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On weakly bisequential spaces

CHUAN LIU

Abstract. Weakly bisequential spaces were introduced by A.V. Arhangel’skii [1], in this paper. We discuss the relations between weakly bisequential spaces and metric spaces, countably bisequential spaces, Fréchet-Urysohn spaces.

Keywords: bisequential spaces, filter base, s-map
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1. Introduction

Let $X$ be a topological space. A filter base ($\omega$-filter base) is defined to be a family $\xi$ of nonempty sets such that if $A, B \in \xi$ (for countable subfamily $\mu \subset \xi$), there is a $C \in \xi$ such that $C \subset A \cap B$ ($C \subset \cap \mu$). A filter base $\xi$ converges to a point $x$ in a space $X$ (accumulates at the point $x$) if each neighborhood base of $x$ contains an element of $\xi$ (respectively, if $x \in \cap \{\bar{P} : P \in \xi\}$). We say that a filter base $\xi$ meshes with a filter base $\eta$ if every $A \in \xi$ intersects every $B \in \eta$. A space $X$ is said to be bisequential (countably bisequential, weakly bisequential) at a point $x \in X$ if for any filter base (countable filter base, $\omega$-filter base) in $X$ accumulating at $x$ there is a countable filter base $\mu$ in $X$ that converges to $x$ and meshes with $\xi$. A space is called bisequential (countable bisequential, weakly bisequential) if it is bisequential (countably bisequential, weakly bisequential) at each point.

A space $X$ is called Fréchet-Urysohn if given $A \subset X$, $x \in X$, and $x \in \bar{A}$, there exists a sequence $\{x_n : n \in N\} \subset A$ which converges to $x$.

A map $f : X \to Y$ is weakly bi-quotient if, whenever $y \in Y$ and $U$ is a cover of $f^{-1}(y)$ by open subsets of $X$, then countably many $f(U)$ with $U \in \mathcal{U}$, cover a neighborhood base of $y$ in $Y$.

Let $S_\kappa$ be a quotient space of the topological sum of $\kappa$ many convergent sequences by identifying all limit points to a point. $S_\omega$ is called sequential fan.

All the maps in this paper are continuous and onto, spaces are regular $T_1$. Readers may refer to [1], [2] and [3] for unstated notations and definitions.

The following diagrams indicate the relation between weakly bisequential spaces (bi-quotient maps) and other spaces (maps).

- bisequential $\rightarrow$ weakly bisequential $\rightarrow$ Fréchet-Urysohn.
- bisequential $\rightarrow$ countably bisequential $\rightarrow$ Fréchet-Urysohn.
- bi-quotient $\rightarrow$ weakly bi-quotient $\rightarrow$ pseudo-open.
- bi-quotient $\rightarrow$ countably bi-quotient $\rightarrow$ pseudo-open.
2. Main results

The following proposition is quite similar to the Proposition 3.2 in [7].

**Proposition 2.1.** The following properties of a map \( f : X \to Y \) are equivalent:

(a) \( f \) is weakly bi-quotient;

(b) if an \( \omega \)-filter base \( \mathcal{F} \) accumulates at \( y \) in \( Y \), then \( f^{-1}(\mathcal{F}) \) accumulates at some \( x \in f^{-1}(y) \).

**Proof:** (a)\(\to\)(b). Suppose that \( f^{-1}(\mathcal{F}) \) does not accumulate at any \( x \in f^{-1}(y) \). For \( x \in f^{-1}(y) \), there is a \( F_x \in \mathcal{F} \) and a nbd \( V_x \) of \( x \) such that \( V_x \cap f^{-1}(F_x) = \emptyset \). \( \{V_x : x \in f^{-1}(y)\} \) is an open cover for \( f^{-1}(y) \). Since \( f \) is weakly biquotient, there exists a countable family \( \mathcal{U}' = \{V_{x_i} : i \in N\} \subset \{V_x : x \in f^{-1}(y)\} \) such that \( y \in \text{int}f(\cup \mathcal{U}') \). Let \( \{F_{x_i} : i \in N\} \subset \mathcal{F} \) such that \( V_{x_i} \cap f^{-1}(F_{x_i}) = \emptyset \) for \( i \in N \). So \( f(V_{x_i}) \cap F = \emptyset \) for all \( i \in N \), where \( F \subset \cap \{F_{x_i} : i \in N\} \). Then \( f(\cup \mathcal{U}') \cap F = \emptyset \), but \( y \in \bar{F} \) and \( f(\cup \mathcal{U}') \) is a nbd of \( y \), a contradiction.

