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Cauchy-Neumann problem for a class of nondiagonal parabolic systems with quadratic nonlinearities II. Local and global solvability results

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Cauchy-Neumann problem for a class of nondiagonal parabolic systems with quadratic nonlinearities

II. Local and global solvability results

A. Arkhipova

Abstract. We prove local in time solvability of the nonlinear initial-boundary problem to nonlinear nondiagonal parabolic systems of equations (multidimensional case). No growth restrictions are assumed on generating the system functions.

In the case of two spatial variables we construct the global in time solution to the Cauchy-Neumann problem for a class of nondiagonal parabolic systems. The solution is smooth almost everywhere and has an at most finite number of singular points.

Keywords: boundary value problem, nonlinear parabolic systems, solvability

Classification: 35J65

This article is a continuation of the author’s work [9]. Here we prove two independent results. These are local and global in time solvability theorems for a nonlinear initial boundary-value problem to nondiagonal parabolic systems.

In §1 (Theorem 1) local classical solvability is stated for general situations, that is, we do not assume any structural restriction and growth conditions on forming system and boundary condition functions. A related result for *quasilinear* parabolic systems under the Dirichlet and Neumann boundary conditions was proved in [1], [2].

Global in time weak solvability of the Cauchy-Neumann problem for parabolic systems studied in [9] is proved in §2 (Theorem 2). We consider a variational structure of an elliptic operator and consider only the case of two spatial variables. These systems have a *nondiagonal* main matrix and *quadratic* nonlinearity in the gradient. Note that the global solvability result is essentially based on the extendibility theorem (Theorem 1, [9]) and the local solvability theorem (Theorem 1 of the present paper).

This investigation is a generalization of the author’s results [3], [4] where global in time weak solvability of the Cauchy-Dirichlet problem was stated for the same class of parabolic systems.

Here we make use of the notation of the Part I of the paper (see [9]).

1. Local in time classical solvability

Let \( \Omega \) be a domain in \( \mathbb{R}^n, n \geq 2 \), with sufficiently smooth boundary \( \partial \Omega \). For a fixed \( T_1 > 0 \) and \( Q = \Omega \times (0, T_1) \) we consider a solution \( u : Q \to \mathbb{R}^N \),
\[ u = (u^1, \ldots, u^N), \quad N > 1, \text{ of the parabolic system} \]

\[ u_t^k - A_{kl}^{\alpha\beta}(z, u, u_x)u_{x\beta x\alpha}^l + b^k(z, u, u_x) = 0, \quad z = (x, t) \in Q, \; k = 1, \ldots, N. \]

The function \( u \) satisfies the initial condition

\[ u|_{t=0} = 0, \]

and nonlinear boundary condition

\[ \Phi^k(z, u, u_x) + \psi^\alpha_{kl}(z, u)u_{x\alpha}^l + g^k(z, u)|_{\Gamma} = 0, \quad \Gamma = \partial \Omega \times (0, T_1), \quad k \leq N. \]

We define the sets

\[ \mathcal{M} = \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{nN}, \quad \mathcal{N} = \Gamma \times \mathbb{R}^N \times \mathbb{R}^{nN}, \quad \mathcal{N}^0 = \Gamma \times \mathbb{R}^N, \]

and suppose that the functions \( A_{kl}^{\alpha\beta}, b^k, \Phi^k, \psi^\alpha_{kl}, g^k \) are smooth enough on \( \mathcal{M}, \mathcal{N} \) and \( \mathcal{N}^0 \), respectively. (More exactly, see conditions \( A_1, \ldots, A_5 \) below.)

Suppose that the matrix \( A = \{A_{kl}^{\alpha\beta}\}_{k,l \leq N} \) satisfies on \( \mathcal{M} \) the condition

\[ A_{kl}^{\alpha\beta}(z, u, p)\xi^k_{\alpha} \xi^l_{\beta} \geq \nu |\xi|^2, \quad \forall \xi \in \mathbb{R}^{nN}, \quad \nu = \text{const} > 0. \]

We introduce the functions

\[ \Phi^k(z, u, p) = \frac{1}{2} \int_0^1 \frac{\partial \Phi^k(z, u, sp)}{\partial p^l_{\beta}} ds \cdot p^l_{\beta} + \Phi^k(z, u, 0) = \zeta^k_{kl}(z, u, p)p^l_{\beta} + \Phi^k(z, u, 0) \]

and suppose that

\[ \zeta^k_{kl}(z, u, p) + \psi^\beta_{kl}(z, u) \cos(n, x_{\beta})\eta^k \eta^l \geq \nu_0 |\eta|^2, \quad \forall \eta \in \mathbb{R}^N, \quad \nu_0 = \text{const} > 0, \]

where \( n = n(x) \) is the outward normal vector to \( \Omega \) at a point \( x \in \partial \Omega, (z, u, p) \in \mathcal{N} \).

We rewrite (1.3) in the form

\[ (\zeta^k_{kl}(z, u, u_x) + \psi^\beta_{kl}(z, u))u_{x\beta}^l + G^k(z, u)|_{\Gamma} = 0, \quad k \leq N, \]

where \( G^k(x, t, u) = g^k(x, t, u) + \Phi^k(x, t, u, 0) \).

The compatibility condition is written in the following form

\[ G^k(x, 0, 0) = 0, \quad x \in \partial \Omega, \quad k = 1, \ldots, N. \]
We intend to prove the existence of a smooth solution to (1.1)–(1.3) (or (1.1), (1.2), (1.7)) on some interval \([0, T_0]\), where \(T_0 \leq T_1\).

Note that in the case \(A = A(z, u)\) and condition (1.7) in the form

\[ A_{kl}^\alpha \beta (z, u) u^l_{,x\beta} \cos(m, x\alpha) + G^k(z, u)\big|_\Gamma = 0, \]

local in time classical solvability of (1.1), (1.2) follows from [1], [2]. To prove the existence of a solution we use the contraction method.

