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## On Mazurkiewicz sets

MARTA N. CHARATONIK, WŁODZIMIERZ J. CHARATONIK

*Abstract.* A Mazurkiewicz set  $M$  is a subset of a plane with the property that each straight line intersects  $M$  in exactly two points. We modify the original construction to obtain a Mazurkiewicz set which does not contain vertices of an equilateral triangle or a square. This answers some questions by L.D. Loveland and S.M. Loveland. We also use similar methods to construct a bounded noncompact, nonconnected generalized Mazurkiewicz set.

*Keywords:* Mazurkiewicz set, GM-set, double midset property

*Classification:* Primary 54C99, 54F15, 54G20; Secondary 54B20

By a Mazurkiewicz set (shortly M-set) we mean a subset  $X$  of the plane such that every straight line intersects  $X$  in exactly two points. It was constructed in [3] using transfinite induction. The notion was generalized in two directions: to generalized Mazurkiewicz sets (GM-sets) and to sets with the double midset property (DMP-sets). Let us recall that a subset  $X$  of the plane is a GM-set if it contains at least two points and each line that separates two points of  $X$  intersects  $X$  in exactly two points. A subset  $X$  of the plane is a DMP-set if it contains at least two points and the perpendicular bisector of every segment joining two points in  $X$  intersect  $X$  in exactly two points. It follows from the definitions that every M-set is a GM-set and every GM-set is an DMP-set. For more information about these notions see [2]. In the same article the authors ask some questions related to the subject. Here we answer some of them in a more general case constructing, using transfinite induction, an M-set with some additional geometrical properties. Namely, the M-set that does not contain vertices of an equilateral triangle or vertices of a square, and whose image under the inversion with respect to the unit circle is a bounded, noncompact, nonconnected GM-set.

We will need some denotation. The symbol  $\mathfrak{c}$  denotes the cardinal number continuum, i.e. the first ordinal number whose cardinality is the cardinality of reals. All the constructions are going to be done in the complex plane  $\mathbb{C}$ . Given  $x, y \in \mathbb{C}$  the symbol  $l(x, y)$  denotes the line through  $x$  and  $y$  if  $x \neq y$ , and  $l(x, x) = \{x\}$ . If both  $x$  and  $y$  are distinct from 0, and  $x \neq y$ , then  $c(x, y)$  is the circle that contains  $x$ ,  $y$  and 0. Moreover we put  $c(x, x) = \{x\}$ . For a subset  $A \subset \mathbb{C}$  we put  $L(A) = \bigcup \{l(x, y) : x, y \in A\}$  and  $C(A) = \bigcup \{c(x, y) : x, y \in A\}$ .

We denote by  $B$  the open unit disk in the plane, i.e.  $B = \{z \in \mathbb{C} : |z| < 1\}$ .

**Theorem.** *There is an M-set  $A$  satisfying the following conditions:*

- (1)  $A$  does not contain vertices of an equilateral triangle;
- (2)  $A$  does not contain vertices of a right isosceles triangle;
- (3)  $A \cap \text{cl } B = \emptyset$ ;
- (4) any circle that contains 0 and is not contained in  $\text{cl } B$  intersects  $A$  at exactly two points.

PROOF: Given two different points  $a, b \in \mathbb{C}$  define  $P(a, b)$  as the set of all points  $x \in \mathbb{C}$  such that the triangle with vertices  $a, b, x$  is an equilateral one or a right isosceles one. Thus  $P(a, b)$  has exactly eight points. In particular we have  $P(0, 1) = \{i, -i, 1 + i, 1 - i, \frac{1}{2} + \frac{\sqrt{2}}{2}i, \frac{1}{2} + \frac{-\sqrt{2}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{-\sqrt{3}}{2}i\}$ . Additionally we put  $P(x, x) = \emptyset$ . For a set  $A \subset \mathbb{C}$  let  $P(A) = \bigcup \{P(x, y) : x, y \in A\}$ .

Let  $\{l_\alpha : \alpha \leq \mathfrak{c}\}$  be the set of all straight lines in the plane, and let  $\{c_\alpha : \alpha \leq \mathfrak{c}\}$  be the set of all circles passing through 0 and not contained in  $\text{cl } B$ . We will define, for  $\alpha < \mathfrak{c}$ , the set  $A_\alpha$ , and  $A = \bigcup \{A_\alpha : \alpha < \mathfrak{c}\}$  will be the required M-set. Assume that, for some  $\alpha < \mathfrak{c}$ , the sets  $A_\beta$  for  $\beta < \alpha$ , have been defined satisfying the following conditions:

- (1 $_\beta$ )  $\text{card}(A_\beta) < \mathfrak{c}$ ;
- (2 $_\beta$ ) for every  $\gamma < \beta$  we have  $A_\gamma \subset A_\beta$ ;
- (3 $_\beta$ )  $A_\beta \cap l_\beta$  is a two point set;
- (4 $_\beta$ )  $A_\beta \cap c_\beta$  is a two point set;
- (5 $_\beta$ )  $A_\beta \cap P(A_\beta) = \emptyset$ ;
- (6 $_\beta$ )  $A_\beta$  contains no three colinear points;
- (7 $_\beta$ ) there is no circle in the plane that contains three different points of  $A_\beta$  and the point 0;
- (8 $_\beta$ )  $A_\beta \cap \text{cl } B = \emptyset$ .