(b)\(\to\)(a). Suppose that \( f \) is not weakly biquotient, then there is an open cover \( \mathcal{U} \) of \( f^{-1}(y) \) for some \( y \in Y \) such that for any countable subfamily \( \lambda \) of \( \mathcal{U} \), \( y \notin \text{int}f(\cup \mathcal{U}) \). Let \( \mathcal{F} = \{Y - f(\cup \lambda) : \lambda \subset \mathcal{U}, |\lambda| \leq \omega\} \), then \( \mathcal{F} \) is an \( \omega \)-filter base accumulating at \( y \). By (b), \( f^{-1}(\mathcal{F}) \) accumulates at some \( x \in f^{-1}(y) \). Let \( U \in \mathcal{U} \) with \( x \in U \), let \( \lambda = \{U\} \). \( U \cap (f^{-1}(Y - f(U))) \neq \emptyset \), hence \( f(U) \cap (Y - f(U)) \neq \emptyset \), a contradiction. \( \square \)

Similar to the proof of Theorem 3.D.2 in [7], we have the following:

**Theorem 2.1.** A topological space \( Y \) is a weakly bisequential space if and only if it is a weakly bi-quotient image of a metrizable space.

**Corollary 2.1.** A weakly bisequential space is Fréchet-Urysohn [1].

**Theorem 2.2.** A closed image \( X \) of a metric space is a closed s-image of a metric space if and only if \( X \) is weakly bisequential.

**Proof:** It is easy to see that a closed s-mapping is weakly bi-quotient, so \( X \) is weakly bisequential. (In fact, a pseudo-open Lindelöf map is weakly bi-quotient).

Now we prove that a weakly bisequential closed image of a metric space is a closed s-image of a metric space. First, we prove that \( S_{\omega_1} \) is not weakly bisequential.

We write \( S_{\omega_1} = \{\infty\} \cup \{S_\alpha : \alpha < \omega_1\} \), where \( S_\alpha \) is a sequence converging to \( \infty \). Let \( H_\alpha = \cup \{S_\beta : \beta < \alpha\} \) for \( \alpha < \omega_1 \), \( \infty \in H_\alpha \). Suppose \( S_{\omega_1} \) is weakly bisequential, then there exists a decreasing sequence \( \{A_n : n \in N\} \) such that \( \{A_n : n \in N\} \) meshes with \( \{H_\alpha : \alpha < \omega_1\} \). We may choose \( x_n \in A_n \cap S_{\omega_n} - \{x_1, \ldots, x_{n-1}\} \) recursively, then \( x_n \to \infty \), a contradiction.

\( X \) is a closed image of a metric space, so it is a Fréchet-Urysohn space with a \( \sigma \)-hereditarily closure preserving \( k \)-network ([4]). \( X \) contains no closed copy of \( S_{\omega_1} \), hence \( X \) is a Fréchet-Urysohn and \( \aleph \)-space ([5]), and thus it is a closed s-image of a metric space ([6]). \( \square \)
Next, we discuss some relations between weakly bisequential spaces and other topological spaces.

