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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 41 (2000), No. 4, 811--815

Persistent URL: <http://dml.cz/dmlcz/119212>

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## Locally minimal topological groups and their embeddings into products of $o$ -bounded groups

TARAS BANAKH

*Abstract.* It is proven that an infinite-dimensional Banach space (considered as an Abelian topological group) is not topologically isomorphic to a subgroup of a product of  $\sigma$ -compact (or more generally,  $o$ -bounded) topological groups. This answers a question of M. Tkachenko.

*Keywords:*  $\omega$ -bounded group,  $\sigma$ -bounded group,  $o$ -bounded group, Weil complete group, locally minimal group, Lie group

*Classification:* 22A05, 54H11

In this paper we answer in the negative the following question of M. Tkachenko posed in [Tk, Problem 3.1]: *Does every second countable topological group embed into a product of  $\sigma$ -compact groups?* Namely, we show that an infinite-dimensional Banach space (considered as an Abelian topological group) admits no such an embedding. In fact, we prove a bit more: no infinite-dimensional Banach space admits an embedding into a product of  $o$ -bounded groups.

Let us recall some definitions, see [Tk]. All topological groups considered in this note are Hausdorff. A subset  $B$  of a topological group  $G$  is defined to be *totally bounded* if for every neighborhood  $U$  of the origin in  $G$  there exists a finite set  $F \subset G$  such that  $B \subset (F \cdot U) \cap (U \cdot F)$ . A topological group  $G$  is defined to be  *$\sigma$ -bounded* if  $G$  is a countable union  $G = \bigcup_{n=1}^{\infty} B_n$  of totally bounded subsets.

A topological group  $G$  is defined to be  $\aleph_0$ -*bounded* if for every neighborhood  $U$  of the origin in  $G$  there exists a subset  $F \subset G$  with  $|F| \leq \aleph_0$  and  $G = F \cdot U$ , see [Gu]. It is known that each second countable group is  $\aleph_0$ -bounded and each  $\aleph_0$ -bounded group embeds into a product of second countable groups ([Gu]).

A topological group  $G$  is called  *$o$ -bounded* if for every sequence  $(U_n)_{n \in \omega}$  of neighborhoods of the origin in  $G$  there exists a sequence  $(F_n)_{n \in \omega}$  of finite subsets in  $G$  such that  $G = \bigcup_{n \in \omega} F_n \cdot U_n$ , see [Tk, 3.9], [He].

According to [Tk] for a topological group  $G$  we have the implications

$$(\sigma\text{-bounded}) \Rightarrow (o\text{-bounded}) \Rightarrow (\aleph_0\text{-bounded}),$$

no of which can be reversed. The considered three classes of groups are closed with respect to the operations of taking subgroups and continuous homomorphic images. M. Tkachenko asked in [Tk, Problem 3.1] if every  $\aleph_0$ -bounded group embeds isomorphically into a product of  $\sigma$ -bounded groups.

The following theorem answers this question in the negative.

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Research supported in part by grant INTAS-96-0753.

**Main Theorem.** *An infinite-dimensional Banach space (considered as an Abelian topological group) admits no isomorphic embedding into a product of  $o$ -bounded groups.*

As a by-product of the proof we get a characterization of Lie groups in terms of embeddings into products of  $o$ -bounded groups as well as a theorem on equivalence of  $o$ -boundedness and  $\sigma$ -boundedness for groups which are continuous homomorphic images of second countable Weil complete groups. A topological group is called *Weil complete* if it is complete in its left (equivalently, right) uniformity.

### Interplay between $o$ -boundedness and $\sigma$ -boundedness

The main result of this section is

**Equivalence Theorem.** *Suppose that a topological group  $G$  is a continuous homomorphic image of a second countable Weil complete group. The group  $G$  is  $o$ -bounded if and only if it is  $\sigma$ -bounded.*

PROOF: By hypothesis there exists a surjective continuous group homomorphism  $h : H \rightarrow G$ , where  $H$  is a second countable Weil complete group. Let  $d$  be any left-invariant complete metric on  $H$  and  $B(\varepsilon) = \{x \in H : d(x, e) \leq \varepsilon\}$ ,  $\varepsilon > 0$ , denote the closed  $\varepsilon$ -ball around the neutral element  $e$  of the group  $H$ .