Put  $N_\alpha = \bigcup \{A_\beta : \beta < \alpha\}$ . Then

- $\text{card}(P(N_\alpha)) < \mathfrak{c}$ ,
- $\text{card}(l_\alpha \cap (\bigcup \{l(x, y) : x, y \in N_\alpha\})) < \mathfrak{c}$ ,
- $\text{card}(c_\alpha \cap (\bigcup \{c(x, y) : x, y \in N_\alpha\})) < \mathfrak{c}$ ,
- $\text{card}(l_\alpha \cap N_\alpha) \leq 2$ ,
- $\text{card}(c_\alpha \cap N_\alpha) \leq 2$ .

Thus we can choose points  $x_\alpha, y_\alpha, z_\alpha, t_\alpha$  that satisfy the following conditions, where  $G_\alpha = \text{cl } B \cup P(N_\alpha) \cup L(N_\alpha) \cup C(N_\alpha)$ .

- $x_\alpha, y_\alpha \in l_\alpha \setminus c_\alpha$ ,
- $z_\alpha, t_\alpha \in c_\alpha \setminus l_\alpha$ ,
- if  $\text{card}(l_\alpha \cap N_\alpha) = 2$ , then  $\{x_\alpha, y_\alpha\} = l_\alpha \cap N_\alpha$ ,
- if  $\text{card}(c_\alpha \cap N_\alpha) = 2$ , then  $\{z_\alpha, t_\alpha\} = c_\alpha \cap N_\alpha$ ,
- if  $\text{card}(l_\alpha \cap N_\alpha) = 1$ , then  $\{x_\alpha\} = l_\alpha \cap N_\alpha$  and  $y_\alpha \notin G_\alpha$ ,
- if  $\text{card}(c_\alpha \cap N_\alpha) = 1$ , then  $\{z_\alpha\} = c_\alpha \cap N_\alpha$  and  $t_\alpha \notin G_\alpha$ ,
- if  $\text{card}(l_\alpha \cap N_\alpha) = 0$ , then  $x_\alpha, y_\alpha \notin G_\alpha$ ,
- if  $\text{card}(c_\alpha \cap N_\alpha) = 0$ , then  $z_\alpha, t_\alpha \notin G_\alpha$ .

Finally put  $A_\alpha = N_\alpha \cup \{x_\alpha, y_\alpha, z_\alpha, t_\alpha\}$ . One can verify that, by the construction, conditions  $(1_\alpha)$ – $(8_\alpha)$  are satisfied. Putting  $A = \bigcup \{A_\alpha : \alpha < \mathfrak{c}\}$  we see that  $A$  is the required M-set. This finishes the proof.  $\square$

**Remark 1.** In [2, Questions 2 and 3, p. 488] the authors asked if there is a DMP-set in the plane that does not contain vertices of a square (Question 2) and if there is a DMP-set in the plane that does not contain vertices of an equilateral triangle (Question 3). Because every M-set is a DMP-set, the Theorem answers both questions.

Denote by  $h : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$  the inversion with respect to the unit circle, i.e.  $h(z) = 1/\bar{z}$ . Observe that  $h(h(z)) = z$ .

**Proposition.** *Let  $A$  be an M-set that satisfies conditions (3) and (4) of the Theorem. Then  $h(A)$  is a GM-set.*

PROOF: First observe that  $h(A) \subset B$  by condition (3). Let  $l$  be a line that separates two points of  $h(A)$ . If  $0 \in l$ , then  $h(l \setminus \{0\}) = l \setminus \{0\}$ . If  $0 \notin l$ , then  $h(l)$  is a circle passing through 0 and not contained in  $B$ . In any case  $h(l) \cap A$  is a two point set by (4), and therefore  $h(h(l)) \cap h(A) = l \cap h(A)$  is a two point set, as required.  $\square$

**Remark 2.** In [2, Question 6, p. 490] the authors ask the following question. Is there a bounded GM-set which is not a simple closed curve? Is a bounded GM-set necessarily closed? Connected? Since the constructed set  $h(A)$  is a bounded GM-set homeomorphic to an M-set, it is neither closed (M-sets are not bounded, so not compact) nor connected (M-sets are zerodimensional, see [1, Theorem 2, p. 553]). Thus the Proposition answers in the negative all of the three questions. It also answers more particular Question 7 and partially Question 8.

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