From the definition, we know that bisequential spaces are weakly bisequential. Weakly bisequential spaces are Fréchet-Urysohn ([1]). Also, it is well known that countably bisequential spaces are Fréchet-Urysohn. What is the relation between countably bisequential spaces and weakly bisequential spaces? In fact, we have the following examples:

**Proposition 2.2.** There exists a weakly bisequential space which is not countably bisequential.

**Proof:** The sequential fan $S_\omega$ is such a space, since every countable Fréchet-Urysohn space is weakly bisequential ([1]), so it is weakly bisequential. But it is not countably bisequential. Suppose not, we write $S_\omega = \{\infty\} \cup \{S_n : n \in N\}$, where $S_n$ is a sequence converging to $\infty$. Let $H_n = \cup\{S_i : i \geq n\}$. Then $\{H_n : n \in N\}$ is a decreasing sequence accumulating at $\infty$ and we choose a sequence $\{x_k\}$ such that $x_k \in H_k \cap S_{n_k}$ for each $k \in N$ and $\{x_k\}$ converges to $\infty$, this is a contradiction. 

**Proposition 2.3.** There exists a countably bisequential space which is not weakly bisequential.

**Proof:** Let $X$ be the $\Sigma$-product of $\{D_\alpha : \alpha < \omega_1\}$, where $D_\alpha = \{0, 1\}$ for each $\alpha < \omega_1$. It is well known that $X$ is countably bisequential. But $X$ is not weakly bisequential ([1]).

Simon [8] gave an example that the product of two compact Fréchet-Urysohn spaces is not Fréchet-Urysohn. We prove that the spaces in Simon’s example are weakly bisequential. So, not every product of compact weakly bisequential spaces is Fréchet-Urysohn.

Let $\mathcal{P}$ be an almost disjoint family in $\omega$, let $\Omega = \omega \cup \{P : P \in \mathcal{P}\}$. Endow $\Omega$ with a topology as follow: each singleton in $\omega$ is open, for $P \in \mathcal{P}$, a neighborhood base of $P$ is $\{P\} \cup \{P - A : A \in [P]<\omega\}$. Then $\Omega$ is a locally compact space. Let $\Omega'$ be the one point compactification of $\Omega$, we write $\Omega' = \Omega \cup \{\infty\}$.

**Theorem 2.3.** $\Omega'$ is weakly bisequential if it is Fréchet-Urysohn.

**Proof:** Let $\mathcal{F}$ be an $\omega$-filter base in $\Omega'$ accumulating at $\infty$, let $\mathcal{F}' = \mathcal{F} \cap \omega$, $\mathcal{F}'' = \mathcal{F} \cap \mathcal{P}$.

Case 1. $\mathcal{F}'$ is an $\omega$-filter base in $\{\infty\} \cup \omega$ accumulating at $\infty$.

By [1, Theorem 6], $\{\infty\} \cup \omega$ is weakly bisequential. So there is a countable decreasing sequence $\{A_n : n \in N\}$ which converges to $\infty$ and meshes with $\mathcal{F}'$. Hence $\{A_n : n \in N\}$ meshes with $\mathcal{F}$.

Case 2. $\mathcal{F}'$ is not an $\omega$-filter base in $\{\infty\} \cup \omega$ accumulating at $\infty$.

Then $\mathcal{F}''$ is an $\omega$-filter base in $\{\infty\} \cup \mathcal{P}$ accumulating at $\infty$. By [7, Example 10.15], $\{\infty\} \cup \mathcal{P}$ is bisequential, so there is a countable decreasing family
\( \{ A_n : n \in N \} \) which converges to \( \infty \) and meshes with \( \mathcal{F}'' \). Hence it meshes with \( \mathcal{F} \).

So \( \Omega' \) is weakly bisequential. \( \square \)

**Theorem 2.4.** There are two compact weakly bisequential spaces \( X \) and \( Y \) such that \( X \times Y \) is not Fréchet-Urysohn.