We introduce the following notation

\[ \langle v \rangle(x, Q) = \sup_{(x,t),(x',t') \in Q, x \neq x'} \frac{|v(x,t) - v(x',t)|}{|x - x'|^\alpha}, \quad \langle v \rangle(t, Q) = \sup_{(x,t),(x',t') \in Q, t \neq t'} \frac{|v(x,t) - v(x,t')|}{|t - t'|^\beta}, \]

\[ [v]_Q^\alpha = \langle v \rangle(x, Q) + \langle v \rangle(t, Q), \quad \alpha, \beta \in (0,1). \]

\(\|u\|_{m, D}\) denotes the norm of \(u\) in the space \(L^m(D), m \in [1, \infty]\).

Here and below we write \(\mathcal{B}(Q)\) instead of \(\mathcal{B}(Q; \mathbb{R}^N)\) for brevity.

\(\mathcal{H}^{\alpha,\alpha/2}(\overline{Q})\) is the space of all continuous in \(\overline{Q}\) functions with finite norm

\[ \|v\|_{\mathcal{H}^{\alpha,\alpha/2}(\overline{Q})} = \|v\|_{\infty,Q} + \langle v \rangle(x, Q) + \langle v \rangle(t, Q). \]

(So \(\mathcal{H}^{\alpha,\alpha/2}(\overline{Q}) = C^{\alpha,\alpha/2}(\overline{Q})\)).

\(\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})\) is the space of functions \(u\) continuous on \(\overline{Q}\) with derivatives \(u_t, u_x, u_{xx}\) and finite norm:

\[ \|u\|_{\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})} = \|u\|_{\infty,Q} + \|u_x\|_{\infty,Q} + \|u_{xx}\|_{\infty,Q} + \|u_t\|_{\infty,Q} + [u_t]_{Q}^\alpha + [u_{xx}]_{Q}^{(1+\alpha)/2}. \]

We also consider the space \(\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\overline{\Gamma})\) of functions \(v\) that are continuous and have continuous derivatives \(v_x\) on \(\overline{\Gamma}\). Here we define this space as the trace space for \(\mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q})\) [5]. Let \(\partial \Omega\) be a \(C^{2+\alpha}\) surface. We denote by \(V_1, \ldots, V_m \subset \mathbb{R}^n\) a system of neighborhoods with the following properties: (1) \(\bigcup_{j=1}^m V_j \supset \partial \Omega\), (2) there exists a system of \(C^{2+\alpha}\) diffeomorphisms \(P_j\) on \(V_j, j = 1, \ldots, m\), such that \(P_j(V_j \cap \Omega) = B_1^+, P_j(V_j \cap \partial \Omega) = \sigma\). Here \(B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}, B_1^+ = B_1 \cap \{x_n > 0\}, \sigma = B_1 \cap \{x_n = 0\}\).

For a function \(v \in \mathcal{H}^{2+\alpha,1+\alpha/2}(\overline{Q}), x \in V_j \cap \partial \Omega, t \in [0, T_1]\), we define the function \(v^{(y)}(y, t) = v(P_j^{-1}(y), t)\) on \(\mathcal{Q}^+ = B_1^+ \times [0, T_1]\).

Let \(y' = (y_1, \ldots, y_{n-1}), \Sigma = \sigma \times [0, T_1]\), we put

\[ \|v\|_{\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\overline{\Gamma})} = \sup_{j \leq m} \|v^{(y')}\|_{\mathcal{H}^{1+\alpha,(1+\alpha)/2}(\Sigma)}. \]
where

\[(1.10) \quad \|w(y', t)\|_{H^{1+\alpha, (1+\alpha)/2}(\Sigma)} = \|w\|_{\infty, \Sigma} + \|w_{y'}\|_{\infty, \Sigma} + [w_{y'}]^{(\alpha)} + \langle w \rangle^{(1+\alpha)/2}.
\]

It is obvious that definition (1.9),(1.10) depends on the fixed atlas \(\{V_j, P_j\}_{j=1}^m\) but all the norms are equivalent.

Now we fix an \(\alpha_0 > 0\) and some \(T \in (0, T_1]\) and introduce the space

\[X_T = \left\{ v : Q \to \mathbb{R}^N, v \in H^{2+\alpha_0, 1+\alpha_0/2}(\Omega_T) \mid v|_{t=0} = 0 \right\}, \quad Q^T = \Omega \times (0, T).
\]

For a fixed \(v \in X_T\) we put

\[
\Delta A_{kl}^{\alpha\beta}(x, t, v, v_x) = A_{kl}^{\alpha\beta}(x, t, v) - A_{kl}^{\alpha\beta}(x, 0, 0, 0),
\]

\[
\Delta \varphi_{kl}^{\beta}(x, t, v, v_x) = \varphi_{kl}^{\beta}(x, t, v) - \varphi_{kk}^{\beta}(x, 0, 0, 0),
\]

\[
\Delta \psi_{kl}^{\beta}(x, t, v) = \psi_{kl}^{\beta}(x, t, v) - \psi_{kl}^{\beta}(x, 0, 0)
\]

and consider the linear problem

\[(1.11) \quad w_k^l - A_{kl}^{\alpha\beta}(x, 0, 0, 0)w_{x\beta x\alpha}^l + b^k(x, t, v, v_x) + \Delta A_{kl}^{\alpha\beta}(x, t, v, v_x)v_{x\beta x\alpha}^l = 0, \quad (x, t) \in Q^T,
\]

\[
\left(\varphi_{kl}^{\beta}(x, 0, 0, 0) + \psi_{kl}^{\beta}(x, 0, 0)\right)w_{x\beta}^l + G^k(x, t, v) + (\Delta \varphi_{kl}^{\beta}(x, t, v, v_x)
\]

\[
+ \Delta \psi_{kl}^{\beta}(x, t, v))v_{x\beta}^l = 0, \quad \Gamma^T = \partial \Omega \times (0, T), \quad k = 1, \ldots, N,
\]

\[
w\mid_{t=0} = 0.
\]