The “if” part of the theorem is trivial. To prove the “only if” part, suppose  $G$  is an  $o$ -bounded group. We claim that the image  $h(U)$  of some neighborhood  $U$  of the identity in  $H$  is left-bounded in  $H$ , i.e., for every neighborhood  $W$  of the identity in  $G$  there is a finite subset  $F \subset G$  with  $h(U) \subset F \cdot W$ .

Assume that it is not so. To get a contradiction, we shall show that the group  $G$  is not  $o$ -bounded. For this we shall construct by induction a sequence  $(\varepsilon_n)_{n=1}^\infty \subset (0, 1]$  of real numbers and a sequence  $(U_n)_{n=1}^\infty$  of neighborhoods of the origin in  $G$  such that

- (1)  $h(B(\varepsilon_n/2)) \not\subset F \cdot U_n \cdot U_n^{-1}$  for any finite set  $F \subset G$ ;
- (2)  $h(B(\varepsilon_{n+1})) \subset U_n$ ;
- (3)  $\varepsilon_n \leq \varepsilon_{n-1}/2$ .

Let  $\varepsilon_1 = 1$  and assume that for some  $n$  numbers  $\varepsilon_1, \dots, \varepsilon_n$  and neighborhoods  $U_1, \dots, U_{n-1}$  satisfying the conditions (1)–(3) have been constructed. By our assumption, the set  $h(B(\varepsilon_n/2))$  is not left bounded in  $G$ . Hence, there exists a neighborhood  $W \subset G$  of the origin such that  $h(B(\varepsilon_n/2)) \not\subset F \cdot W$  for any finite set  $F \subset G$ . Let  $U_n$  be a neighborhood of the origin in  $G$  such that  $U_n \cdot U_n^{-1} \subset W$ . Clearly, the condition (1) is satisfied. Finally, using the continuity of  $h$ , choose any  $\varepsilon_{n+1}$  to satisfy  $0 < \varepsilon_{n+1} \leq \varepsilon_n/2$  and  $h(B(\varepsilon_{n+1})) \subset U_n$ . This finishes the inductive construction of the sequences  $(\varepsilon_n)_{n=1}^\infty$  and  $(U_n)_{n=1}^\infty$ .

It rests to verify that  $\bigcup_{n=1}^\infty F_n \cdot U_n \neq G$  for any sequence  $(F_n)_{n=1}^\infty$  of finite subsets of  $G$ . For this, given such a sequence  $(F_n)$ , we shall construct inductively a sequence  $(x_n)_{n=1}^\infty$  of points of the group  $H$  such that the following conditions are satisfied for every  $n \geq 1$ :

- (4)  $h(x_n \cdot B(\varepsilon_n)) \cap F_{n-1} \cdot U_{n-1} = \emptyset$ ;
- (5)  $x_{n+1} \in x_n \cdot B(\varepsilon_n/2)$ .

Let  $x_0 = e$  and assume that for some  $n \geq 0$  the points  $x_0, \dots, x_n$  satisfying (4) and (5) have been defined. It follows from (1) that  $h(x_n \cdot B(\varepsilon_n/2)) \not\subset F_n \cdot U_n \cdot U_n^{-1}$  and hence there exists a point  $x_{n+1} \in x_n \cdot B(\varepsilon_n/2)$  with  $h(x_{n+1}) \notin F_n \cdot U_n \cdot U_n^{-1}$ . Multiplying this by  $U_n$ , we get  $h(x_{n+1}) \cdot U_n \cap F_n \cdot U_n = \emptyset$ . Then by (2),  $h(x_{n+1}) \cdot B(\varepsilon_{n+1}) \cap F_n \cdot U_n = \emptyset$ . This finishes the construction of the sequence  $(x_n)_{n=1}^\infty$ .

