapping $f$ between complex Banach spaces so that it may be written in the form either $f = g \circ T$ or $f = T \circ g$, where $g$ is another holomorphic mapping and $T$ a (linear bounded) operator belonging to certain classes of operators.

A rather thorough study of the factorization of the form $f = g \circ T$, where $T$ is in a closed injective operator ideal, was carried out by the authors in [10]. In the present paper we analyze the case $f = T \circ g$.

If $f = T \circ g$, with $T$ in the ideal of compact operators, and $g$ is holomorphic on a Banach space $E$ then, since $g$ is locally bounded, $f$ will be “locally compact” in the sense that every $x \in E$ has a neighborhood $V_x$ such that $f(V_x)$ is relatively compact. It is proved in [2] that the converse also holds: every locally compact holomorphic mapping $f$ can be written in the form $f = T \circ g$, with $T$ a compact operator. Similar results were given in [20] for the ideal of weakly compact operators, in [15] for the Rosenthal operators, and in [19] for the Asplund operators. We extend this type of factorization to every closed surjective operator ideal.

Throughout, $E$, $F$ and $G$ will denote complex Banach spaces, and $\mathbb{N}$ will be the set of natural numbers. We use $B_E$ for the closed unit ball of $E$, and $B(x, r)$ for the open ball of radius $r$ centered at $x$. If $A \subset E$, then $\bar{\Gamma}(A)$ denotes the absolutely convex, closed hull of $A$, and if $f$ is a mapping on $E$, then

$$\|f\|_A := \sup\{|f(x)| : x \in A\}.$$  

We denote by $\mathcal{L}(E, F)$ the space of all operators from $E$ into $F$, endowed with the usual operator norm. A mapping $P : E \to F$ is a $k$-homogeneous (continuous)
polynomial if there is a $k$-linear continuous mapping $A : E \times \underbrace{E \times \cdots \times E}_k \to F$ such that $P(x) = A(x, \ldots, x)$ for all $x \in E$. The space of all such polynomials is denoted by $\mathcal{P}^{(k)E,F}$. A mapping $f : E \to F$ is holomorphic if, for each $x \in E$, there are $r > 0$ and a sequence $(P_k)$ with $P_k \in \mathcal{P}^{(k)E,F}$ such that

$$f(y) = \sum_{k=0}^{\infty} P_k(y - x)$$

uniformly for $\|y - x\| < r$. We use the notation

$$P_k = \frac{1}{k!} d^k f(x),$$

while $\mathcal{H}(E,F)$ stands for the space of all holomorphic mappings from $E$ into $F$.

We say that a subset $A \subset E$ is circled if for every $x \in A$ and complex $\lambda$ with $|\lambda| = 1$, we have $\lambda x \in A$.

For a general introduction to polynomials and holomorphic mappings, the reader is referred to [5], [16], [17]. The definition and general properties of operator ideals may be seen in [18].

An operator ideal $U$ is said to be injective ([18, 4.6.9]) if, given an operator $T \in \mathcal{L}(E,F)$ and an injective isomorphism $i : F \to G$, we have that $T \in U$ whenever $iT \in U$. The ideal $U$ is surjective ([18, 4.7.9]) if, given $T \in \mathcal{L}(E,F)$ and a surjective operator $q : G \to E$, we have that $T \in U$ whenever $Tq \in U$. We say that $U$ is closed ([18, 4.2.4]) if for all $E$ and $F$, the space $U(E,F) := \{T \in \mathcal{L}(E,F) : T \in U\}$ is closed in $\mathcal{L}(E,F)$.

Given an operator $T \in \mathcal{L}(E,F)$, a procedure is described in [4] to construct a Banach space $Y$ and operators $k \in \mathcal{L}(E,Y)$ and $j \in \mathcal{L}(Y,F)$ so that $T = jk$. We shall refer to this construction as the DFJP factorization. It is shown in [12, Propositions 1.6 and 1.7] (see also [8, Proposition 2.2] for simple statement and proof) that given an operator $T \in \mathcal{L}(E,F)$ and a closed operator ideal $U$,

(a) if $U$ is injective and $T \in U$, then $k \in U$;

(b) if $U$ is surjective and $T \in U$, then $j \in U$.

