Giorgio Nordo; Boris A. Pasynkov
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Perfect compactifications of functions

GIORGIO NORDO*, BORIS A. PASYNKOV†

Abstract. We prove that the maximal Hausdorff compactification $\chi f$ of a $T_2$-compactifiable mapping $f$ and the maximal Tychonoff compactification $\beta f$ of a Tychonoff mapping $f$ (see [P]) are perfect. This allows us to give a characterization of all perfect Hausdorff (respectively, all perfect Tychonoff) compactifications of a $T_2$-compactifiable (respectively, of a Tychonoff) mapping, which is a generalization of two results of Skljarenko [S] for the Hausdorff compactifications of Tychonoff spaces.

Keywords: Hausdorff (Tychonoff) mapping, compactification of a mapping, maximal Hausdorff (Tychonoff) compactification of a mapping, perfect compactification of a mapping

Classification: Primary 54C05, 54C10, 54C20, 54C25; Secondary 54D15, 54D30, 54D35

1. Introduction

In 1961, E.G. Skljarenko introduced the notion of the perfect compactification of a Tychonoff space. Given a Tychonoff space $X$, we say that a compactification $\gamma X$ of $X$ is perfect if $cl_{\gamma X}(bd_X(U)) = bd_{\gamma X}(\langle U \rangle_{\gamma X})$ for every open set $U$ of $X$, where $\langle U \rangle_{\gamma X}$ denotes the maximal extension of $U$ relatively to $\gamma X$, that is the maximal open set of $\gamma X$ whose trace on $X$ is $U$.

In [S], Skljarenko, using proximal techniques, gave some characterizations of the perfect compactifications and he proved that $\gamma X$ is a perfect compactification of $X$ if and only if the canonical map $\varphi_\gamma : \beta X \to \gamma X$ is monotone (i.e. every its fibre is connected) and so — in particular — that the Stone-Čech compactification $\beta X$ is a perfect compactification of $X$.

Further results concerning this class of compactifications were given by Diamond in [D].

Recently, the first author [N] has generalized the notion of perfectness from a Hausdorff compactification of a Tychonoff space to a generic extension of an arbitrary space simplifying the treatment in a more general setting and obtaining several new characterizations.

Since it is clear now what is the compactification of a continuous mapping and since the notion of a topological space is the simplest case of the notion of a

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continuous mapping (a space is its mapping to the one-point space), it is natural
to extend to continuous mappings some results concerning compactifications of
spaces.

The study of compactifications (= perfect extensions) of a continuous mapping
was started in 1953 by Whyburn [W].

In [P], using techniques of partial topological products, Pasynkov described a
general way to obtain all Tychonoff (i.e. completely regular, $T_0$-) compactifications
of Tychonoff mappings between arbitrary spaces and he proved that the poset
$TK(f)$ of all the Tychonoff compactifications of a Tychonoff mapping $f : X \to Y$
admits the maximal compactification $\beta f : \beta f X \to Y$ which is the exact analogue
of the Stone-Čech compactification of a Tychonoff space (since if $|Y| = 1$, $X$
becomes a Tychonoff space and the domain $\beta f X$ of $\beta f$ coincides with $\beta X$).

The following similar result is obtained in [BN]:
If a continuous mapping $f : X \to Y$ is $T_2$-compactifiable (i.e. $f$ has some Hausdorff
compactification) then it has the maximal compactification $\chi f : \chi f X \to Y$ in the
poset $HK(f)$ of all Hausdorff compactifications of $f$.

Let us note in this connection that — unlike the corresponding case for spaces
— there exist Hausdorff compact mappings which are not Tychonoff ([HI], [C]).
Thus, it is necessary to consider the cases of Tychonoff and $T_2$-compactifiable
mappings separately. It would be interesting

\textit{to find wide enough conditions when every Hausdorff compactification of a Ty-
chonoff mapping is Tychonoff.}

In this paper, we generalize to continuous mappings two extrinsic characteri-
izations of perfect compactifications of spaces obtained by Skljarenko in [S].

We will prove that:

1. the maximal Hausdorff (maximal Tychonoff) compactification $\chi f$ (respec-
tively $\beta f$) of a $T_2$-compactifiable (Tychonoff) mapping $f$ is a perfect exten-
sion of $f$ (Theorems 3.1, 3.9);
2. a Hausdorff (Tychonoff) compactification $b f$ of a $T_2$-compactifiable (Ty-
chonoff) mapping $f$ is a perfect extension of $f$ if and only if the canonical
morphism of $\chi f$ (respectively $\beta f$) to $b f$ is monotone (Theorems 3.6, 3.11).