It follows from (3) and (5) that the sequence  $(x_n)_{n=1}^\infty$  is Cauchy with respect to the metric  $d$  and thus converges to some point  $x_\infty \in H$ . We claim that  $x_\infty \in x_n \cdot B(\varepsilon_n)$  for every  $n \geq 1$ . Indeed, using (5) and (3), we get

$$d(x_\infty, x_n) \leq \sum_{i=n}^\infty d(x_i, x_{i+1}) \leq \sum_{i=n}^\infty \frac{\varepsilon_i}{2} \leq \sum_{i=n}^\infty \frac{\varepsilon_n}{2 \cdot 2^{i-n}} = \varepsilon_n.$$

Then by (4),  $h(x_\infty) \notin F_n \cdot U_n$  for every  $n \geq 1$  which implies  $h(x_\infty) \notin \bigcup_{n=1}^\infty F_n \cdot U_n$  and  $G \neq \bigcup_{n=1}^\infty F_n \cdot U_n$ .

This contradiction shows that the image  $h(U)$  of some symmetric neighborhood  $U = U^{-1}$  of the identity in  $H$  is left bounded in  $G$ . Then for every points  $x, y \in G$  the set  $x \cdot h(U)$  is left bounded while the set  $x \cdot h(U) \cap h(U) \cdot y$  is totally bounded. Now fix a dense countable subset  $(d_n)_{n \in \omega}$  in  $H$ . Then  $H = \bigcup_{i,j \in \omega} (d_i \cdot U) \cap (U \cdot d_j)$  and consequently,  $G = h(H) = \bigcup_{i,j \in \omega} (h(d_i) \cdot h(U)) \cap (h(U) \cdot h(d_j))$  is a countable union of totally bounded subsets. □

### Locally minimal groups

Recall that a topological group  $G$  is called *minimal* if  $G$  admits no strictly weaker Hausdorff group topology.

We define a topological group  $G$  to be *locally minimal* if there exists a neighborhood  $U$  of the origin in  $G$  such that  $G$  admits no strictly weaker Hausdorff group topology for which  $U$  is a neighborhood of the origin.

Clearly, each minimal group is locally minimal. It can be easily shown that each locally compact group is locally minimal. There are also non-locally compact locally minimal groups:

**Proposition 1.** *A normed linear space (considered as an Abelian topological group) is locally minimal.*

PROOF: Let  $X$  be a normed linear space and let  $B$  denote the unit open ball in  $X$  with the center at the origin. Suppose  $\tau$  is a weaker Hausdorff group topology on  $X$  such that  $B$  is a neighborhood of the origin in  $(X, \tau)$ . To prove our proposition it suffices to verify that for every  $n \in \mathbb{N}$  the set  $\frac{1}{n}B = \{x \in X : \|x\| < \frac{1}{n}\}$  is a neighborhood of the origin in  $(X, \tau)$ . Let  $U \subset B$  be an open neighborhood of the origin in  $(X, \tau)$ . By the continuity of the group operation on  $(X, \tau)$ , the set  $V = \{x \in X : nx \in U\}$  is open in  $(X, \tau)$ . We claim that  $V \subset \frac{1}{n}B$ . Indeed, assuming

the converse, we would find  $x \in V$  with  $\|x\| \geq \frac{1}{n}$ . Then  $\|nx\| \geq 1$ , a contradiction with  $nx \in U \subset B$ .  $\square$

We call a topological group  $G$  a *group without small subgroups* if there exists a neighborhood of the origin in  $G$  containing no non-trivial subgroup. It is easy to see that each normed space is a group without small subgroups. Locally minimal groups without small subgroups have the following remarkable property.