We write \(\Delta A_{v, x} = \{\Delta A_{kl}^{\alpha\beta}(x, t, v, v_x)v_{x\beta x\alpha}^l\}_{k \leq N}, \Delta \varphi_{v, x} = \{\Delta \varphi_{kl}^{\beta}(x, t, v, v_x)v_{x\beta}^l\}_{k \leq N}, \Delta \psi_{v, x} = \{\Delta \psi_{kl}^{\beta}(x, t, v, v_x)v_{x\beta}^l\}_{k \leq N}, \)

\(G = \{G^k(x, t, v)\}_{k \leq N}\) for brevity.

We assume that the complementing conditions hold for problem (1.11).

If the data are smooth enough then according to the linear theory there exists a unique solution \(w \in X_T\) of (1.11) [5, Chapter VII, Theorem 10.1] and the following estimate is valid:

\[
\|w\|_{\chi^T} \leq c_0\left\{\|b\|_{H^{\alpha_0, \alpha_0/2}(\Omega_T)} + \|\Delta A \cdot v_{x}x\|_{H^{\alpha_0, \alpha_0/2}(\Omega_T)}
\]

\[
+ \|\Delta \varphi \cdot v_x\|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Gamma_T)} + \|\Delta \psi \cdot v_x\|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Gamma_T)}
\]

\[
+ \|G\|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Gamma_T)}\right\}.
\]

The constant \(c_0\) in (1.12) depends on the parameters \(\nu, \nu_0\) from conditions (1.5), (1.6), \(\|A(x, 0, 0, 0)\|_{C^{\alpha}_0(\bar{\Omega})}, \|\varphi(x, 0, 0, 0)\|_{C^{1+\alpha_0}(\partial \Omega)}, \|\psi(x, 0, 0)\|_{C^{1+\alpha_0}(\bar{\Omega})}, C^{2+\alpha_0}\) characteristics of \(\partial \Omega\) and \(T_1\), but it does not depend on the fixed \(T\) and any characteristic of the function \(v\).
Thus, problem (1.11) defines the map $F: X_T \to X_T$,

$$w = F(v), \quad \forall v \in X_T. \quad (1.13)$$

We shall prove that if $T < T_1$ is small enough then there exists a fixed point $u$ of $F$. The function $u \in X_T$ is a solution of the problem (1.1)–(1.3).

Now we impose precise conditions on the data.

Let $M > 0$ be an arbitrary fixed number.

$A_1$. On $\mathcal{M}_M = \{(x, t, u, p) \in \mathcal{M} \mid |u| + |p| \leq M\}$
- the functions $A = \{A_{k,l}^{\alpha \beta}\}_{\alpha, \beta \leq n, k, l \leq N}$ are continuous and have continuous derivatives $A_u, A_p$;
- the functions $A, A_u, A_p$ are Hölder continuous in $x, t, u, p$ with the exponents $\alpha_0, \alpha_0/2, \alpha_0, \alpha_0$, respectively.

$A_2$. On $\mathcal{M}_M$
- the functions $b = \{b^k(x, t, u, p)\}_{k \leq N}$ are continuous with derivatives $b_u$ and $b_p$;
- the functions $b$ are Hölder continuous in $x, t$ with the exponents $\alpha_0, \alpha_0/2$, respectively,
- $b_u, b_p$ are Hölder continuous in $x, t, u, p$ with the exponents $\alpha_0, \alpha_0/2, \alpha_0, \alpha_0$, respectively.

$A_3$. On $\mathcal{N}_M = \{(x, t, u, p) \in \mathcal{N} \mid |u| + |p| \leq M\}$
- the functions $\Phi = \{\Phi^k(x, t, u, p)\}_{k \leq N}$ are continuous with derivatives $\Phi_p$,
- the functions $\kappa = \{\kappa_{kl}^{\beta}(x, t, u, p)\}_{\beta \leq n, k, l \leq N}$ have continuous derivatives $\kappa_x, \kappa_u, \kappa_p$,
- the functions $\kappa, \kappa_u, \kappa_p$ are Hölder continuous in $t$ with the exponent $(1 + \alpha_0)/2$,
- the derivatives $\kappa_x$ are Hölder continuous in $x, t$ with the exponents $\alpha_0$ and $\alpha_0/2$ correspondently,
- the derivatives $\kappa_{xu}, \kappa_{xp}, \kappa_{uu}, \kappa_{up}, \kappa_{pp}$ exist and are Hölder continuous in $x, t, u, p$ with the exponents $\alpha_0, \alpha_0/2, \alpha_0, \alpha_0$.

$A_4$. On $\mathcal{N}_M^0 = \{(x, t, u) \in \mathcal{N}^0 \mid |u| \leq M\}$
- the function $G(x, t, u) = \Phi(x, t, u, 0) + g(x, t, u), \ G = \{G^k\}_{k \leq N}$, is continuous with derivatives $G_x, G_u, G_{xu}, G_{uu}$,
- the functions $G, G_u$ are Hölder continuous in $t$ with the exponent $(1 + \alpha_0)/2$,
- the function $G_x$ is Hölder continuous in $x$ and $t$ with the exponent $\alpha_0, \alpha_0/2$,
- $G_{xu}, G_{uu}$ are Hölder continuous in $x, t, u$ with the exponents $\alpha_0, \alpha_0/2, \alpha_0$.  

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Remark 1. All Hölder constants $h$ in conditions $\mathbb{A}_1$–$\mathbb{A}_5$ depend on $M$, i.e., $h = h(M)$.