\section{Preliminaries}

Throughout the paper, the word “space” will mean “topological space”.

If $X$ is a space, $\tau(X)$ will denote the set of all the open subsets of $X$ while
$\sigma(X)$ will denote the set of all the closed subsets of $X$.

As usual, for any pair of spaces $X$ and $Y$, $C(X,Y)$ denotes the set of all
continuous mappings from $X$ to $Y$ and $C^*(X)$ is the set of all continuous real
bounded functions on $X$.

Undefined notions are used as in [E].
Definitions ([N], [S]). Let \( Y \) be an extension of a space \( X, U \in \tau(X) \) and \( x \in Y \setminus X \).

We say that the pair \( (x, U) \) is perfect if \( x \in cl_Y(bd_X(U)) \) provided \( x \in bd_Y((U)_Y) \), where \( (U)_Y = \bigcup \{ V \in \tau(Y) : V \cap X = U \} \) is the maximal extension of \( U \) in \( Y \), i.e. the maximum open set of \( Y \) whose trace on \( X \) is \( U \).

We say that \( Y \) is a perfect extension of \( X \) relatively to \( x \) if for every \( W \in \tau(X) \) the pair \( (x, W) \) is perfect.

We say that \( Y \) is a perfect extension of \( X \) if it is a perfect extension of \( X \) relatively to every point of its remainder \( Y \setminus X \).

Definition ([N], [S]). Let \( Y \) be an extension of \( X \) and \( x \in Y \setminus X \). We say that \( Y \setminus X \) cuts \( X \) at \( x \) if there exists some neighborhood \( O \) of \( x \) in \( Y \) and a pair \( U, V \) of disjoint open sets of \( X \) such that \( O \cap X = U \cup V \) and \( x \in cl_Y(U) \cap cl_Y(V) \).

The following characterization is given in [N].

Proposition 2.1. Let \( Y \) be an extension of a space \( X \) and \( x \in Y \setminus X \). Then \( Y \) is a perfect extension of \( X \) relatively to \( x \) if and only if \( Y \setminus X \) does not cut \( X \) at \( x \).

Now, we define our framework.

For any fixed space \( Y \), we consider the category \( \text{Top}_Y \), where

\[
Ob(\text{Top}_Y) = \{ f \in C(X,Y) : X \in Ob(\text{Top}) \}
\]

is the class of the objects and, for every pair \( f : X \to Y, g : Z \to Y \) of objects,

\[
M(f, g) = \{ \lambda \in C(X,Z) : g \circ \lambda = f \}
\]

is the class of the morphisms from \( f \) to \( g \), whose generic representant is denoted for short by \( \lambda : f \to g \).

A morphism \( \lambda : f \to g \) from \( f : X \to Y \) to \( g : Z \to Y \) will be called surjective (resp. dense) if \( \lambda(X) = Z \) (resp. if \( \lambda(X) \) is dense in \( Z \)).

If \( \lambda : f \to g \) is a surjective morphism, we say that \( g \) is the image of \( f \) (by the morphism \( \lambda \)) and we write \( g = \lambda(f) \).

Moreover, we say that a morphism \( \lambda : f \to g \) from \( f : X \to Y \) to \( g : Z \to Y \) is an embedding (resp. a homeomorphism) if the mapping \( \lambda : X \to Z \) is an embedding.

A mapping \( g : Z \to Y \) is called an extension of \( f : X \to Y \) if some dense embedding \( \lambda : f \to g \) is fixed (as usual \( X \) and \( f \) are identified with \( \lambda(X) \) and \( g|\lambda(X) \) respectively).

A morphism \( \lambda : g \to h \) between two extensions \( g : Z \to Y \) and \( h : W \to Y \) of a mapping \( f : X \to Y \) will be called canonical if \( \lambda|_X = id_X \).

Now, let us recall some other definitions.

Definitions. A mapping \( f : X \to Y \) is said to be \( T_0 \) ([P]) if for any \( x, x' \in X \) such that \( x \neq x' \) and \( f(x) = f(x') \) there exist either a neighborhood of \( x \) in \( X \) which does not contain \( x' \) or a neighborhood of \( x' \) in \( X \) not containing \( x \).
A mapping \( f : X \to Y \) is said to be Hausdorff (or \( T_2 \)) \([P]\) if for every \( x, x' \in X \) such that \( x \neq x' \) and \( f(x) = f(x') \) there are disjoint neighborhoods of \( x \) and \( x' \) in \( X \).

