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On very weak solutions of a class of nonlinear elliptic systems

Menita Carozza, Antonia Passarelli di Napoli

Abstract. In this paper we prove a regularity result for very weak solutions of equations of the type \(- \text{div} \ A(x, u, Du) = B(x, u, Du)\), where \(A, B\) grow in the gradient like \(t^{p-1}\) and \(B(x, u, Du)\) is not in divergence form. Namely we prove that a very weak solution \(u \in W^{1,r}\) of our equation belongs to \(W^{1,p}\). We also prove global higher integrability for a very weak solution for the Dirichlet problem

\[
\begin{cases}
- \text{div} A(x, u, Du) = B(x, u, Du) & \text{in } \Omega,

u - u_o \in W^{1,r}(\Omega, \mathbb{R}^m).
\end{cases}
\]

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1. Introduction

Let us consider equations of the type

\[
- \text{div} A(x, u, Du) = B(x, u, Du),
\]

where \(x \in \Omega\), a bounded open subset of \(\mathbb{R}^n\), \(n \geq 2\), \(u : \Omega \rightarrow \mathbb{R}^m\), \(m \geq 1\) and \(A : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \rightarrow \mathbb{R}\) and \(B : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \rightarrow \mathbb{R}^n\) are Carathéodory functions such that

(H1) \(\quad |A(x, u, z)| \leq c_1 + c_2|u|^{p-1} + c_3|z|^{p-1},\)

(H2) \(\quad \langle A(x, u, z), z \rangle \geq |z|^p - c_4|u|^p - c_5\)

and

(H3) \(\quad |B(x, u, z)| \leq c_6 + c_7|u|^{p-1} + c_8|z|^{p-1},\)

where \(c_i, i = 1, \ldots, 8\), and \(c\) are positive constants.

The previous assumptions allow us to give the following
Definition 1.1. A mapping \( u \in W^{1,r}_{\text{loc}}(\Omega, \mathbb{R}^m) \), max\{1, p - 1\} \leq r < p, is called a very weak solution of the equation (1.1) if

\[
\int_{\Omega} [A(x, u, Du)D\Phi - B(x, u Du)\Phi] \, dx = 0
\]

for all \( \Phi \in W^{1,r-p+1}_{\text{loc}}(\Omega, \mathbb{R}^m) \) with compact support.

The main result is the following

Theorem 1.2. Let the assumptions (H1)–(H3) hold. Then there exists an exponent \( r_1 = r_1(m, n, p) \), max\{1, p - 1\} < r_1 < p, such that if \( u \in W^{1,r}_{\text{loc}}(\Omega, \mathbb{R}^m) \), \( r_1 \leq r < p \), is a very weak solution of the equation (1.1), then \( u \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^m) \).

The theory of very weak solutions of equations of type (1.1) with the right hand-side in divergence form has been initiated by T. Iwaniec and C. Sbordone in \([IS]\). For that type of equations they proved that if \( r \) is sufficiently close to \( p \), then a very weak solution really is a solution (see \([I]\), \([IS]\)). The main tool they used is the Hodge decomposition and later other authors used the same technique to approach similar problems (see \([GLS]\), \([M1]\)). In our case (the right hand-side of (1.1) is not in divergence form) the Hodge decomposition seems to be not useful. In proving Theorem 1.2 we follow the techniques of Lewis (see \([Le]\), \([M2]\)) using the theory about the Hardy-Littlewood maximal function and the \( A_p \)-weights. A fundamental tool in our proof is the choice of a suitable test function, involving level sets of maximal function defined by using a Lemma due to Acerbi and Fusco (see \([AF]\) and Lemma 2.5 below). Another fundamental tool is a well known Hedberg estimate (see \([H]\) and Lemma 2.6 below).

Remark 1.3. With the same techniques we can reobtain Theorem 1.2 for equations of the following type

\[
- \text{div}(w(x) A(x, u, Du)) = w(x) B(x, u, Du)
\]

with \( w(x) \) an \( A_p \)-weight (see \([Mu]\) and Definition 2.1).

Remark 1.4. Note that the Euler-Lagrange system of the functional

\[
I(u) = \int_{\Omega} [|Du|^p + |u|^p + a(x)] \, dx
\]

is of type (1.1). Then Theorem 1.2 says also that a weak minimum of the functional (1.2) (see \([IS]\), \([M2]\)) really is a minimum. Instead for the general functional

\[
I(u) = \int_{\Omega} f(x, u, Du) \, dx,
\]
where \( f \) grows as \( |Du|^p \), the Euler-Lagrange system has the right hand-side not in divergence form but growing with respect to the gradient as \( t^p \). So that, unfortunately, Theorem 1.2 does not recover the previous general case.