We put

\begin{equation}
H_0 = \|b(x,0,0,0)\|_{\infty,\Omega} + [b(x,t,0,0)]_{Q}^{(\alpha_0)} + \|G_x(x,t,0)\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\Gamma)} + \|G_v(x,t,0)\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\Gamma)} + \langle G(x,t,0)\rangle_{t,\Gamma}^{(1+\alpha_0)/2},
\end{equation}

and note that $H_0$ depends on $T_1$ but it does not depend on $T$ and $M$.

Now we formulate the main local result.

**Theorem 1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^{2+\alpha_0}$-smooth boundary $\partial \Omega$, $\alpha_0 \in (0,1)$ is a fixed number. Suppose that conditions (1.5), (1.6), and (1.8) hold and linear problem (1.11) satisfies the complementing conditions. Then there exist numbers $M_0$ and $T_0 = T_0(M_0) \in (0,T_1]$ such that if assumptions $\mathbb{A}_1$–$\mathbb{A}_5$ hold with $M = M_0$ then problem (1.1)–(1.3) is uniquely solvable in $X_T$ for any fixed $T < T_0$. Numbers $M_0$ and $T_0(M_0)$ depend on the given problem data.

We split the proof of Theorem 1 into the following lemmas.

**Lemma 1.** There exist numbers $M_0$ and $\hat{T} = \hat{T}(M_0) \leq T_1$, such that for any $T \leq \hat{T}$ the map $F$ transforms $\mathcal{B}_{M_0}$ in $\mathcal{B}_{M_0}$, where $\mathcal{B}_{M_0} = \{v \in X_T| \|v\|_{X_T} \leq M_0\}$. The numbers $M_0$ and $\hat{T}$ depend on $H_0$ (see (1.14)) and on the same values as the constant $c_0$ in (1.12).

**Lemma 2.** Let $M_0$ and $\hat{T}$ be fixed as in Lemma 1. There exists a positive number $\theta = \theta(M_0)$ such that for every $v_1, v_2 \in \mathcal{B}_{M_0} \subset X_T$, $T \leq \hat{T}$, we have

\begin{equation}
\|F(v_1) - F(v_2)\|_{X_T} \leq \theta \left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right) \|v_1 - v_2\|_{X_T}.
\end{equation}

Indeed, if these lemmas are proved then we fix $T'$ from the condition

$$\theta \left[(T')^{\alpha_0/2} + (T')^{(1-\alpha_0)/2}\right] = 1.$$ 

For $T < T_0 = \min\{\hat{T}, T'\}$, the mapping $F$ is a contraction in $\mathcal{B}_{M_0} \subset X_T$, which implies the existence of a unique $u \in \mathcal{B}_{M_0}$ such that $u = F(u)$. Certainly, $u$ is the solution to (1.1), (1.2), (1.7) or (1.1)–(1.3) and Theorem 1 is proved.
Proof of Lemma 1: Fix $T \leq T_1$, $M > 0$ and $v \in \mathcal{B}_M \subset X_T$ arbitrary. For all summands $J_k$ in the braces of (1.12), we shall derive the following inequalities:

\begin{equation}
J_k \leq h_k(M) \left( T^\alpha/2 + T^{(1-\alpha)/2} \right) + H_k, \quad k = 1, \ldots, 5.
\end{equation}

In what follows, we denote by $h_k(M)$, $h(M)$ different but nondecreasing in $M$ functions. They may depend on the data and $T_1$ but not on the fixed $T$. All parameters $H_k$ and $H$ are independent of $M$ and $T$.

1. Estimation of $J_1 = \|b\|_{\mathcal{H}_{\alpha^0,\alpha^0/2}(Q^T)}$. We split $J_1$ in

$$J_1 = \|b\|_{\infty,Q^T} + \langle b \rangle_{x,Q^T}^{(\alpha_0)} + \langle b \rangle_{t,Q^T}^{(\alpha_0/2)} = j_1 + j_2 + j_3.$$

First of all, note that $v|_{t=0} = 0$ and

\begin{equation}
\begin{aligned}
\|v\|_{\infty,Q^T} &\leq \|v_t\|_{\infty,Q^T}, \\
\|v_x\|_{\infty,Q^T} &\leq \langle v_x \rangle_{t,Q^T}^{(1+\alpha_0)/2} T^{(1+\alpha_0)/2}, \\
\|v_{xx}\|_{\infty,Q^T} &\leq \langle v_{xx} \rangle_{t,Q^T}^{(\alpha_0/2)} T^{\alpha_0/2}.
\end{aligned}
\end{equation}

It is evident that

$$j_1 \leq \|b(x, t, v, v_x) - b(x, 0, 0, 0)\|_{\infty,Q^T} + \|b(x, 0, 0, 0)\|_{\infty,Q^T}$$

$$\leq h(M) \left( T^{\alpha_0/2} + \|v\|_{\infty,Q^T} + \|v_x\|_{\infty,Q^T} \right) + H$$

$$\leq h(M) \left( T^{\alpha_0/2} + T + T^{(1-\alpha_0)/2} \right) + H.$$

To estimate $j_2 = \langle b \rangle_{x,Q^T}^{(\alpha_0)}$ we write the inequalities

$$|b(x, t, v(x, t), v_x(x, t)) - b(x', t, v(x', t), v_x(x', t)|$$

$$\leq |b(x, t, 0, 0) - b(x', t, 0, 0)| + \int_0^1 \frac{\partial b(x, t, sv(x, t), sv_x(x, t))}{\partial s} ds$$

$$- \int_0^1 \frac{\partial b(x, t, sv(x', t), sv_x(x', t))}{\partial s} ds \leq H |\Delta x|^\alpha_0$$

$$+ \int_0^1 |b_v(x, t, sv(x, t), sv_x(x, t)) - b_v(x', t, sv(x', t), sv_x(x', t))| |v(x, t)| ds$$

$$+ \int_0^1 |b_v(x', t, sv(x', t), sv_x(x', t))| ds |v(x, t) - v(x', t)|$$
To justify inequality (\(*\)), we have used (1.17) and the following inequalities:

\[
|v(x,t) - v(x',t)| \leq \langle v_x \rangle_{t, Q_T}^{(1+\alpha_0)/2} T^{(1+\alpha_0)/2} |\Delta x|,
\]

\[
|v_x(x,t) - v_x(x',t)| \leq \langle v_{xx} \rangle_{t, Q_T}^{\alpha_0/2} T^{\alpha_0/2} |\Delta x|.
\]