We shall say that \( f : X \to Y \) is compact if it is perfect (i.e. closed and all its fibres are compact).

A mapping \( f : X \to Y \) is said to be completely regular \([P]\) if for every \( F \in \sigma(X) \) and \( x \in X \setminus F \) there exists a neighborhood \( O \) of \( f(x) \) in \( Y \) and a continuous mapping \( \varphi : f^{-1}(O) \to [0, 1] \) such that \( \varphi(x) = 1 \) and \( \varphi(F \cap f^{-1}(O)) \subseteq \{0\} \).

A completely regular, \( T_0 \) mapping is called Tychonoff (or \( T_{3\frac{1}{2}} \)) \([P]\).

The following lemma is evident.

**Lemma 2.2.** Every morphism defined on a Hausdorff mapping is a Hausdorff mapping too.

The next lemma from \([P]\) will be useful in the following.

**Lemma 2.3.** Let \( f : X \to Y \) be a Hausdorff mapping, \( y \in Y \) and let \( K_1, K_2 \) be two disjoint compact subsets of \( X \) such that \( K_1 \cup K_2 \subseteq f^{-1}(\{y\}) \). Then \( K_1 \) and \( K_2 \) have disjoint neighborhoods in \( X \).

**Corollary 2.4.** If \( f : X \to Y \) is a Hausdorff compact mapping, \( y \in Y \) and \( K_1, K_2 \) are closed disjoint subsets of \( f^{-1}(\{y\}) \) then \( K_1 \) and \( K_2 \) have disjoint neighborhoods in \( X \).

**Definition.** A restriction \( f|_{X'} : X' \to Y \) to \( X' \subseteq X \) of a mapping \( f : X \to Y \) is called a closed submapping of \( f \) if \( X' \) is a closed subset of \( X \).

Obviously every closed submapping of a compact mapping is compact too.

Many well-known statements which hold in the category \( \mathbf{Top} \) have their analogue (and hence a generalization) in \( \mathbf{Top}_Y \). The following properties were given in \([P]\).

**Proposition 2.5.** Let \( \lambda \) and \( \mu \) be morphisms from a mapping \( f : X \to Y \) to a Hausdorff mapping \( h : Z \to Y \) and \( D \) be a dense subset of \( X \). Then, if \( \lambda|_D = \mu|_D \), the morphisms \( \lambda \) and \( \mu \) coincide.

**Proposition 2.6.** The composition of two compact Hausdorff mappings is compact Hausdorff.

**Proposition 2.7.** Every image \( \lambda(k) \) of a compact mapping \( k : X \to Y \) (under a morphism \( \lambda \)) is compact.

**Proposition 2.8.** Every compact submapping \( h|_{X'} : X' \to Y \) of a Hausdorff mapping \( h : X \to Y \) is a closed submapping of \( h \).

**Proposition 2.9.** Every morphism \( \lambda : k \to h \) from a compact mapping \( k : X \to Y \) to a Hausdorff mapping \( h : Z \to Y \) is a perfect mapping.
**Definition.** We say that a mapping \( c : X^c \to Y \) is a *compactification* of a mapping \( f : X \to Y \) if it is a compact extension of \( f \).

**Definitions.** Let \( c : X^c \to Y \) and \( d : X^d \to Y \) be compactifications of a mapping \( f : X \to Y \). We say that:

- \( c \) is *projectively larger than* \( d \) (relatively to \( f \)) and we write that \( c \geq_f d \) (or \( c \geq d \), for short) if there exists some canonical morphism \( \lambda : c \to d \);
- \( c \) is *equivalent to* \( d \) (relatively to \( f \)) and we write that \( c \equiv_f d \) (shortly, \( c \equiv d \)) if there exists a canonical homeomorphism \( \lambda : c \to d \).

In [BN], the following useful result is obtained:

**Proposition 2.10.** Let \( c : X^c \to Y \) and \( d : X^d \to Y \) be Hausdorff compactifications of a mapping \( f : X \to Y \). Then \( c \equiv_f d \) if and only if \( c \geq d \) and \( d \geq c \).

**Definition.** A Hausdorff mapping \( f : X \to Y \) will be called *\( T_2 \)-compactifiable* (or *Hausdorff compactifiable*) if it has some Hausdorff compactification.