Moreover, we consider the boundary value problem

\[
\begin{cases}
- \text{div } A(x, u, Du) = B(x, u, Du) & \text{ in } \Omega \\
 u - u_o \in W^{1,r}(\Omega, \mathbb{R}^m),
\end{cases}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) with Lipschitz boundary and \( A \) and \( B \) verify the assumptions (H1)–(H3). We will prove the global higher integrability of \( Du \), with \( u \) solution of the problem (1.3). More precisely, we will prove the following:

**Theorem 1.5.** Let (H1)–(H3) hold and assume \( u_o \in W^{1,p}(\Omega, \mathbb{R}^m) \). Then there exists an exponent \( r_1 = r_1(m, n, p) \), \( \max\{1, p - 1\} < r_1 < p \) such that if \( u \in W^{1,r}(\Omega, \mathbb{R}^m) \), \( r_1 \leq r < p \), is a very weak solution of the Dirichlet problem (1.3), then \( u \in W^{1,p}(\Omega, \mathbb{R}^m) \).

**2. Preliminaries**

In this section we introduce notations, definitions and preliminary results.

Let \( B(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \} \) and \( |B(x, r)| \) denote its Lebesgue measure. For a measurable function \( f \) on \( \mathbb{R}^n \) we set

\[
f_{x,r} = \int_{B(x,r)} |f(y)| \, dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.
\]

Denote the Hardy-Littlewood maximal function of \( f \) by

\[
Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| \, dy
\]

and set

\[
M^k f(x) = M^{k-1}(Mf)(x) \quad \text{for } k \geq 2.
\]

**Definition 2.1.** For \( 1 < p < \infty \), we say that a nonnegative measurable function \( a \in L^1_{\text{loc}}(\mathbb{R}^n) \) is in the Muckenhoupt class \( A_p \), or is an \( A_p \)-weight if and only if the quantity

\[
A_p(a) = \sup_{x \in \mathbb{R}^n, r > 0} \left( \int_{B(x,r)} a \right) \left( \int_{B(x,r)} a^{-\frac{1}{p-1}} \right)^{p-1}
\]

is finite.

Now let us list some lemmas useful in the sequel.
Lemma 2.2. Let $1 < p < \infty$. There exists a positive constant $c = c(n, p)$ such that for any $0 < 2\delta < p - 1$, the function $(Mf)^{-\delta}$ is an $A_p$-weight and the quantity $A_p((Mf)^{-\delta})$ is less or equal to $c$ for all $f \in L^1(\mathbb{R}^n)$, $f \neq 0$.

For the proof see [Do], [Le] and [T].

We also recall the following well known theorem about $A_p$-weights (see [Mu])

Theorem 2.3. For $1 < p < \infty$ and $a \in A_p$, there exists a positive constant $c = c(p, n, A_p(a))$ such that

$$\int_{\mathbb{R}^n} a(x)(Mf(x))^p \, dx \leq c \int_{\mathbb{R}^n} a(x)|f(x)|^p \, dx$$

for all $f \in L^p(\mathbb{R}^n, a)$.

Moreover we will use the following lemmas.

Lemma 2.4. Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$, $r > 0$ and $B = B(x_0, r)$. If $f \in W^{1,p}(B)$ then there exists $c = c(n, p)$ such that for any $x \in B$

$$|f(x) - f_{x_0,r}| \leq c \ r M(|Df| \chi_B)(x),$$

where $\chi_B$ is the characteristic function of $B$.

Lemma 2.5. Let $\lambda > 0$, $1 < q < \infty$, $x_0 \in \mathbb{R}^n$ and $r > 0$. Suppose $f \in W^{1,q}(\mathbb{R}^n)$, supp $f \subset B(x_0, r)$ and

$$F(\lambda) = \{x : M(|Df|)(x) \leq \lambda\} \cap B(x_0, 2r) \neq \phi.$$

Then $f/F(\lambda)$ has an extension to $\mathbb{R}^n$, denoted by $v = v(\cdot, \lambda)$, such that

(i) $v = f$ on $F(\lambda)$,
(ii) supp $v \subset B(x_0, 2r)$,
(iii) $v \in W^{1,\infty}(\mathbb{R}^n)$ with $\|v\|_{\infty} \leq c \lambda r$ and $\|Dv\|_{\infty} \leq c\lambda$.

Proof: See [AF] and [Le].

The following lemma is a result due to Hedberg (see [H]).

Lemma 2.6. Let $u$ be a function in $W^{1,p}_0(\Omega)$ and $\Omega$ a bounded open subset of $\mathbb{R}^n$. Set

$$I(|Du|)(x) = \int_{\Omega} |Du|(y)|x - y|^{1-n} \, dy.$$

Then, the following estimate holds

$$u(x) \leq c I(|Du|)(x) \leq c M(|Du|)(x) \text{ a.e.}$$

where $c$ is a positive constant depending on the dimension $n$ and on the Lebesgue measure of $\Omega$.

Proof: See [H] and [GT].