**Proof:** Let \( X \) and \( Y \) be the spaces in Simon’s example ([8]). By the theorem above, \( X, Y \) are weakly bisequential, but \( X \times Y \) is not Fréchet-Urysohn. \( \square \)

**Proposition 2.4.** There exists a compact, weakly bisequential space which is not bisequential.

**Proof:** In fact, both \( X \) and \( Y \) in Theorem 2.4 are not bisequential. Suppose one of \( X \) and \( Y \) is bisequential, then so is \( \alpha_3 ([2]) \). So the product \( X \times Y \) is Fréchet-Urysohn ([2]), a contradiction. \( \square \)

**Theorem 2.5.** Let \( X \) be a discrete space and \( X^* = X \cup \{ \infty \} \) the one point compactification of \( X \). Then \( X^* \) is weakly bisequential if and only if it is bisequential.

**Proof:** We only prove sufficiency. If the cardinality of \( X \) is non-measurable then, by [7, Example 10.15], \( X^* \) is bisequential. If the cardinality of \( X \) is measurable, by [7, Lemma 10.14], there is an ultrafilter \( \mathcal{F} \) such that \( \cap \mathcal{F} = \emptyset \). But \( \cap \mathcal{F}' \in \mathcal{F} \) for every countable \( \mathcal{F}' \subset \mathcal{F} \). \( \mathcal{F} \) is an \( \omega \)-filter base accumulating at \( \infty \) [7, Lemma 10.14], then there is a sequence \( \{ A_n : n \in N \} \) which converge to \( \infty \) and meshes with \( \mathcal{F} \). \( \{ A_n : n \in N \} \subset \mathcal{F} \) because \( \mathcal{F} \) is an ultrafilter. \( \cap \{ A_n : n \in N \} \in \mathcal{F} \), so \( \cap \{ A_n : n \in N \} \cap X \neq \emptyset \), hence \( \{ A_n : n \in N \} \) does not converge to \( \infty \), a contradiction. \( \square \)

**Proposition 2.5** (\( \exists \) measurable cardinal). There is a compact, countably bisequential space that is not weakly bisequential.

**Proof:** Let \( X^* \) be the space in Example 10.15 in [7]. Then \( X^* \) is not bisequential. By Theorem 2.5, \( X^* \) is not weakly bisequential. \( \square \)

A space \( X \) is called weakly quasi-first countable ([9]) if for each \( i \in N \), there exists a mapping \( B^i : N \times X \to \mathcal{P}(X) \), where \( \mathcal{P}(X) \) denotes the power set of \( X \), such that the following hold:

(i) fix \( i \in N \) for each \( n \in N \) and \( x \in X \), \( B^i(n + 1, x) \subset B^i(n, x) \), and \( \{ x \} = \cap \{ B^i(n, x) : n \in N \} \); and

(ii) a subset \( V \) of \( X \) is open if and only if for each \( y \in V \) and for each \( i \in N \) there exists \( n(i) \) with \( B^i(n(i), y) \subset V \).

If \( B^i = \emptyset \) for \( i \in N \), then \( X \) is called weakly first countable. Obviously, weakly first countable is weakly quasi-first countable.
Theorem 2.6. A Fréchet-Urysohn, weakly quasi-first countable space $X$ is weakly bisequential.

Proof: For $x \in X$, let $\mathcal{F}$ be an $\omega$-filter base accumulating at $x$. Since $X$ is weakly quasi-first countable, there is a family of subsets of $X$, say, $\{B^i(n, x) : n \in N, i \in N\}$ satisfying (i) and (ii).

Claim 1. There exists $i_0 \in N$ such that $\{B^{i_0}(n, x) : n \in N\}$ meshes with $\mathcal{F}$.

Suppose not; then for each $i \in N$, there exist $n(i)$ and $F_i \in \mathcal{F}$ such that $B^i(n(i), x) \cap F_i = \emptyset$. Let $F \in \mathcal{F}$ where $F \subseteq \cap\{F_i : i \in N\}$. Then $F \cap B^i(n(i), x) = \emptyset$ for all $i \in N$. Since $X$ is Fréchet-Urysohn and $x$ is an accumulating point of $F$, there is $\{x_n : n \in N\} \subseteq F$, $x_n \rightarrow x$. $\{x_n : n \in N\} \cap B^i(n(i), x) = \emptyset$, it is easy to see that $\{x_n : n \in N\}$ is closed, a contradiction.