All Hausdorff compactifications of any \( T_2 \)-compactifiable mapping form a set up to their equivalence (see [BN]).

**Definition.** If \( f : X \to Y \) is a \( T_2 \)-compactifiable mapping, \( HK(f) \) will denote the set of all Hausdorff compactifications of \( f \) (up to the equivalence \( \equiv_f \)).

So, by 2.10, it follows that \((HK(f), \geq)\) is a poset and, for any pair of Hausdorff compactifications \( c, d \in HK(f) \), we can write \( c = d \) instead of \( c \equiv_f d \), that is, we do not distinguish between equivalent Hausdorff compactifications.

In [BN], the following is proved:

**Theorem 2.11.** For any \( T_2 \)-compactifiable mapping \( f : X \to Y \), there is in the poset \((HK(f), \geq)\) a maximal Hausdorff compactification \( \chi_f : \chi_f X \to Y \) of \( f \).

From 2.5 it follows — in particular — that for any Hausdorff compactification \( bf : X^b \to Y \) of a \( T_2 \)-compactifiable mapping \( f : X \to Y \) there exists a unique canonical morphism \( \lambda_b : \chi_f \to bf \).

The following useful property can be found in [P].

**Proposition 2.12.** Let \( bf : X^b \to Y \) and \( bg : Z^b \to Y \) be Hausdorff compactifications of \( f : X \to Y \) and \( g : Z \to Y \) respectively, \( \lambda : f \to g \) be a perfect morphism and \( \tilde{\lambda} : bf \to bg \) be a morphism such that \( \tilde{\lambda}|_X = \lambda \). Then \( \tilde{\lambda}(X^b \setminus X) \subseteq Z^b \setminus Z \).

In [P], Pasynkov proved that any TychoFF mapping \( f : X \to Y \) has a TychoFF (and hence Hausdorff) compactification.

**Definition.** For any TychoFF mapping \( f : X \to Y \), we will denote by \( TK(f) \) the set of all TychoFF compactifications of \( f \) (up to the equivalence \( \equiv_f \)).

In [P], it is shown that, for any TychoFF mapping \( f : X \to Y \), there exists in \((TK(f), \geq)\) a maximal TychoFF compactification \( \beta f : \beta f X \to Y \) of \( f \).
Definition. For any mapping $g : T \rightarrow Y$ and any $U \in \tau(Y)$, let $C^*(U,g) = C^*(g^{-1}(U))$.

The following characterization of $\beta f$ is given in [P].

Theorem 2.13. For any Tychonoff compactification $bf : X^b \rightarrow Y$ of a Tychonoff mapping $f : X \rightarrow Y$, the following conditions are equivalent:

1. $bf = \beta f$;
2. for every $U \in \tau(Y)$ and $\varphi \in C^*(U,f)$, there exists a unique extension $\bar{\varphi} \in C^*(U,bf)$;
3. for every compact Tychonoff mapping $k : Z \rightarrow Y$ and every morphism $\lambda : f \rightarrow k$ there exists a morphism $\bar{\lambda} : bf \rightarrow k$ which extends $\lambda$.

Proposition 2.14. ([P]). For any Tychonoff compactification $bf : X^b \rightarrow Y$ of a Tychonoff mapping $f : X \rightarrow Y$ there exists a unique (perfect) canonical morphism $\mu_b : \beta f \rightarrow bf$ and it results $\mu_b(\beta f X \setminus X) = X^b \setminus X$.

3. Perfectness of the maximal compactifications of a mapping

In [S], Skljarenko proved that a compactification $\gamma X$ of a Tychonoff space $X$ is perfect if and only if the canonical map $\varphi : \beta X \rightarrow \gamma X$ is monotone (that is, every its fibre is connected) and hence — in particular — that the Stone-Čech compactification $\beta X$ of $X$ is a perfect compactification of $X$.

In the following we will obtain similar (and more general) results for compactifications of a mapping.

Definition. Let $\bar{f} : \bar{X} \rightarrow Y$ be an extension of a mapping $f : X \rightarrow Y$. We say that $\bar{f}$ is a perfect extension of $f$ if its domain $\bar{X}$ is a perfect extension of the space $X$.

Theorem 3.1. The maximal Hausdorff compactification $\chi f : \chi f X \rightarrow Y$ of a $T_2$-compactifiable mapping $f : X \rightarrow Y$ is a perfect extension of $f$.