We arrive at the inequality

\[
j_2 \leq H + h(M) \left( T^{\alpha_0} + T^{(1+\alpha_0)/2} \right).
\]

\(j_3\) is estimated in a similar way:

\[
|b(x,t,v(x,t), v_x(x,t)) - b(x',t', v(x',t'), v_x(x',t'))| \leq |b(x,t,0,0) - b(x',0,0)|
\]

\[
+ \left| \int_0^1 \left[ \frac{db(x,t,v(x,t), v_x(x,t))}{ds} - \frac{db(x',t', v(x',t'), v_x(x',t'))}{ds} \right] ds \right|
\]

\[
\leq H |\Delta t|^{\alpha_0/2} + h(M) \left( \|v\|_{Q_T} + \|v_x\|_{Q_T} \right) \left( |\Delta t|^{\alpha_0/2} + |v(x,t) - v(x',t')|^{\alpha_0} + |v_x(x,t) - v_x(x',t')|^{\alpha_0} + h(M) \right)
\]

\[
\leq H |\Delta t|^{\alpha_0/2} + h(M) \left( T + T^{(1+\alpha_0)/2} \right) \left( |\Delta t|^{\alpha_0/2} + |\Delta t|^{\alpha_0} + |\Delta t|^{\alpha_0 \cdot (1+\alpha_0)/2} \right)
\]

\[
+ M \left\{ |\Delta t| + |\Delta t|^{(1+\alpha_0)/2} \right\}.
\]

It follows that \(j_3 \leq H + h(M) T^{\alpha_0/2}\) and we get (1.16) for \(k = 1\), where \(H_1\) is defined by \(\|b(x,0,0,0)\|_{Q_T}\) and \([b(x,t,0,0)]_{Q_T}^{(\alpha_0)}\).
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2. Estimation of $J_2 = \|\Delta A \cdot v_{xx}\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\overline{\Omega}^T)}$. We have

$$J_2 = \|\Delta A \cdot v_{xx}\|_{\infty, \mathcal{Q}^T} + \langle \Delta A \cdot v_{xx}\rangle_{x, \mathcal{Q}^T}^{(\alpha_0)} + \langle \Delta A \cdot v_{xx}\rangle_{t, \mathcal{Q}^T}^{(\alpha_0/2)} = i_1 + i_2 + i_3.$$ 

It is easy to see that

$$i_1 \leq h(M)\|v_{xx}\|_{\infty, \mathcal{Q}} \leq h(M)T^{\alpha_0/2}.$$ 

Further,

$$i_2 \leq \langle \Delta A \rangle_{x, \mathcal{Q}^T}^{(\alpha_0)}\|v_{xx}\|_{\infty, \mathcal{Q}^T} + \|\Delta A\|_{\infty, \mathcal{Q}^T}\langle v_{xx}\rangle_{x, \mathcal{Q}},$$

where

$$\langle A_{k\ell}^{\alpha\beta}(x, t, v, v_x) - A_{k\ell}^{\alpha\beta}(x, 0, 0, 0)\rangle_{x, \mathcal{Q}^T}^{(\alpha_0)} \leq \langle A_{k\ell}^{\alpha\beta}(x, t, v, v_x)\rangle_{x, \mathcal{Q}^T}^{(\alpha_0)} + H \leq h(M) + H,$$

$$\|\Delta A\|_{\infty, \mathcal{Q}} \leq h(M)T^{\alpha_0/2}.$$ 

Whence,

$$i_2 \leq h(M)\|v_{xx}\|_{\infty, \mathcal{Q}^T} + h(M)T^{\alpha_0/2} \leq h(M)T^{\alpha_0/2}.$$ (1.17)

At last,

$$i_3 \leq \langle \Delta A \rangle_{t, \mathcal{Q}^T}^{(\alpha_0/2)}\|v_{xx}\|_{\infty, \mathcal{Q}^T} + \|\Delta A\|_{\infty, \mathcal{Q}^T}\langle v_{xx}\rangle_{t, \mathcal{Q}^T}^{(\alpha_0/2)} \leq h(M)T^{\alpha_0/2}.$$ 

Consequently, $J_2 \leq h(M)T^{\alpha_0/2}$ and (1.18) is proved for $k = 2$.

3. Estimation of $J_3 = \|\Delta \mathcal{A} v_x\|_{\mathcal{H}^{1+\alpha_0, (1+\alpha_0)/2}(\mathcal{G}^T)}$ and $J_4 = \|\Delta v v_x\|_{\mathcal{H}^{1+\alpha_0, (1+\alpha_0)/2}(\mathcal{G}^T)}$.