We say that $U$ is factorizable if, for every $T \in \mathcal{U}(E,F)$, there are a Banach space $Y$ and operators $k \in \mathcal{L}(E,Y)$ and $j \in \mathcal{L}(Y,F)$ so that $T = jk$ and the identity $I_Y$ of the space $Y$ belongs to $U$.

We now give a list of closed operator ideals which are injective, surjective or factorizable. We recall the definition of the most commonly used, and give a reference for the others.

An operator $T \in \mathcal{L}(E,F)$ is (weakly) compact if $T(B_E)$ is a relatively (weakly) compact subset of $F$; $T$ is (weakly) completely continuous if it takes weak Cauchy sequences in $E$ into (weakly) convergent sequences in $F$; $T$ is Rosenthal if every sequence in $T(B_E)$ has a weak Cauchy subsequence; $T$ is unconditionally converging if it takes weakly unconditionally Cauchy series in $E$ into unconditionally convergent series in $F$. 
The results on this list may be found in [18] and the other references given, for the injective and surjective case. The factorizable case may be seen in [12].

If $\mathcal{U}$ is an operator ideal, the dual ideal $\mathcal{U}^d$ is the ideal of all operators $T$ such that the adjoint $T^*$ belongs to $\mathcal{U}$. Easily, we have:

$\mathcal{U}$ is closed injective $\implies \mathcal{U}^d$ is closed surjective

$\mathcal{U}$ is closed surjective $\implies \mathcal{U}^d$ is closed injective

The list above might therefore be completed with some more dual ideals.

Moreover, to each $T \in \mathcal{L}(E, F)$ we can associate an operator $T^q : E^{**}/E \to F^{**}/F$ given by $T^q(x^{**} + E) = T^{**}(x^{**}) + F$. Let $\mathcal{U}^q := \{T \in \mathcal{L}(E, F) : T^q \in \mathcal{U}\}$. Then, if $\mathcal{U}$ is injective (resp. surjective, closed), so is $\mathcal{U}^q$ ([8, Theorem 1.6]).

Remark 1. There is another notion of factorizable operator ideal which may be used. We say that $\mathcal{U}$ is DFJP factorizable ([8, Definition 2.3]) if, for every $T \in \mathcal{U}$, the identity of the intermediate space in the DFJP factorization of $T$ belongs to $\mathcal{U}$. Clearly, every DFJP factorizable operator ideal is factorizable. The following example shows that the converse is not true. Let $\mathcal{A}$ be the ideal of all the operators that factor through a subspace of $c_0$. Clearly, $\mathcal{A}$ is factorizable. Consider the operator $T : \ell_2 \to \ell_2$ given by $T((x_n)) := (x_n/n)$. We have $T \in \mathcal{A}$. The intermediate space in the DFJP factorization is an infinite dimensional reflexive space. Clearly, the identity map on it does not belong to $\mathcal{A}$.

All the factorizable ideals on the table above are DFJP factorizable ([8]). Note also that, if $\mathcal{U}$ is DFJP factorizable, then so are $\mathcal{U}^d$ and $\mathcal{U}^q$ ([8]).
2. Surjective factorization

In this section, we study the factorizations in the form $T \circ g$, with $T \in \mathcal{U}$, where $\mathcal{U}$ is a closed surjective operator ideal.

**Lemma 2** ([13, Proposition 2.9]). Given a closed surjective operator ideal $\mathcal{U}$, let $S \in \mathcal{L}(E, F)$ and suppose that for every $\epsilon > 0$ there are a Banach space $D_\epsilon$ and an operator $T_\epsilon \in \mathcal{U}(D_\epsilon, F)$ such that

$$S(B_E) \subseteq T_\epsilon(B_{D_\epsilon}) + \epsilon B_F.$$ 

Then, $S \in \mathcal{U}$.