Proof: Suppose by contradiction that $\chi f$ is not a perfect extension of $f$. By 2.1, there exists some $x \in \chi f X \setminus X$ such that $\chi f X \setminus X$ cuts $X$ at $x$, i.e. there are a neighborhood $U$ of $x$ in $\chi f X$ and a pair $U_0$, $U_1$ of disjoint open subsets of $X$ such that $x \in cl_{\chi f X}(U_0) \cap cl_{\chi f X}(U_1)$ and $U \cap X = U_0 \cup U_1$. Note that $G = cl_U(U_0) \cap cl_U(U_1) \subseteq \chi f X \setminus X$.

Let $X'$ be the disjoint union of $\chi f X \setminus U$ and $U_i' = cl_U(U_i)$ (for $i = 0, 1$). The copy of $G$ lying in $U_i'$ will be denoted by $G_i$ and the copy of a point $t \in G$ lying in $G_i$ will be denoted by $t_i$ (for $i = 0, 1$). In particular, we have $x_i \in U_i'$ (for $i = 0, 1$). Set $\lambda(t) = t$ for $t \in X' \setminus (G_0 \cup G_1)$ and $\lambda(t_i) = t$ for $t_i \in G_i$ (for $i = 0, 1$). Hence, $\lambda(x_i) = x$ (for $i = 0, 1$) and $X \subseteq X'$, $\lambda|_X = id_X$.

Let $\theta$ consist of inverse images of all open sets of $\chi f X$ by the mappings $\lambda$ and $\lambda_i \equiv \lambda|_{U_i'}$ (for $i = 0, 1$). Evidently, $\theta$ is a topology on $X'$, $U_i'$ is open in $X'$ (for $i = 0, 1$), $\lambda$ is continuous and $\lambda : X' \setminus (G_0 \cup G_1) \rightarrow \chi f X \setminus G$ is a homeomorphism.
In particular, $\lambda|_X$ is the identical homeomorphism of $X$. Since $\lambda^{-1}\{\{t\}\}$ consists of two points for $t \in G$, all fibres of $\lambda$ are compact.

Since $X' \setminus U'_i$ is closed in $X'$, the corestriction of $\lambda$ to this set is a homeomorphism and $\lambda(X' \setminus U'_i) = (\chi f X \setminus U) \cup \operatorname{cl}_U(U_j)$ (where $j = 1$ when $i = 0$ and $j = 0$ when $i = 1$) is closed in $\chi f X$ (for $i = 0, 1$), $\lambda$ is closed and so perfect. Evidently, $X$ is dense in $X'$ and $\lambda$ is Hausdorff.

Thus, $bf = \chi f \circ \lambda$ is a compact Hausdorff mapping (by 2.6) and $bf|_X = f$. So, $bf$ is a Hausdorff compactification of $f$ and $\lambda$ is a canonical morphism from $bf$ to $\chi f$, i.e. $bf \geq \chi f$.

Moreover, $\lambda$ is not 1–1 because $x = \lambda(x_0) = \lambda(x_1)$. Thus $bf > \chi f$ which is a contradiction to the maximality of $\chi f$. \hfill $\square$

To obtain an extrinsic characterization of the perfect Hausdorff compactification, we need two lemmas.

**Lemma 3.2.** Let $Y_1$ and $Y_2$ be extensions of a space $X$, $x \in Y_1 \setminus X$ and $f : Y_2 \to Y_1$ a continuous mapping closed at $x$ such that $f|_X = id_X$ and $f^{-1}\{\{x\}\}$ is connected. Then, if $Y_2$ is a perfect extension of $X$ relatively to any point of $F = f^{-1}\{\{x\}\}$, $Y_1$ is a perfect extension of $X$ relatively to $x$.

**Proof:** First, we observe that $f^{-1}\{\{x\}\} \neq \emptyset$ as otherwise by the closedness of $f$ at $x$, there exists some neighborhood $N$ of $x$ such that $f^{-1}(N) \subseteq \emptyset$.

Now, suppose — by contradiction — that $Y_1$ is not a perfect extension of $X$ relatively to $x$. By 2.1, $Y_1 \setminus X$ cuts cut $X$ at $x$, i.e. there exist a neighborhood $O$ of $x$ in $Y_1$ and disjoint open sets $U, V$ of $X$ such that $O \cap X = U \cup V$ and $x \in \operatorname{cl}_{Y_1}(U) \cap \operatorname{cl}_{Y_1}(V)$.