Finally, we need the theorem (see [G] and [Gi])
**Theorem 2.7.** Let \( R > 0, \ q > 1 \) and \( g \in L^q(B(x_0, R)) \) be such that

\[
\int_{B(x, \frac{r}{8})} |g|^q \, dx \leq c \left( \int_{B(x, r)} |g| \, dx \right)^q + \vartheta \int_{B(x, r)} |g|^q \, dx + \tilde{c}
\]

for \( 0 < \vartheta < 1 \) and \( x \in B(x_0, R/2), 0 < r \leq R/8 \).

Then there exists \( c' = c'(n, \vartheta, c, q) \) and \( \eta = \eta(n, \vartheta, c, q) > 0 \) such that if \( \tau = q(1 + \eta) \) then

\[
\left( \int_{B(x, R/4)} |g|^\tau \, dx \right)^{1/\tau} \leq c' \left( \int_{B(x, R/2)} |g|^q \, dx \right)^{1/q} + \tilde{c}.
\]

**3. Main results**

**Proof of Theorem 1.2.** Let \( B = B(x_0, R) \subset \Omega \) for some \( R \leq 1 \). For fixed \( y_0 \in B(x_0, R/2) \) and \( 0 < \rho < R/8 \), let \( B_\rho = B(y_0, \rho) \) and \( \varphi \in C^\infty_0(B_\rho) \) be such that \( \varphi = 1 \) on \( B_\rho \), \( 0 \leq \varphi \leq 1 \) on \( B_\rho \) and \( |D\varphi| \leq c \rho^{-1} \).

With \( u_4 = \int_{B_\rho} u(x) \, dx \), we set \( \tilde{u} = (u - u_4)\varphi \), \( E(\lambda) = \{ x \in \mathbb{R}^n : M(|D\tilde{u}|) \leq \lambda \} \) and \( F(\lambda) = E(\lambda) \cap B_\rho \).

Since \( \text{supp} \ \tilde{u} \subset B_\rho \), we observe that for \( x \in \mathbb{R}^n - B_{3\rho} \)

\[
(3.1) \quad M(|D\tilde{u}|)(x) \leq c \rho^{-n} \int_{B_\rho} |D\tilde{u}|(y) \, dy,
\]

where \( c \) is a constant depending only on the dimension \( n \), and setting

\[
\lambda_0 = c \rho^{-n} \int_{B_\rho} |D\tilde{u}|(y) \, dy,
\]

\( F(\lambda) \) is not empty for \( \lambda > \lambda_0 \) and thanks to Lemma 2.5 we can extend the function \( \tilde{u}|_{F(\lambda)} \) to whole \( \mathbb{R}^n \).

Let \( v \) be the extension of \( \tilde{u}|_{F(\lambda)} \). \( v \) satisfies the conditions (i)–(iii) (see Lemma 2.5) so that we can consider \( v \) as a particular test function in Definition 1.1. By (H1) and (H3) we get

\[
\int_{F(\lambda)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] \, dx
\]

\[
= \int_{B_\rho - F(\lambda)} [B(x, u, Du) v - A(x, u, Du) Dv] \, dx
\]

\[
\leq c \lambda \int_{B_\rho - F(\lambda)} [|Du|^{p-1} + |u|^{p-1} + 1] + \rho||Du||^{p-1} + |u|^{p-1} + 1 \right| dx.
\]
Multiplying both sides of the previous inequality by $\lambda^{-(1+\delta)}$, where $\delta = p - r$ will be chosen at the end of the proof, and integrating from $\lambda_0$ to $+\infty$, we have

\[
(3.2) \quad \int_{\lambda_0}^{+\infty} \lambda^{-(1+\delta)} \, d\lambda \int_{B_{4\rho}} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] \chi_{\{M(|D\tilde{u}|) \leq \lambda\}} \, dx
\]

\[
\leq c \int_{\lambda_0}^{+\infty} \lambda^{-\delta} \, d\lambda \int_{B_{4\rho} - F(\lambda)} [(|Du|^{p-1} + |u|^{p-1} + 1) + \rho(|Du|^{p-1} + |u|^{p-1} + 1)] \, dx.
\]

Interchanging the order of integration, the left hand side of (3.2) becomes

\[
\int_{B_{4\rho} - E(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] \, dx \int_{M(|D\tilde{u}|)} \lambda^{-(1+\delta)} \, d\lambda
\]

\[
+ \int_{\lambda_0}^{+\infty} \lambda^{-(1+\delta)} \, d\lambda \int_{E(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] \, dx
\]

\[
(3.3) \quad = \frac{1}{\delta} \int_{B_{4\rho} - E(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
+ \frac{\lambda_0^{-\delta}}{\delta} \int_{E(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] \, dx
\]

\[
\equiv \frac{1}{\delta} J_1 + \frac{\lambda_0^{-\delta}}{\delta} J_2.
\]

Let us recall that supp $\tilde{u} \subset B_{2\rho}$, $\tilde{u} = u$ on $B_{\rho}$ and $B_{4\rho} - E(\lambda_0) = B_{4\rho} - F(\lambda_0)$, so we have

\[
J_1 = \int_{B_{4\rho}} [A(x, u, Du)] D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
- \int_{F(\lambda_0)} [A(x, u, Du)] D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
(3.4) \quad = \int_{B_{2\rho} - B_{\rho}} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
- \int_{F(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
+ \int_{B_{\rho}} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx.
\]