So there is $i_0 \in N$ such that $\{B^{i_0}(n, x) : n \in N\}$ converges to $x$ and meshes with $\mathcal{F}$, hence $X$ is weakly bisequential. □

Remark 2.1. It is natural to ask whether every weakly bisequential space is quasi-weakly first countable, the answer is ‘No’. The one point compactification of a discrete space $Y$ whose cardinality is $2^\omega$ is such a space. $Y$ is bisequential [7, Example 10.15] but not first countable. So $Y$ is not weakly quasi-first countable because of the following Corollary 2.2.

A space $X$ is called an $\alpha_4$ space if for every point $x \in X$ and any countable family $\{S_n : n \in N\}$ of sequences converging to $x$ one can find a sequence $S$ converging to $x$ which meets infinitely many $S_n$.

A subset $B$ of $X$ is called a sequential neighborhood of $x \in X$ if for every sequence converging to $x$ is eventually in $B$.

Theorem 2.7. A space $X$ is weakly first countable if and only if $X$ is a weakly quasi-first countable, $\alpha_4$ space.

Proof: Necessity is obvious. We only prove sufficiency.

For $x \in X$, let $\mathcal{F}_x$ be the family $\{B^i(n, x) : n \in N, i \in N\}$ that satisfies (i) and (ii) in the definition of weakly quasi-first countable. Let

$\mathcal{B}_x = \{\cup \mathcal{F} : \mathcal{F} \subseteq \mathcal{F}_x, |\mathcal{F}| < \omega, \text{ and } \cup \mathcal{F} \text{ is a sequential neighborhood of } x\}$.

We can see that $\mathcal{B}_x$ is countable, let $\mathcal{B} = \cup \{\mathcal{B}_x : x \in X\}$.

We will prove that $\mathcal{B}$ is a weak base for $X$.

Let $U$ be a subset of $X$, for each $x \in U$. If there is a $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$, then $U$ is open.

In fact, $U$ is a sequential neighborhood for each $x \in U$, hence $U$ is sequential open. But $X$ is a sequential space [9], so $U$ is open.

Let $V$ be an open subset of $X$, we prove that for $x \in V$, there is $B \in \mathcal{B}_x$ such that $B \subseteq V$.

Let $\mathcal{P} = \{F \in \mathcal{F}_x : F \subseteq V\}$, and we rewrite $\mathcal{P} = \{F_n : n \in N\}$. 

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Claim 2. There is \( m \in N \) such that \( \cup\{ F_n \in \mathcal{P} : n \leq m \} \) is a sequential neighborhood of \( x \).

Suppose not, there is a sequence \( \{ x^{(1)}(n) \} \) with \( x^{(1)}(n) \to x \) and \( \{ x^{(1)}(n) \} \cap F_1 = \emptyset \). Since \( F_1 \cup F_2 \) is not a sequential neighborhood of \( x \), then there is a sequence \( x^{(2)}(n) \to x \) and \( \{ x^{(2)}(n) \} \cap (F_1 \cup F_2) = \emptyset \). Continuing this way, we get countably many convergent sequences \( \{ x^{(i)}(n) \}, (i \in N) \) with \( x^{(i)}(n) \to x \) and \( \{ x^{(i)}(n) \} \cap \{ F_j : j \leq i \} = \emptyset \). \( X \) is an \( \alpha_4 \)-space, so there is a sequence \( S = \{ y_n : n \in N \} \) which converges \( x \) and meets infinitely many \( \{ x^{(i)}(n) \} \). We prove that \( S \) is eventually in some finite union of a subfamily of \( \mathcal{P} \).