We denote by $\mathcal{C}_\mathcal{U}(E)$ the collection of all $A \subset E$ so that $A \subseteq T(B_Z)$ for some Banach space $Z$ and some operator $T \in \mathcal{U}(Z, E)$ (see [21]).

The following probably well-known properties of $\mathcal{C}_\mathcal{U}$ will be needed:

**Proposition 3.** Let $\mathcal{U}$ be a closed surjective operator ideal. Then:

(a) if $A \in \mathcal{C}_\mathcal{U}(E)$ and $B \subset A$, then $B \in \mathcal{C}_\mathcal{U}(E)$;
(b) if $A_1, \ldots, A_n \in \mathcal{C}_\mathcal{U}(E)$, then $\cup_{i=1}^n A_i \in \mathcal{C}_\mathcal{U}(E)$ and $\sum_{i=1}^n A_i \in \mathcal{C}_\mathcal{U}(E)$;
(c) if $A \subset E$ is bounded and, for every $\epsilon > 0$, there is a set $A_\epsilon \in \mathcal{C}_\mathcal{U}(E)$ such that $A \subseteq A_\epsilon + \epsilon B_E$, then $A \in \mathcal{C}_\mathcal{U}(E)$.
(d) if $A \in \mathcal{C}_\mathcal{U}(E)$, then $\bar{\Gamma}(A) \in \mathcal{C}_\mathcal{U}(E)$;

**Proof:** (a) is trivial and (b) is easy. Both are true without any assumption on the operator ideal $\mathcal{U}$.

(c) For $A \subset E$ bounded, consider the operator

$$T : \ell_1(A) \longrightarrow E \quad \text{given by} \quad T((\lambda_x)_{x \in A}) = \sum_{x \in A} \lambda_x x.$$ 

Given $\epsilon > 0$, there is $A_\epsilon \in \mathcal{C}_\mathcal{U}(E)$ such that $A \subseteq A_\epsilon + \epsilon B_E$. Therefore,

$$A \subseteq T(B_{\ell_1(A)}) \subseteq \bar{\Gamma}(A) \subseteq \Gamma(A) + \epsilon B_E \subseteq \Gamma(A_\epsilon) + 2\epsilon B_E.$$ 

Clearly, $\Gamma(A_\epsilon) \in \mathcal{C}_\mathcal{U}(E)$. Hence, $T \in \mathcal{U}$ (by Lemma 2), and $A \in \mathcal{C}_\mathcal{U}(E)$.

(d) If $A \in \mathcal{C}_\mathcal{U}(E)$, there is a space $Z$ and $T \in \mathcal{U}(Z, E)$ such that $A \subseteq T(B_Z)$. Therefore, for all $\epsilon > 0$,

$$\bar{\Gamma}(A) \subseteq \overline{T(B_Z)} \subseteq T(B_Z) + \epsilon B_E.$$ 

Now, it is enough to apply part (c). \hfill \Box

We shall denote by $\mathcal{H}_\mathcal{U}(E, F)$ the space of all $f \in \mathcal{H}(E, F)$ such that each $x \in E$ has a neighborhood $V_x$ with $f(V_x) \in \mathcal{C}_\mathcal{U}(F)$. Easily, a polynomial $P \in \mathcal{P}^{(kE, F)}$ belongs to $\mathcal{H}_\mathcal{U}(E, F)$ if and only if $P(B_E) \in \mathcal{C}_\mathcal{U}(F)$. The set of all such polynomials will be denoted by $\mathcal{P}_\mathcal{U}^{(kE, F)}$.