By (3.2), (3.3) and (3.4) we obtain

\[
\frac{1}{\delta} \int_{B_{\rho}} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]

\[
\leq \frac{1}{\delta} \int_{F(\lambda_0)} [A(x, u, Du) D\tilde{u} - B(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx
\]
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\[ + \frac{1}{\delta} \int_{B_2 \rho - B_\rho} [B(x, u, Du) \tilde{u} - A(x, u, Du) \tilde{u}] M(|D\tilde{u}|)^{-\delta} \, dx \]

\[ + \frac{\lambda_0^{-\delta}}{\delta} \int_{E(\lambda_0) \cap B_2 \rho} [B(x, u, Du) \tilde{u} - A(x, u, Du) \tilde{u}] \, dx \]

\[ + c \int_{0}^{+\infty} \lambda^{-\delta} \, d\lambda \int_{B_4 \rho - F(\lambda)} [|Du|^{p-1} + |u|^{p-1} + 1] \, dx. \]

Moreover, since \( \lambda_0^{-\delta} \leq M(|D\tilde{u}|)^{-\delta} \) on \( E(\lambda_0) \), using (H1),(H2),(H3) and multiplying by \( \delta \) we obtain

\[ \int_{B_2 \rho} (|Du|^p) M(|D\tilde{u}|)^{-\delta} \, dx \]

\[ \leq c \int_{E(\lambda_0) \cap B_2 \rho} |(D\tilde{u} + \tilde{u})||Du|^{p-1} + |u|^{p-1} + 1)M(|D\tilde{u}|)^{-\delta} \, dx \]

\[ + c \int_{B_2 \rho - B_\rho} (|D\tilde{u}| |Du|^{p-1} + |D\tilde{u}| |u|^{p-1} + |D\tilde{u}|)M(|D\tilde{u}|)^{-\delta} \, dx \]

\[ + c \int_{B_2 \rho} (|\tilde{u}| |Du|^{p-1} + |\tilde{u}| |u|^{p-1} + |\tilde{u}|)M(|D\tilde{u}|)^{-\delta} \, dx \]

\[ + c \delta \int_{\lambda_0}^{+\infty} \lambda^{-\delta} \, d\lambda \int_{B_4 \rho} (|Du|^{p-1} + |u|^{p-1} + 1) \chi_{\{M(|D\tilde{u}|) > \lambda\}} \, dx. \]

We write the previous relation as

\[ (3.5) \quad I_0 \leq c[I_1 + I_2 + I_3] + c\delta I_4. \]

To simplify the presentation we will estimate the integrals \( I_i, i = 1, 2, 3, 4 \) at the end of this section.

**Conclusion.**

By the estimates of the integrals \( I_i \) below, we get

\[ I_0 \leq c \left( \eta^{1-\delta} + \delta^{1-\delta} + \frac{\delta}{1-\delta} \right) \int_{B_4 \rho} |Du|^{p-\delta} \, dx \]

\[ + c(\eta^{1-p} + \eta^{1-\delta} + \delta^{-\delta}) \rho^n \left( \int_{B_4 \rho} |Du|^t \right)^{\frac{p-\delta}{t}} \]

\[ + c\delta^{-\delta} \int_{B_2 \rho \setminus B_\rho} |Du|^{p-\delta} \, dx + c\rho^n. \]

Observe that by Lemma 2.4

\[ |u(x) - u_{4\rho}| \leq c\rho[M(|Du|_{\chi_{B_4 \rho}})] \quad \text{for any} \quad x \in B_4 \rho \]
and then
\begin{equation}
|D\tilde{u}| \leq |Du| + cM(|Du|\chi_{B_{4\rho}}).
\end{equation}

Since \( \tilde{u} = u \) on \( B_{\rho} \), we see that for \( x \in B_{\rho/2} \)
\[
M(|D\tilde{u}|) \leq M(|Du|\chi_{B_{\rho}}) + c\int_{B_{4\rho}} |D\tilde{u}| \, dx
\]
\[
\leq M(|Du|\chi_{B_{\rho}}) + c\int_{B_{4\rho}} [M(|Du|\chi_{B_{4\rho}})] \, dx.
\]