If not, pick \( n_1 \in N \) such that \( B^1(n_1) \subset U \). Since \( B^1(n_1) \) is not a sequential neighborhood of \( x \), there is subsequence \( S_1 \subset S \), \( S_1 \cap B(n_1) = \emptyset \) and \( S - S_1 \) is eventually in \( B^1(n_1) \), choose \( y_{m_1} \in S_1 \). Pick \( n_2 \in N \) such that \( B^2(n_2) \subset U \). Since \( S_1 \) is not eventually in \( B^2(n_2) \), there is a subsequence \( S_2 \subset S_1 \) such that \( S_2 \cap B^2(n_2) = \emptyset \) and \( S - S_2 \) is eventually in \( B^2(n_2) \). Pick \( y_{m_2} \in S_2 - \{ y_{m_1} \} \). Suppose that \( B^i(n_i), S_i, y_{m_i} \ (i \leq j) \) have been selected in such a way that \( S_k \subset S_l \) if \( k < l \), \( S_i \) is infinite for \( i \leq j \). \( S_i \cap B^i(n_i) = \emptyset \), \( S_{i+1} \subset S_i \) is eventually in \( B^i(n_i) \). Since \( S \) is not contained in any finite union of subfamily of \( \mathcal{P} \), choose \( B^{j+1}(n_{j+1}) \subset U \), \( S_j \) is not eventually in \( B^{j+1}(n_{j+1}) \), there is an infinite subsequence \( S_{j+1} \) of \( S_j \) such that \( S_{j+1} \cap B^{j+1}(n_{j+1}) = \emptyset \). Pick \( y_{m_{j+1}} \in S_{j+1} - \{ y_{m_i} : i \leq j \} \).

We can get a subsequence \( S' = \{ y_{m_i} \} \) converging to \( x \). From the construction above, for each \( i \in N \), \( S' \cap B^i(n_i) = \emptyset \), so it is not difficult to see that \( S' \) is closed, a contradiction.

But from the selection of \( S \), \( S \) cannot be eventually in any finite union of \( \mathcal{P} \). A contradiction. So the claim has been proved.

So the finite union of \( \mathcal{P} \) in claim 2 is an element of \( \mathcal{B}_x \). Hence \( \mathcal{B} \) is a weak base for \( X \), and \( X \) is weakly first countable.

Corollary 2.2. Let \( X \) be a countably bisequential space. Then \( X \) is first countable if \( X \) is weakly quasi-first countable.

Proof: Every countably bisequential space is an \( \alpha_4 \) space. Thus \( X \) is weakly first countable by Theorem 2.7. It is well known that weakly first countable, Fréchet Urysohn spaces are first countable.

3. Questions

Question 3.1. Let \( X \) and \( Y \) be weakly bisequential. Is \( X \times Y \) weakly bisequential provided \( X \times Y \) is Fréchet-Urysohn?

Let \( \mathcal{P} \) be a cover for \( X \). \( \mathcal{P} \) is called a cs* - network if for any \( x \in X \), \( x \in U \) with \( U \) open and a sequence \( S \) converging to \( x \), there is a \( P \in \mathcal{P} \) such that \( x \in P \), \( P \subset U \) and \( P \) contains a subsequence of \( S \).

Question 3.2. Let \( X \) be a weakly bisequential space with a point-countable \( k \)-network. Does \( X \) have a point-countable cs*-network?
Remark 3.1. If the answer to Question 3.2 is positive, then we can give an affirmative answer to the Question 10.2 in [3].

Question 3.3. Let $X$ be a Fréchet-Urysohn space with a point-countable k-network. Is $X$ weakly bisequential if it contains no closed copy of $S_{\omega_1}$?

Question 3.4. Let $X$ be a Fréchet-Urysohn space with countable network. Is $X$ weakly bisequential?

Question 3.5. Is it possible to characterize weak bisequentiality in terms of the Fréchet-Urysohn property?

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