The following result is an easy consequence of the Hahn-Banach theorem and the Cauchy inequality.
**Lemma 4** ([20, Lemma 3.1]). Given $f \in \mathcal{H}(E, F)$, a circled subset $U \subset E$, and $x \in E$, we have

$$\frac{1}{k!} d^k f(x)(U) \subseteq \Gamma(f(x + U))$$

for every $k \in \mathbb{N}$.

**Proposition 5.** Let $\mathcal{U}$ be a closed surjective operator ideal, and $f \in \mathcal{H}(E, F)$. The following assertions are equivalent:

(a) $f \in \mathcal{H}_\mathcal{U}(E, F)$;

(b) there is a zero neighborhood $V \subset E$ such that $f(V) \in \mathcal{C}_\mathcal{U}(F)$;

(c) for every $k \in \mathbb{N}$ and every $x \in E$, we have that $d^k f(x) \in \mathcal{P}_\mathcal{U}^{(kE, F)}$;

(d) for every $k \in \mathbb{N}$, we have that $d^k f(0) \in \mathcal{P}_\mathcal{U}^{(kE, F)}$.

**Proof:** (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) follow from Lemma 4.

(d) $\Rightarrow$ (a). Let $x \in E$. There is $\epsilon > 0$ such that

$$f(y) = \sum_{k=0}^{\infty} \frac{1}{k!} d^k f(0)(y)$$

uniformly for $y \in B(x, \epsilon)$ ([17, §7, Proposition 1]). By Proposition 3(b), for each $m \in \mathbb{N}$, we have

$$\left\{ \sum_{k=0}^{m} \frac{1}{k!} d^k f(0)(y) : y \in B(x, \epsilon) \right\} \subset \mathcal{C}_\mathcal{U}(F).$$

Using the uniform convergence on $B(x, \epsilon)$, and Proposition 3(c), we conclude that $f(B(x, \epsilon)) \subset \mathcal{C}_\mathcal{U}(F)$.

(a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are trivial. $\square$

If $A$ is a closed convex balanced, bounded subset of $F$, $F_A$ will denote the Banach space obtained by taking the linear span of $A$ with the norm given by its Minkowski functional.

**Theorem 6.** Let $\mathcal{U}$ be a closed surjective operator ideal, and $f \in \mathcal{H}(E, F)$. The following assertions are equivalent:

(a) $f \in \mathcal{H}_\mathcal{U}(E, F)$;

(b) there is a closed convex, balanced subset $K \subset \mathcal{C}_\mathcal{U}(F)$ such that $f$ is a holomorphic mapping from $E$ into $F_K$;

(c) there is a Banach space $G$, a mapping $g \in \mathcal{H}(E, G)$ and an operator $T \in \mathcal{U}(G, F)$ such that $f = T \circ g$.

**Proof:** (a) $\Rightarrow$ (b) follows the ideas in the proof of [2, Proposition 3.5] and [20, Theorem 3.7].
For each $m \in \mathbb{N}$ and $x \in E$, define

$$A_m(x) := \left\{ \lambda y : y \in B\left(x, \frac{1}{m}\right) \text{ and } |\lambda| \leq 1 \right\}$$

and

$$U_m := \bigcup \left\{ B\left(x, \frac{1}{m}\right) : \|x\| \leq m \text{ and } \|f\|_{A_m(x)} \leq m \right\}.$$  

For each $x \in E$ there is a neighborhood of the compact set $\{\lambda x : |\lambda| \leq 1\}$ on which $f$ is bounded. Hence, there is $m \in \mathbb{N}$ so that $\|f\|_{A_m(x)} \leq m$, which shows that $E = \bigcup_{m=1}^{\infty} U_m$.