On the other hand, setting
\[
H = \{ x \in B_{\rho/2} : M(|Du|\chi_{B_{\rho}})(x) \geq c\int_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})(x) \, dx \}
\]
we have
\[
M(|D\tilde{u}|)(x) \leq cM(|Du|\chi_{B_{\rho}})(x) \text{ on } H.
\]
Then
\[
\int_{B_{\rho}} |Du|^p M(|D\tilde{u}|)^{-\delta} \geq c \int_{B_{\rho}} M(|Du|\chi_{B_{\rho}})^p M(|D\tilde{u}|)^{-\delta}
\]
\[
\geq c \int_{H} M(|Du|\chi_{B_{\rho}})^p M(|D\tilde{u}|)^{-\delta} \geq c \int_{H} M(|Du|\chi_{B_{\rho}})^p M(|Du|\chi_{B_{\rho}})^{-\delta} \, dx
\]
\[
= c \int_{B_{\rho/2}} M(|Du|\chi_{B_{\rho}})^{p-\delta} \, dx - c \int_{B_{\rho/2}\setminus H} M(|Du|\chi_{B_{\rho}})^{p-\delta} \, dx
\]
\[
\geq c \int_{B_{\rho/2}} |Du|^{p-\delta} - c\rho^n \left( \int_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}}) \, dx \right)^{p-\delta}
\]
\[
\geq c \int_{B_{\rho/2}} |Du|^{p-\delta} - c\rho^n \left( \int_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})^t \, dx \right)^{\frac{p-\delta}{t}}
\]
\[
\geq c \int_{B_{\rho/2}} |Du|^{p-\delta} - c\rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}},
\]
where we applied Lemma 2.2 and Muckenhoupt’s Theorem in the first and last inequality, in previous estimate. Since we will apply Sobolev-Poincaré inequality in the estimates of \( I_i \), we have to choose \( (p - \delta)_* \leq t \leq p - \delta \), where as usual \( (p - \delta)_* = \frac{n(p-\delta)}{n + p - \delta} \). Then we have
\[
I_0 = \int_{B_{\rho}} |Du|^p M(|D\tilde{u}|)^{-\delta}
\]
\begin{equation}
\geq c \int_{B_{\rho/2}} |Du|^{p-\delta} - c\rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}}.
\end{equation}
From inequalities (3.6) and (3.8) it follows that
\[
\int_{B_{\rho/2}} |Du|^{p-\delta} \, dx
\leq c \left( \eta^{1-\delta} + \delta^{1-\delta} + \frac{\delta}{1-\delta} \right) \int_{B_{4\rho}} |Du|^{p-\delta} \, dx
\]
\[
+ c(\eta^{1-p} + \delta^{-\delta} + \eta^{1-p}) \rho^n \left( \int_{B_{4\rho}} |Du|^t \right)^{\frac{p-\delta}{t}}
\]
\[
+ c\delta^{-\delta} \int_{B_{2\rho} \setminus B_{\rho/2}} |Du|^{p-\delta} \, dx + c\rho^n.
\]
Now, applying the “hole filling”, we add the quantity
\[
c \delta^{-\delta} \int_{B_{\rho/2}} |Du|^{p-\delta} \, dx
\]
to both sides of the previous inequality and we get
\[
\int_{B_{\rho/2}} |Du|^{p-\delta} \, dx
\leq \frac{c}{c\delta^{-\delta} + 1} \left( \eta^{1-\delta} + \delta^{-\delta} + \delta^{1-\delta} + \frac{\delta}{1-\delta} \right) \int_{B_{4\rho}} |Du|^{p-\delta} \, dx
\]
\[
+ \hat{c} \left( \int_{B_{4\rho}} |Du|^t \right)^{\frac{p-\delta}{t}} + \tilde{c}.
\]
Notice that there exist 0 < \delta_1 < 1 and 0 < \eta_1 < 1 such that if 0 < \delta < \delta_1 and 0 < \eta < \eta_1,
\[
\frac{c}{c\delta^{-\delta} + 1} \left( \eta^{1-\delta} + \delta^{-\delta} + \delta^{1-\delta} + \frac{\delta}{1-\delta} \right) \leq \vartheta < 1.
\]
From the estimates above we have for 0 < \delta < \delta_1 and 0 < \eta < \eta_1
\[
\int_{B_{\rho/2}} |Du|^{p-\delta} \, dx
\leq \vartheta \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + \hat{c} \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}} + \tilde{c},
\]
where \hat{c} depends on \( m, n, p \) but not on \( \delta \).

The result follows from Theorem 2.6 with an argument similar to the one of [GLS].
Now let us estimate the integrals \( I_i, \ i = 1, 2, 3, 4. \)

**Estimate of \( I_1. \)**

\[
I_1 = \int_{E(\lambda_0) \cap B_{2\rho}} (|D\tilde{u}| + |\tilde{u}|)(|Du|^{p-1} + |u|^{p-1} + 1)M(|D\tilde{u}|)^{-\delta} \, dx \\
\leq c \int_{E(\lambda_0) \cap B_{2\rho}} (|Du|^{p-1} + |u|^{p-1} + 1)M(|D\tilde{u}|)^{1-\delta} \, dx
\]

by Lemma 2.6.