For a fixed atlas $\{V_j, P_j\}_{j=1}^m$ we choose a neighborhood $V_j$ and a mapping $P_j$ and then express $\Delta \mathcal{A} v_x$ in the local coordinate system $(y_1, \ldots, y_n)$. We shall write $y = y(x)$ and $x = x(y)$ for $y = P_j(x)$ and $x = P_j^{-1}(y)$, respectively,

$$\hat{v}(y, t) = v(x(y), t), \quad y \in B_1^+, \quad t \in [0, T].$$

In the new coordinates

$$\Delta x_{k\ell}^{\beta}(x, t, v(x, t), v_x(x, t))v_{x\beta}^{l}(x, t) = \left[ x_{k\ell}^{\beta}ight] \left(x(y), t, \hat{v}(y, t), \hat{v}_y(y, t) \frac{\partial y}{\partial x}\right)$$

$$\left. - x_{k\ell}^{\beta}(x(y), 0, 0, 0) \hat{v}_{y\gamma}^l(y, t) \frac{\partial y}{\partial x_{\beta}}\right|_{t=0} = 0 \quad \text{in} \quad B_1^+.$$ 

Putting

$$\hat{x}_{k\ell}^{\gamma}(y, t, \hat{v}(y, t), \hat{v}_y(y, t)) = \left[ x_{k\ell}^{\beta}\right] \left(x(y), t, \hat{v}(y, t), \hat{v}_y(y, t) \frac{\partial y}{\partial x}\right) \frac{\partial y_{\gamma}}{\partial x_{\beta}},$$

we have $\Delta \mathcal{A} v_x|_{x=x(y)} = \Delta \hat{x} \hat{v}_y.$
According to definition (1.9), (1.10), we have to estimate the expression

$$\hat{J}_3 = \|\Delta \hat{x} \hat{v}_y\|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Sigma T)} = \|\Delta \hat{x} \hat{v}_y\|_{\infty, \Sigma T} + \|\Delta \hat{x} \hat{v}_y\|_{\infty, \Sigma T} + \|\Delta \hat{x} \hat{v}_y\|_{\infty, \Sigma T}$$

$$+ \left\langle (\Delta \hat{x} \hat{v}_y)_{\Sigma T} \right\rangle + \left\langle \hat{\Delta} x \hat{y} \hat{y}' \right\rangle_{\Sigma T}.$$

Here $\Sigma T = \sigma \times (0, T)$, $\sigma = \{y' \in \mathbb{R}^{n-1} \mid |y'| < 1\}$.

We have the following estimates for $\hat{v}$:

$$\|\hat{v}\|_{\infty, \Sigma T} \leq M T, \quad \hat{v}_{\Sigma T} \leq h(M) \left( T^{1-\alpha_0/2} + T^{(1+\alpha_0)/2} \right),$$

$$\|\hat{v}_y\|_{\infty, \Sigma T} \leq h(M) T^{(1+\alpha_0)/2}, \quad \hat{v}_{\Sigma T} \leq h(M) T^{\alpha_0/2},$$

$$\left\langle \hat{v}_y \right\rangle_{\Sigma T} \leq h(M), \quad \|\hat{v}_y\|_{\infty, \Sigma T} \leq h(M) \left( T^{(1+\alpha_0)/2} + T^{\alpha_0/2} \right),$$

where $h(M)$ depends on the same parameters as in (1.17), (1.18) and $C^{2+\alpha_0}$ characteristics of the maps $y = y(x)$ and $x = x(y)$.

Now for $\hat{J}_3$ we deduce the estimate

$$\hat{J}_3 \leq h(M) T^{(1+\alpha_0)/2} + h(M) T^{\alpha_0/2} \left( \|\Delta \hat{x}\|_{\infty, \Sigma T} + \|\Delta \hat{x} \hat{y}\|_{\infty, \Sigma T} \right)$$

$$+ \|\Delta \hat{x}\|_{\infty, \Sigma T} h(M) T^{\alpha_0/2} + \left\langle (\Delta \hat{x})_{\Sigma T} \hat{v}_y \right\rangle \left\langle \hat{v}_y \right\rangle \left\langle \hat{v}_y \right\rangle_{\Sigma T}$$

$$+ \left\langle \hat{\Delta} x \hat{y} \hat{y}' \right\rangle_{\Sigma T}.$$
From the compatibility condition (1.8) it follows that $\hat{G}|_{t=0} = \hat{G}_y'|_{t=0} = 0$.

It is easy to see that

\[
\hat{J}_5 \leq \|\hat{G}\|_{\infty, \Sigma^T} + \|\hat{G}_y'|_{\infty, \Sigma^T} + \|\hat{G}_\hat{v}|_{\infty, \Sigma^T} \|\hat{v}'\|_{\infty, \Sigma^T}
+ [\hat{G}_y']_{\Sigma^T}^{(\alpha_0)} + [\hat{G}_\hat{v}]_{\Sigma^T}^{(\alpha_0)} \|v\|_{\infty, \Sigma^T} + \|\hat{G}_\hat{v}|_{\infty, \Sigma^T} \|\hat{v}|_{\Sigma^T}^{(\alpha_0)}
+ \langle \hat{G}_t \rangle_{t, \Sigma^T}^{(1+\alpha_0)/2} \leq h(M)T^{\alpha_0/2} + [\hat{G}_y']_{\Sigma^T}^{(\alpha_0)} + \langle \hat{G}_t \rangle_{t, \Sigma^T}^{(1+\alpha_0)/2}.
\]

(1.20)

To estimate $[\hat{G}_y']_{\Sigma^T}^{(\alpha_0)}$ we consider the expression

\[
\langle \hat{G}_y' \rangle_{y', \Sigma^T}^{(\alpha_0)} = \sup_{y', y'' \in \sigma} \frac{|\hat{G}_y'(y', t, \hat{v}(y', 0, t)) - \hat{G}_y'(y'', t, \hat{v}(y'', 0, t))|}{\|\Delta y'|^{\alpha_0}}
\leq \sup_{\{...\}} \left| \int_0^1 \frac{d}{ds} \hat{G}_y'(y', t, s\hat{v}(y', 0, t)) - \frac{d}{ds} \hat{G}_y'(y'', t, s\hat{v}(y'', 0, t)) \right| ds \cdot \|\Delta y'|^{\alpha_0}
+ \langle \hat{G}_y'(y', t, 0) \rangle_{y', \Sigma^T}^{(\alpha_0)}
\leq h \left( \|\hat{v}\|_{\infty, \Sigma^T} + \langle \hat{v} \rangle_{y', \Sigma^T}^{(\alpha_0)} \right) + H \leq h(M) \left( T + T^{(1+\alpha_0)/2} \right) + H.
\]