We claim that $F \cap \operatorname{cl}_{Y_2}(U) \cap \operatorname{cl}_{Y_2}(V) = \emptyset$. In fact, if there exists some $t \in F \cap \operatorname{cl}_{Y_2}(U) \cap \operatorname{cl}_{Y_2}(V)$, by continuity of $f$, $W = f^{-1}(O)$ is a neighborhood of $t$ in $Y_2$ and, from $f|_X = id_X$ and $O \cap X = U \cup V$, it follows that $W \cap X = U \cup V$. But this means that $Y_2 \setminus X$ cuts $X$ at $t \in F$ and by 2.1, $Y_2$ is not a perfect extension of $X$ relatively to $t \in F$, which is a contradiction.

Moreover, $x \in O$ implies $F \subseteq W \subseteq \operatorname{cl}_{Y_2}(W) = \operatorname{cl}_{Y_2}(W \cap X) = \operatorname{cl}_{Y_2}(U) \cup \operatorname{cl}_{Y_2}(V)$. So, $(\operatorname{cl}_{Y_2}(U) \cap F) \cup (\operatorname{cl}_{Y_2}(V) \cap F) = F$ and, as $F$ is connected, one of these two closed sets must be empty. Suppose that $\operatorname{cl}_{Y_2}(U) \cap F = \emptyset$. Since $f : Y_2 \to Y_1$ is closed at $x$, there is some neighborhood $N$ of $x$ in $Y_1$ such that $f^{-1}(N) \subseteq Y_2 \setminus \operatorname{cl}_{Y_2}(U)$.

So, $\operatorname{cl}_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ and $U \cap N = U \cap X \cap N = U \cap f^{-1}(X \cap N) \subseteq \operatorname{cl}_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ imply $U \cap N = \emptyset$. This contradicts $x \in \operatorname{cl}_{Y_1}(U)$.

Thus, it is proved that $Y_1$ is a perfect extension of $X$ relatively to $x$. \hfill $\square$

We recall that a mapping is called *monotone* if every its fibre is connected.

**Corollary 3.3.** Let $Y_1$ and $Y_2$ be extensions of a space $X$ and $f : Y_2 \to Y_1$ be a continuous, closed and monotone mapping such that $f|_X = id_X$. Then, if $Y_2$ is a perfect extension of $X$, $Y_1$ is a perfect extension of $X$ too.
**Definition.** Let $S$ be a subspace of a space $T$. We say that $S$ is *normally situated* (strongly normal in the terminology of [A]) in $T$ if every pair of disjoint closed sets of $S$ can be separated by a pair of disjoint open sets of $T$.

**Remark.** It follows from Corollary 2.4 that every fibre of a compact Hausdorff mapping is normally situated in its domain.

**Lemma 3.4.** Let $Y_1$ and $Y_2$ be extensions of $X$, $x \in Y_1 \setminus X$ and $f : Y_2 \to Y_1$ be a continuous mapping closed at $x$, such that $F = f^{-1}(\{x\})$ is normally situated in $Y_2$ and $f|X = id_X$. If $Y_1$ is a perfect extension of $X$ relatively to $x$ then $F$ is connected.

**Proof:** Suppose, by contradiction, that $F$ is not connected, i.e. that there are disjoint non-empty closed sets $C_1, C_2$ of $F$ such that $C_1 \cup C_2 = F$.

Since $F$ is normally situated in $Y_2$, there are disjoint open sets $U_1, U_2$ of $Y_2$ such that $C_i \subseteq U_i$ (for $i = 1, 2$). So $F \subseteq U_1 \cup U_2$ and, by the closedness of $f$, there exists an open neighborhood $O$ of $x$ in $Y_1$ such that $f^{-1}(O) \subseteq U_1 \cup U_2$.

We may suppose that $f^{-1}(O) = U_1 \cup U_2$.

Since $X$ is dense in $Y_2$, $V_i = U_i \cap X$ for $i = 1, 2$ are non-empty disjoint open sets of $X$ and $O \cap X = f^{-1}(O) \cap X = V_1 \cup V_2$.

On the other hand, $x \in cl_{Y_1}(V_1) \cap cl_{Y_1}(V_2)$ because (for $i = 1, 2$) $U_i \subseteq cl_{Y_2}(U_i) = cl_{Y_2}(U_i \cap X) = cl_{Y_2}(V_i)$ and $x \in f(U_i) \subseteq f(cl_{Y_2}(V_i)) \subseteq cl_{Y_1}(f(V_i)) = cl_{Y_1}(V_i)$.