Let us suppose \( 0 < \eta \leq \frac{1}{2} \) and \( |Du| \geq \eta^{-1}\lambda_0 \), then at \( x \in E(\lambda_0) \) we have

\[
|Du|^{p-1}M(|D\tilde{u}|)^{1-\delta} \leq \eta^{1-\delta}|Du|^{p-\delta}.
\]

On the other hand, if \( x \in E(\lambda_0) \) and \( |Du| < \eta^{-1}\lambda_0 \) we get

\[
|Du|^{p-1}M(|D\tilde{u}|)^{1-\delta} \leq \eta^{1-p}\lambda_0^{p-\delta}.
\]

Then by (3.10), (3.11) in \( E(\lambda_0) \cap B_{2\rho} \) we have

\[
|Du|^{p-1}M(|D\tilde{u}|)^{1-\delta} \leq c(\eta^{1-p}\lambda_0^{p-\delta} + \eta^{1-\delta}|Du|^{p-\delta}).
\]

By the definition of \( \lambda_0 \) and formula (3.7), we note that

\[
\eta^{1-p}\lambda_0^{p-\delta} \leq c \eta^{1-p} \left( \int_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}}) \, dx \right)^{p-\delta} \\
\leq c \eta^{1-p} \left( \int_{B_{4\rho}} M(|Du|\chi_{B_{4\rho}})^t \, dx \right)^{p-\delta} \left( \frac{t}{t} \right).
\]

where \( (p - \delta)^* = \frac{n(p-\delta)}{n+p-\delta} \leq t < p - \delta. \) Finally, by the estimates above and the Hardy-Littlewood theorem we get

\[
I_1 \leq c \eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c \eta^{1-p}\rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}} \\
+ \int_{E(\lambda_0) \cap B_{2\rho}} (|u|^{p-1} + 1)M(|D\tilde{u}|)^{1-\delta} \, dx.
\]
On the other hand, for $0 < \eta \leq \frac{1}{2}$ and $|u| \geq \eta^{-1} \lambda_0$, we have for $x \in E(\lambda_0)$

$$|u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \leq |u|^{p-\delta} \eta^{1-\delta} \lambda_0^{-1} M(|D\tilde{u}|)^{1-\delta} \leq \eta^{1-\delta} |u|^{p-\delta}.$$ 

If $|u| < \eta^{-1} \lambda_0$, we have

$$|u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \leq c \eta^{1-p} \lambda_0^{-1} \lambda_0^{1-\delta} = c \eta^{1-p} \lambda_0^{p-\delta}.$$ 

Therefore, by estimate (3.12) above,

$$\int_{E(\lambda_0) \cap B_{2\rho}} |u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \leq c \eta^{1-p} \rho^n \left( \int_{B_{4\rho}} |D\tilde{u}|^t \, dx \right)^{\frac{p-\delta}{t}} + c \eta^{1-\delta} \int_{E(\lambda_0) \cap B_{2\rho}} |u|^{p-\delta} \leq \int_{B_{4\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx$$

with $t < p - \delta$. Moreover using Young inequality we have that

$$\int_{E(\lambda_0) \cap B_{2\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx \leq \int_{B_{4\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx$$

$$\leq c \eta^{1-\delta} \int_{B_{4\rho}} M(|D\tilde{u}|)^{p-\delta} \, dx + c \eta^{\frac{(1-\delta)^2}{p-1}} \rho^n$$

$$\leq c \eta^{1-\delta} \int_{B_{4\rho}} [M^2(|D\tilde{u}|^t)]^{p-\delta} \, dx + c \eta^{\frac{1}{p-1}} \rho^n$$

$$\leq c \eta^{1-\delta} \int_{B_{4\rho}} |D\tilde{u}|^{p-\delta} \, dx + c \eta^{\frac{1}{p-1}} \rho^n.$$ 

Therefore

$$I_1 \leq c \eta^{1-p} \rho^n \left( \int_{B_{4\rho}} |D\tilde{u}|^t \, dx \right)^{\frac{p-\delta}{t}} + c \eta^{1-\delta} \int_{B_{4\rho}} |D\tilde{u}|^{p-\delta} \, dx + c \eta^{\frac{1}{p-1}} \rho^n.$$ 

**Estimate of $I_2$.**

We have now to estimate the integral

$$I_2 \leq \int_{B_{2\rho} \setminus B_{\rho}} |D\tilde{u}| |Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx$$

$$+ \int_{B_{2\rho} \setminus B_{\rho}} |D\tilde{u}| |u|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx$$

$$+ \int_{B_{2\rho} \setminus B_{\rho}} |D\tilde{u}| M(|D\tilde{u}|)^{-\delta} \, dx = (J + JJ + JJJ).$$
Let $D_1$ be the set of all $x \in B_{2\rho} \setminus B_{\rho}$ such that

$$M(|D\tilde{u}|)(x) \leq \delta M(|Du|_{B_{4\rho}})(x)$$

and set $D_2 = (B_{2\rho} - B_{\rho}) - D_1$. Then

$$J \leq \int_{D_1} |D\tilde{u}| |Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx + \int_{D_2} |\varphi||Du|^p M(|D\tilde{u}|)^{-\delta} \, dx$$

$$+ \frac{c}{\rho} \int_{D_2} |u - u_{4\rho}| |Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx.$$