In the same way we derive that

\[
\langle \hat{G}_y' \rangle_{t, \Sigma^T}^{(\alpha_0/2)} \leq h(M) \left( T + T^{1-\alpha_0/2} \right) + H.
\]

This implies

\[
[\hat{G}_y']_{\Sigma^T}^{(\alpha_0)} \leq h(M) \left( T^{1-\alpha_0/2} + T^{(1+\alpha_0)/2} \right) + H.
\]

(1.21)

To estimate $[\hat{G}_\hat{v}]_{\Sigma^T}^{(\alpha_0)}$ we argue in the same way.

At last, we shall derive

\[
\langle \hat{G}_t \rangle_{t, \Sigma^T}^{(1+\alpha_0)/2} \leq h(M) \left( T + T^{(1-\alpha_0)/2} \right) + H.
\]

(1.22)
Indeed,

\[|\hat{G}(y', t', \hat{v}(y', 0, t')) - \hat{G}(y', t'', \hat{v}(y', 0, t''))|\]

\[\leq \left| \int_0^1 \left[ \frac{d\hat{G}(y', t', s\hat{v}(y', 0, t'))}{ds} - \frac{d\hat{G}(y', t'', s\hat{v}(y', 0, t''))}{ds} \right] ds \right|\]

\[+ |\hat{G}(y', t', 0) - \hat{G}(y', t'', 0)| \leq \int_0^1 |\hat{G}_\hat{v}(y', t', s\hat{v}(y', 0, t')) - \hat{G}_\hat{v}(y', t'', s\hat{v}(y', 0, t''))| ds \|\hat{v}\|_{\infty, \Sigma T}\]

\[+ \int_0^1 |\hat{G}_\hat{v}(y', t'', s\hat{v}(y', 0, t''))| ds \|\hat{v}\|_{\infty, \Sigma T} |\Delta t|\]

\[+ H|\Delta t|^{(1-\alpha_0)/2} \leq h(M)T \left\{ |\Delta t|^{(1-\alpha_0)/2} + |\Delta t| \right\} + h(M)|\Delta t| + H|\Delta t|^{(1-\alpha_0)/2}\]

\[\leq \left( h(M)T^{(1-\alpha_0)/2} + H \right) |\Delta t|^{(1-\alpha_0)/2}.
\]

This proves (1.22). Now from (1.20)–(1.22) it follows that

\[\hat{J}_5 \leq h(M) \left( T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right) + H,
\]

verifying (1.16) for \(J_5\).

Thus, by (1.12), (1.16), we obtain that

\[(1.23) \quad \|w\|_{X_T} \leq c_0 h(M) \left( T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right) + c_0 H_0,
\]

where \(c_0\) is the constant from (1.12) and \(H_0\) is defined in (1.14).

Now we put \(M_0 = 2c_0 H_0\) and fix \(\hat{T} \leq T_1\) from the condition

\[c_0 h(M_0) \left( \hat{T}^{\alpha_0/2} + \hat{T}^{(1-\alpha_0)/2} \right) \leq c_0 H_0.
\]

Then from (1.23) it follows that for any \(T \leq \hat{T}\) and \(v \in B_{M_0} \subset X_T\), \(\|w\|_{X_T} = \|F(v)\|_{X_T} \leq M_0\). Lemma 1 is proved.

**Proof of Lemma 2:** Let \(M\) and \(\hat{T}(M)\) be fixed as pointed in the statement of Lemma 1. For any \(T \leq \hat{T}\) we fix \(v'\) and \(v''\) in \(B_{M} \subset X_T\). Put \(w' = F(v')\), \(w'' = F(v'')\), \(\hat{w} = w' - w''\), \(\hat{v} = v' - v''\). According to (1.14) we have the following system:

\[\hat{w}_k^l = A_{kl}^{\alpha\beta}(x, 0, 0, 0)w_x^l + \left[ b^k(z, v', v_x') - b^k(z, v'', v_x'') \right]
\]

\[+ \left[ \Delta A_{kl}^{\alpha\beta}(z, v', v_x')(v^l)_{x\beta x\alpha} - \Delta A_{kl}^{\alpha\beta}(z, v'', v_x')(v''^l)_{x\beta x\alpha} \right] = 0,
\]

\(z \in Q_T, \ k \leq N,\)
We write

\[
\begin{align*}
&\left(\dot{x}_{kl}^\beta(x,0,0,0) + \psi_{kl}^\beta(x,0,0)\right) \dot{w}_{x\beta}^l + \left[G^k(z,v') - G^k(z,v'')\right] \\
&+ \left[\Delta \dot{x}_{kl}^\beta(z,v',v') + \Delta \psi_{kl}^\beta(z,v')\right] (v''^l)_{x\beta} - \left(\Delta \dot{x}_{kl}^\beta(z,v'',v_x) + \Delta \psi_{kl}^\beta(z,v'')\right) (v''''^l)_{x\beta}
\end{align*}
\]

(1.24)

We shall prove that every term on the right-hand side of (1.25) is estimated by (1.24). For example,

\[
\text{Problem (1.24) can be written in the short form:}
\]

\[
\begin{align*}
\dot{w}_t^k - A_{kl}^{\alpha\beta}(x,0,0,0) \dot{w}_{x\beta x\alpha}^l + D^k(z) + E^k(z) &= 0, & z \in Q_T, & k \leq N, \\
(1.24^0) \left(\dot{x}_{kl}^\beta(x,0,0,0) + \psi_{kl}^\beta(x,0,0)\right) \dot{w}_{x\beta}^l + Z^k(z) + Y^k(z)|_{\Gamma_T} &= 0,
\end{align*}
\]

where $D$, $E$, $Z$, $Y$ denote the correspondent expressions in the square brackets of (1.24). For example, $D = \{D^k\}_{k=1}^N$, $D^k(z) = b^k(z,v'k(z),v''(z)) - b^k(z,v''(z),v_x''(z))$ and so on.