Thus $Y_1 \setminus X$ cuts $X$ at $x$. This contradicts that $Y_1$ is a perfect extension of $X$ relatively to $x$. Hence, $F$ is connected. \qed

**Corollary 3.5.** Let $Y_1$ and $Y_2$ be extensions of $X$ and $f : Y_2 \to Y_1$ be a continuous closed mapping such that $f|X = id_X$, $f^{-1}(X) = X$ and every its fibre is normally situated in $Y_2$. Then, if $Y_1$ is a perfect extension of $X$, the mapping $f$ is monotone.

**Theorem 3.6.** Let $bf : X^b \to Y$ be a Hausdorff compactification of a mapping $f : X \to Y$ and let $\chi f : \chi f X \to Y$ be the maximal Hausdorff compactification of $f$. Then $bf$ is a perfect extension of $f$ if and only if the canonical morphism $\lambda_b : \chi f \to bf$ is monotone.

**Proof:** Suppose that $bf$ is a perfect compactification of $f$, i.e. that $X^b$ is a perfect extension of $X$. From 2.9, $\lambda_b$ is perfect and, since $\chi f$ is Hausdorff, by 2.2, $\lambda_b$ is Hausdorff, too. Hence (see Remark before Lemma 3.4), every fibre of $\lambda_b$ is normally situated in $\chi f X$. By Corollary 3.5, $\lambda_b$ is monotone.

Conversely, suppose that $\lambda_b : \chi f X \to X^b$ is monotone. Since $\chi f$ is a perfect extension of $f$, i.e. $\chi f X$ if a perfect extension of $X$, 3.3 implies that $X^b$ is a perfect extension of $X$. Hence $bf$ is a perfect extension of $f$. \qed

If $X$ is a Tychonoff space and $|Y| = 1$, every compactification $\gamma X$ of $X$ corresponds to the (Tychonoff) compactifications $\gamma f : \gamma X \to Y$ of $f$, the domain
\(\chi_fX\) of the maximal Hausdorff compactification of \(f\) coincides with the Stone-
Čech compactification \(\beta X\) of \(X\), the canonical morphism \(\lambda : \chi_f \to \gamma f\) becomes
the usual canonical map \(\varphi_\gamma : \beta X \to \gamma X\) and so the previous theorem gives as
corollary the following proposition for spaces proved in [S].

**Theorem 3.7.** A compactification \(\gamma X\) of a Tychonoff space \(X\) is a perfect ex-
tension of \(X\) if and only if the canonical mapping \(\varphi_\gamma : \beta X \to \gamma X\) is monotone.

**Remark.** Let us observe that weaker versions of Theorems 3.1 and 3.6 were
proved by Mazroa [M] by means of the notion of proximity for mappings (see
[No]) only for the particular case of (Tychonoff) compactifications of a surjective
(Tychonoff) mapping between \(T_3\)-spaces.

**Theorem 3.8.** Let \(f : X \to Y\) be a Tychonoff mapping, \(\beta f : \beta f X \to Y\) be its
maximal Tychonoff compactification and \(\chi f : \chi f X \to Y\) be its maximal Hausdorff
compactification. Then the canonical morphism \(\lambda : \chi f \to \beta f\) is monotone.

**Proof:** Since \(\chi f\) is compact and \(\beta f\) is Hausdorff, by 2.9, \(\lambda\) is perfect. From 2.12
it follows that \(\lambda(\chi f X \setminus X) \subseteq \beta f X \setminus X\) and as \(\lambda\) is canonical, \(\lambda^{-1}(X) = X\) and
\(\lambda(\chi f X \setminus X) = \beta f X \setminus X\).

Now, suppose — by contradiction — that \(\lambda : \chi f X \to \beta f X\) is not monotone,
i.e. that there is some \(x \in \beta f X \setminus X\) such that \(\lambda^{-1}(\{x\})\) is not connected. So, there
are non-empty disjoint closed sets \(B, C\) of \(\lambda^{-1}(\{x\})\) such that \(B \cup C = \lambda^{-1}(\{x\})\).
Since \(\lambda^{-1}(\{x\})\) is normally situated in \(\chi f X\) (see Remark before Lemma 3.4),
there are disjoint open sets \(U, V\) of \(\chi f X\) such that \(B \subseteq U\) and \(C \subseteq V\). So, \(U \cup V\)
is an open neighborhood of \(\lambda^{-1}(\{x\})\) and as \(\lambda : \chi f X \to \beta f X\) is closed, there
exists an open neighborhood \(W\) of \(x\) in \(\beta f X\) such that \(\lambda^{-1}(W) \subseteq U \cup V\).