Next, from the definition of $D_1$ and the Hardy-Littlewood maximal theorem, we get

$$\int_{D_1} |D\tilde{u}| |Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx$$

$$\leq \int_{D_1} M(|D\tilde{u}|)^{1-\delta} |Du|^{p-1} \, dx \leq c\delta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx.$$ 

On the other hand, since $M(|Du|_{B_{4\rho}})(x) \geq (|Du|_{B_{4\rho}})(x)$, we have

$$\int_{D_2} |\varphi||Du|^p M(|D\tilde{u}|)^{-\delta} \, dx$$

$$\leq \delta^{-\delta} \int_{D_2} |Du|^{p-\delta} \, dx \leq \delta^{-\delta} \int_{B_{2\rho} - B_{\rho}} |Du|^{p-\delta} \, dx.$$ 

Finally, by Young’s inequality, we obtain

$$\int_{D_2} \frac{|u - u_{4\rho}|}{\rho} |Du|^{p-1} M(|D\tilde{u}|)^{-\delta} \, dx \leq \delta^{-\delta} \int_{D_2} \frac{|u - u_{4\rho}|}{\rho} |Du|^{p-1-\delta} \, dx$$

$$\leq \delta^{-\delta} \int_{D_2} |Du|^{p-\delta} \, dx + c \int_{B_{4\rho}} \left( \frac{|u - u_{4\rho}|}{\rho} \right)^{p-\delta} \, dx$$

$$\leq \delta^{-\delta} \int_{B_{2\rho} - B_{\rho}} |Du|^{p-\delta} \, dx + c \rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}},$$

where $(p - \delta)_* = \frac{n(p-\delta)}{n + p - \delta} \leq t < p - \delta$.

Then, by the previous estimates we can conclude that

$$J \leq c \delta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx$$

$$(3.15) + c \delta^{-\delta} \int_{B_{2\rho} - B_{\rho}} |Du|^{p-\delta} \, dx + c \rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}}.$$
To estimate $JJ$ we remark that by Young inequality and (3.7)

$$JJ \leq \int_{B_{2\rho}\setminus B_{\rho}} |u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \, dx$$

$$\leq c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_{\rho}} M(|D\tilde{u}|)^{p-\delta} \, dx + c\eta \frac{(1-\delta)^2}{p-1} \left( \int_{B_{2\rho}\setminus B_{\rho}} |u|^{p-\delta} \, dx \right)$$

$$\leq c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_{\rho}} [M^2(|Du\chi_{B_{4\rho}}|)]^{p-\delta} \, dx + c\eta \frac{1}{1-p} \left( \int_{B_{2\rho}\setminus B_{\rho}} |u|^{p-\delta} \, dx \right)$$

$$\leq c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta \frac{1}{1-p} \rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}},$$

where $0 < \eta < \frac{1}{2}$. Arguing as in the previous estimate we have

$$JJJ \leq \int_{B_{2\rho}\setminus B_{\rho}} M(|D\tilde{u}|)^{1-\delta} \, dx$$

$$\leq c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_{\rho}} M(|D\tilde{u}|)^{p-\delta} \, dx + c\eta \frac{(1-\delta)^2}{p-1} \rho^n$$

$$\leq c\eta^{1-\delta} \int_{B_{2\rho}\setminus B_{\rho}} [M^2(|Du\chi_{B_{4\rho}}|)]^{p-\delta} \, dx + c\eta \frac{1}{1-p} \rho^n$$

$$\leq c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta \frac{1}{1-p} \rho^n.$$

Then from (3.15), (3.16), (3.17) we get

$$I_2 \leq c(\delta^{1-\delta} + \eta^{1-\delta}) \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta \frac{1}{1-p} \rho^n \left( \int_{B_{4\rho}} |Du|^t \, dx \right)^{\frac{p-\delta}{t}}$$

$$+ c\delta^{\frac{-\delta}{t}} \int_{B_{2\rho}\setminus B_{\rho}} |Du|^{p-\delta} \, dx + c\eta \frac{1}{1-p} \rho^n. (*)$$
Estimate of $I_3$.

Using Lemma 2.6 and Young’s inequality we have that

$$I_3 \leq \int_{B_{2\rho}} (|\tilde{u}| |Du|^{p-1} + |\tilde{u}| u|^{p-1} + |\tilde{u}|) M(|D\tilde{u}|)^{-\delta} \, dx$$

$$\leq \int_{B_{2\rho}} (|\tilde{u}|^{1-\delta} |Du|^{p-1} + |\tilde{u}|^{p-\delta} + |\tilde{u}|^{1-\delta}) \, dx$$

$$\leq c\eta^{1-\delta} \int_{B_{2\rho}} (|D\tilde{u}|)^{p-\delta} \, dx + c(\eta \frac{(1-\delta)^2}{p-1} + 1) \left( \int_{B_{2\rho}} |\tilde{u}|^{p-\delta} \, dx \right) + c\rho^n$$

$$\leq c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c(\eta^{1-p} + 1) \left( \int_{B_{2\rho}} |u|^{p-\delta} \, dx \right) + c\rho^n$$

$$\leq c\eta^{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + c\eta^{1-p} \rho^n \left( \int_{B_{4\rho}} |Du|^{t} \, dx \right)^{\frac{p-\delta}{t}} + c\rho^n,$$

where $0 < \eta < \frac{1}{2}$.