**Proposition 2.** *Let  $G \subset \prod_{i \in \mathcal{I}} G_i$  be a subgroup of a product of topological groups. If  $G$  is a locally minimal group without small subgroup, then there exists a finite subset  $F \subset \mathcal{I}$  such that the projection  $\text{pr}_F : G \rightarrow \prod_{i \in F} G_i$  is an isomorphic embedding.*

PROOF: Let  $U$  be a neighborhood of the origin in  $G$  containing no non-trivial subgroup and  $V$  be a neighborhood of the origin in  $G$  such that  $G$  admits no strictly weaker Hausdorff group topology for which  $V$  remains a neighborhood of the origin. By definition of the product topology on  $\prod_{i \in \mathcal{I}} G_i$ , there exists a finite subset  $F \subset \mathcal{I}$  and a neighborhood  $W$  of the origin  $e$  of the group  $\prod_{i \in F} G_i$  such that  $\text{pr}_F^{-1}(W) \subset U \cap V$ . We claim that the projection  $\text{pr}_F : G \rightarrow \prod_{i \in F} G_i$  is an isomorphic embedding. Observe that  $\text{pr}_F^{-1}(e) \subset U$  is a trivial subgroup of  $G$  (by the choice of  $U$ ) and thus the map  $\text{pr}_F : G \rightarrow \prod_{i \in F} G_i$  is injective. Then  $\tau = \{\text{pr}_F^{-1}(O) : O \text{ is an open subset in } \prod_{i \in F} G_i\}$  is a weaker Hausdorff group topology on  $G$ . Since  $V$  is a neighborhood of the origin in  $(G, \tau)$ , the topology  $\tau$  coincides with the original topology of the group  $G$  and thus the map  $\text{pr}_F : G \rightarrow \prod_{i \in F} G_i$  is an isomorphic embedding.  $\square$

**Problem.** *Investigate the class of locally minimal groups.*

## A characterization of Lie groups

**Characterization Theorem.** *A second countable group  $G$  is a Lie group if and only if the following conditions are satisfied:*

- (1)  $G$  is a locally minimal Weil complete group without small subgroups;
- (2)  $G$  embeds isomorphically into a product of  $\sigma$ -bounded groups.

PROOF: The “only if” part of the theorem is trivial. To prove the “if” part, suppose that a second countable group  $G$  satisfies the conditions (1)–(2). By Proposition 2, the group  $G$  embeds isomorphically into a finite product  $G_1 \times \cdots \times G_n$  of  $\sigma$ -bounded groups. Since subgroups of  $\sigma$ -bounded groups are  $\sigma$ -bounded, we may assume that the projection of  $G$  on each  $G_i$  coincides with  $G_i$ . Then according to Equivalence Theorem, each  $G_i$ , being a continuous homomorphic image of a Weil complete group  $G$ , is  $\sigma$ -bounded. Consequently, the product  $G_1 \times \cdots \times G_n$  as well as its subgroup  $G$  is  $\sigma$ -bounded. Now Weil completeness of  $G$  implies that  $G$  is  $\sigma$ -compact and hence, being second countable, must be locally compact. Since  $G$  has no small subgroups,  $G$  is a Lie group according to the well known Gleason-Montgomery-Zippin Theorem ([Gl], [MZ]).  $\square$

**Question.** *Is Characterization Theorem valid for Raïkov complete groups, i.e., groups complete with respect to the two-sided uniformity?*

### Proof of Main Theorem

Suppose that an infinite-dimensional Banach space  $X$  embeds into a product  $\prod_{i \in \mathcal{I}} G_i$  of  $\mathfrak{o}$ -bounded groups. The groups  $G_i$ , being  $\mathfrak{o}$ -bounded, are  $\aleph_0$ -bounded. Then the subgroup  $X$  of their product  $\prod_{i \in \mathcal{I}} G_i$  is  $\aleph_0$ -bounded ([Gu]). Next, the group  $X$ , being metrizable and  $\aleph_0$ -bounded, is second countable. Thus  $X$  is a second countable Weil complete abelian group without small subgroups (see Proposition 1) which embeds into a product of  $\mathfrak{o}$ -bounded groups. By Characterization Theorem,  $X$  must be a Lie group, a contradiction with the infinite-dimensionality of  $X$ .  $\square$

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(Received December 10, 1999, revised March 24, 2000)