Let $W_m$ be the balanced hull of $U_m$. Since the sets $A_m(x)$ are balanced, we have $|f(x)| \leq m$ for all $x \in W_m$. Let $V_m := 2^{-1} W_m$. We have $E = \bigcup_{m=1}^{\infty} V_m$ and hence

$$f(E) = \bigcup_{m=1}^{\infty} f(V_m). \quad (1)$$

For each $k, m \in \mathbb{N}$, define

$$K_{mk} := \bar{\Gamma}\left(\frac{1}{k!} d^k f(0)(W_m)\right) \in \mathcal{C}_U(F).$$

By Proposition 3, we obtain that the set

$$K_m := \left\{ \sum_{k=0}^{\infty} 2^{-k} z_k : z_k \in K_{mk} \right\}$$

belongs to $\mathcal{C}_U(F)$. Easily, $f(V_m) \subseteq K_m$. Hence $f(V_m) \in \mathcal{C}_U(F)$ for all $m \in \mathbb{N}$. By Proposition 3, we can select numbers $\beta_m > 0$ with $\sum \beta_m < \infty$ so that

$$K := \bar{\Gamma}\left( \bigcup_{m=1}^{\infty} \beta_m f(V_m) \right) \in \mathcal{C}_U(F).$$

It follows from (1) that $f$ maps $E$ into $F_K$.

It remains to show that $f \in \mathcal{H}(E, F_K)$. Let $x \in E$. Easily, there are $\epsilon > 0$ and $r \in \mathbb{N}$ such that $f(B(x, 2\epsilon)) \subseteq rK$. By Lemma 4,

$$\frac{1}{k!} d^k f(x) \left( B(0, 2\epsilon) \right) \subseteq rK \quad (2)$$

for all $k \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$ and $a \in B(0, \epsilon)$, we have

$$f(x + a) - \sum_{k=0}^{n} \frac{1}{k!} d^k f(x)(a) = 2^{-n} \sum_{k=n+1}^{\infty} 2^{n-k} \frac{1}{k!} d^k f(x)(2a).$$
Since $K$ is convex and closed, we get from (2) that
\[
\sum_{k=n+1}^{\infty} 2^{-k} \frac{1}{k!} d^k f(x)(2a) \in rK.
\]
Hence,
\[
f(x + a) - \sum_{k=0}^{n} \frac{1}{k!} d^k f(x)(a) \in 2^{-n} rK,
\]
and so, the $F_K$-norm of the left hand side is less than or equal to $2^{-n} r$, for all $a \in B(0, \epsilon)$. Thus, $f$ is holomorphic.

(b) $\Rightarrow$ (c). It is enough to note that, by Lemma 2, the natural inclusion $F_K \rightarrow F$ belongs to $U$.

(c) $\Rightarrow$ (a). Each $x \in E$ has a neighborhood $V_x$ such that $g(V_x)$ is bounded in $G$. Hence, $f(V_x) = T(g(V_x)) \in C_\mathcal{U}(F)$. \hfill $\square$

**Theorem 7.** Let $\mathcal{U}$ be a closed surjective, factorizable operator ideal and take a mapping $f \in \mathcal{H}(E, F)$. Then $f \in \mathcal{H}_\mathcal{U}(E, F)$ if and only if there are a Banach space $G$, a mapping $g \in \mathcal{H}(E, G)$ and $T \in \mathcal{U}(G, F)$ such that $I_G \in \mathcal{U}$ and $f = T \circ g$.

**Remark 8.** Theorem 7 implies that, if $\mathcal{U}$ is the ideal of weakly compact (resp. Rosenthal, Banach-Saks or Asplund) operators and $f \in \mathcal{H}_\mathcal{U}(E, F)$, then $f$ factors through a Banach space $G$ which is reflexive (resp. contains a copy of $\ell_1$, has the Banach-Saks property or is Asplund).

Moreover, if $\mathcal{U} = \{ T : T^d$ has separable range $\}$, then $G$ is isomorphic to $G_1 \times G_2$, with $G_1^{**}$ separable and $G_2$ reflexive ([22]). If $\mathcal{U} = \{ T : T^* \text{ is Rosenthal} \}$, then $G$ contains no copy of $\ell_1$ and no quotient isomorphic to $c_0$ ([11]).

**References**


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