Estimate of $I_4$.

By using Lemma (2.6) and the Hardy-Littlewood maximal theorem, we get

$$I_4 = \int_{B_{4\rho}} |Du|^{p-1} + |u|^{p-1} \left( \int_{\lambda_0} M(|D\tilde{u}|) \lambda^{-\delta} d\lambda \right) \, dx$$

$$\leq \frac{1}{1-\delta} \int_{B_{4\rho}} |Du|^{p-1} M(|D\tilde{u}|)^{1-\delta} \, dx + \frac{1}{1-\delta} \int_{B_{4\rho}} |u|^{p-1} M(|D\tilde{u}|)^{1-\delta} \, dx$$

$$\leq \frac{c}{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx + \frac{c}{1-\delta} \int_{B_{4\rho}} |u|^{p-\delta} \, dx$$

$$\leq \frac{c}{1-\delta} \int_{B_{4\rho}} |Du|^{p-\delta} \, dx.$$

Proof of Theorem 1.5. First, let us remark that we have only to prove the regularity near the boundary $\partial \Omega$, since the local higher integrability result has been proved in Theorem 1.2. For $z \in \mathbb{R}^n$, let us introduce the following notations:

- $Q_R(z) = \{ x \in \mathbb{R}^n : |x_i - z_i| < R, i = 1, \ldots, n \}$,
- $Q_R^+(z) = \{ x \in Q_R(z) : x_n > 0 \}$,
- $Q_R^-(z) = \{ x \in Q_R(z) : x_n < 0 \}$,
- $\Gamma_R(z) = \{ x \in Q_R(z) : x_n = 0 \}$.
The compactness of $\overline{\Omega}$ implies that it is possible to recover $\partial \Omega$ with a finite number of neighborhoods $V$ of its points. For every such neighborhood $V$, there exists a Lipschitz continuous function $G$, with Lipschitz inverse, such that

$$G(V) = Q_1(0), \quad G(V \cap \Omega) = Q_1^+(0), \quad G(V \cap \mathbb{R}^n \setminus \overline{\Omega}) = Q_1^-(0), \quad G(V \cap \partial \Omega) = \Gamma_1(0).$$

Setting $\tilde{u}(y) = u(G^{-1}(y))$, it is standard to prove that $\tilde{u}$ solves the equation

$$\int_{Q_+} A(x, \tilde{u}, D\tilde{u}) D\Phi \, dx = \int_{Q_+} B(x, \tilde{u}, D\tilde{u}) \Phi \, dx \quad \forall \Phi \in W^{1, \frac{r}{r-p+1}}(Q^+),$$

where $A, B$ are Carathéodory functions which verify the assumptions (H1)–(H3).

Let us consider $x_0 \in \partial \Omega$ and a cube $Q = Q(x_0, R)$ for some $R \leq 1$. For fixed $y_0 \in Q(x_0, R/2)$ and $0 < \rho < R/8$, let $Q_\rho = B(y_0, \rho)$ and $\varphi \in C^\infty_0(Q_{2\rho})$ be such that $\varphi = 1$ on $Q_\rho$, $0 \leq \varphi \leq 1$ on $Q_{2\rho}$ and $|D\varphi| \leq c \rho^{-1}$.

With $(\tilde{u} - \tilde{u}_0)_{4\rho} = \int_{Q_{4\rho}} \tilde{u}(x) - \tilde{u}_0(x) \, dx$, we set $\tilde{w} = ((\tilde{u} - \tilde{u}_0) - (\tilde{u} - \tilde{u}_0)_{4\rho})\varphi$, $E(\lambda) = \{x \in \mathbb{R}^n : M(|D\tilde{w}|) \leq \lambda \}$ and $F_\lambda = E_\lambda \cap Q_{4\rho}$.

Since supp $\tilde{w} \subset Q_{2\rho}$, for $x \in \mathbb{R}^n - Q_{3\rho}$ we observe that

$$M(|D\tilde{w}|)(x) \leq c \rho^{-n} \int_{Q_{2\rho}} |D\tilde{w}|(y) \, dy = \lambda_0.$$

$F(\lambda)$ is not empty for $\lambda > \lambda_0$ and thanks to Lemma 2.5 we can extend the function $\tilde{w}|_{F(\lambda)}$ to whole $\mathbb{R}^n$.

Let $\Phi$ be the extension of $\tilde{w}|_{F(\lambda)}$. $\Phi$ satisfies the conditions (i)–(iii) (see Lemma 2.5) so that we can consider $\Phi$ as a particular test function. After the choice of that test function the proof can be achieved arguing as in Theorem 1.2.

References


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