Since \(\beta f X \setminus W\) is a closed subset of \(\beta f X\) which does not contain the point \(x\)
and \(\beta f : \beta f X \to Y\) is a Tychonoff mapping, there exist an open neighborhood
\(H\) of \(\beta f(x)\) in \(Y\) and a continuous mapping \(\varphi : (\beta f)^{-1}(H) \to [0, 1]\) such that
\((\beta f)^{-1}(H) \cap (\beta f X \setminus W) = (\beta f)^{-1}(H) \setminus W \subseteq \varphi^{-1}(\{0\})\) and \(\varphi(x) = 1\).

Hence, \(W_\beta = W \cap (\beta f)^{-1}(H)\) is an open neighborhood of \(x\) in \(\beta f X\) and
\(W_\chi = \lambda^{-1}(W_\beta)\) is an open set of \(\chi f X\). Obviously, \(W_\beta \subseteq W\) and \(W_\chi \subseteq U \cup V\).

Let us note that \(W_\chi \cap X = \lambda^{-1}(W_\beta) \cap \lambda^{-1}(X) = \lambda^{-1}(W_\beta \cap X) = W_\beta \cap X\).
Now, \(W_1 = U \cap W_\chi\) and \(W_2 = V \cap W_\chi\) are non-empty disjoint open sets of
\(\chi f X\) such that \(W_\chi = W_1 \cup W_2\).

Let \(O_i = W_i \cap X\) (for \(i = 1, 2\)). Since \(X\) is dense in \(\chi f X\), \(O_1\) and \(O_2\) are
non-empty disjoint open sets of \(X\) such that \(O_1 \cup O_2 = W_\chi \cap X = W_\beta \cap X\),
\(B \subseteq \text{cl}_{\chi f X}(O_1)\) and \(C \subseteq \text{cl}_{\chi f X}(O_2)\).

Moreover, since \(\beta f \circ \lambda = \chi f\) and \(\chi f|X = f\), we have \(O_1 \cup O_2 = W_\chi \cap X = \lambda^{-1}(W_\beta) \cap X \subseteq \lambda^{-1}((\beta f)^{-1}(H)) \cap X = (\chi f)^{-1}(H) \cap X = f^{-1}(H)\).

Since both \(B\) and \(C\) are contained in the fibre \(\lambda^{-1}(\{x\})\), we obtain \(x \in \lambda(B) \cap \lambda(C) \subseteq \lambda(\text{cl}_{\chi f X}(O_1)) \cap \lambda(\text{cl}_{\chi f X}(O_2)) \subseteq \text{cl}_{\beta f X}(\lambda(O_1)) \cap \text{cl}_{\beta f X}(\lambda(O_2)) = \text{cl}_{\beta f X}(O_1) \cap \text{cl}_{\beta f X}(O_2)\).
Theorem 3.9. The maximal Tychonoff compactification \(\hat{\beta}f : \hat{\beta}fX \to Y\) of a Tychonoff mapping \(f : X \to Y\) is a perfect extension of \(f\).

Remark. If \(X\) is a Tychonoff space and \(|Y| = 1\) then, for the maximal Tychonoff compactification \(\beta f : \beta fX \to Y\) and for the maximal Hausdorff compactification \(\chi f : \chi fX \to Y\), \(\beta fX\) and \(\chi fX\) coincide with the Stone-Čech compactification \(\beta X\) of \(X\) and so Theorems 3.1 and 3.9 give us as simple corollary the following proposition for spaces proved in [S].

Theorem 3.10. The Stone-Čech compactification of a Tychonoff space \(X\) is a perfect extension of \(X\).

Theorems 3.6, 3.9 and Corollary 3.3 imply

Theorem 3.11. A Tychonoff compactification \(bf\) of a Tychonoff mapping \(f\) is perfect if and only if the canonical morphism \(\mu _b : \beta f \to bf\) is monotone.
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References


Dipartimento di Matematica, Università di Messina, Contrada Papardo, salita Sperone, 31, 98166 Sant’Agata, Messina, Italy
E-mail: nordo@dipmat.unime.it

Chair of General Topology and Geometry, Mechanics and Mathematics Faculty, Moscow State University, Moscow 119899, Russia
E-mail: pasynkov@mech.math.msu.su

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