For the linear problem (1.24^0), the following estimate holds:

\[
\begin{align*}
\|\dot{w}\|_{X_T} &\leq c_0 \left\{ \|D\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)} + \|E\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)} \\
&+ \|Z\|_{\mathcal{H}^{1+\alpha_0,1+\alpha_0/2}(\bar{T}^T)} + \|Y\|_{\mathcal{H}^{1+\alpha_0,1+\alpha_0/2}(\bar{T}^T)} \right\}.
\end{align*}
\]

(1.25)

We shall prove that every term on the right-hand side of (1.25) is estimated by $h(M)(T^{\alpha_0/2} + T^{1-\alpha_0/2})\|\dot{v}\|_{X_T}$ with some nondecreasing function $h(M) > 0$.

1) Estimation of $\|D\|_{\mathcal{H}^{\alpha_0,\alpha_0/2}(\bar{Q}^T)}$.

First of all we note that $\dot{v}|_{t=0} = 0$ and

\[
\begin{align*}
\|\dot{v}\|_{\infty,\bar{Q}^T} &\leq \|\dot{v}\|_{X_T T}, & [\dot{v}]_{Q^T}^{(\alpha_0)} &\leq c(\Omega)\|\dot{v}\|_{X_T} \left(T^{(1+\alpha_0)/2} + T^{1-\alpha_0/2}\right), \\
(1.26) \|\dot{v}_x\|_{\infty,\bar{Q}^T} &\leq \|\dot{v}\|_{X_T T^{1+\alpha_0/2}}, & [\dot{v}_x]_{Q^T}^{(\alpha_0)} &\leq c(\Omega,T_1)\|\dot{v}\|_{X_T T^{\alpha_0/2}}, \\
\|\dot{v}_{xx}\|_{\infty,\bar{Q}^T} &\leq \|\dot{v}\|_{X_T T^{\alpha_0/2}}.
\end{align*}
\]

We write $D^k$ in the form:

\[
D^k(z) = \int_0^1 \frac{\partial b^k(z,\tilde{v},\tilde{v}_x)}{\partial v^m} ds \tilde{v}^m(z) + \int_0^1 \frac{\partial b^k(z,\tilde{v},\tilde{v}_x)}{\partial v^m_{x\gamma}} ds \tilde{v}^m_{x\gamma}(z)
\]
where \( \bar{v} = v'' + s\hat{v}, \) \( \bar{v}_x = v_x'' + s\hat{v}_x. \) By condition \( \tilde{A}_2, \) we obtain the inequality
\[
\|D\|_{\mathcal{H}^0} \leq h(M) \left( \|\bar{v}\|_{\infty, QT} + \|\hat{v}_x\|_{\infty, QT} \right) + \left[ \int_0^1 b_v(...) ds \right]^{(\alpha_0)}_{QT} \|\bar{v}\|_{\infty, QT} + \left[ \int_0^1 b_p(...) ds \right]^{(\alpha_0)}_{QT} \|\hat{v}_x\|_{\infty, QT}.
\]
Here and below, we denote by \( h(M) \) different positive nondecreasing on \( M \) functions. They do not depend on \( T \) but may depend on \( T_1. \)

2) Estimation of \( \|E\|_{\mathcal{H}^0} \).

\( E^k(z) \) can be written in the form
\[
E^k(z) = \Delta A_{k_1}^0 \bar{z} + (A_{k_1}^0 - A_k^0) \bar{z} M, \quad M = [A_{k_1}(z, v', v_x') - A_{k_1}(z, v, v_x')](v''_x)_x, \quad \bar{z} = A_p^0 \|\hat{v}_x\|_{\infty, QT}.
\]
Then, by condition \( \tilde{A}_1 \) and inequalities (1.26), we obtain the estimate
\[
\|E\|_{\mathcal{H}^0} \leq \|\Delta A(z, v', v_x')\|_{\infty, QT} \|\hat{v}_x\|_{\infty, QT} + \left[ \int_0^1 A_v(...) ds \right]^{(\alpha_0)}_{QT} \|\hat{v}\|_{\infty, QT} + \left[ \int_0^1 A_p(...) ds \right]^{(\alpha_0)}_{\infty, QT} \|\hat{v}_x\|_{\infty, QT} + \left[ \int_0^1 A_v(...) ds \right]^{(\alpha_0)}_{\infty, QT} \|\hat{v}\|_{\infty, QT} M + \left[ \int_0^1 A_p(...) ds \right]^{(\alpha_0)}_{\infty, QT} \|\hat{v}_x\|_{\infty, QT} M + \left[ \int_0^1 A_p(...) ds \right]^{(\alpha_0)}_{\infty, QT} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \|\hat{v}_x\|_{\infty, QT}^{(\alpha_0)} \leq h(M) \|\hat{v}\|_{\infty, QT} M.
\]

3) Estimation of \( \|Y\|_{\mathcal{H}^1+\alpha, (1+\alpha)/2} \).

To estimate this expression we straighten a part of \( \partial\Omega \) and obtain the corresponding local estimate of the norm as was done in the proof of Lemma 1. Using the same notation we introduce an atlas \( \{V_j, P_j\}_{j \leq m} \) for \( \overline{\Omega}, \) where \( P_j: V_j \cap \Omega \to B_{1}^{+}, \quad P_j(V_j \cap \partial\Omega) = \sigma, \quad \Sigma_T = \sigma \times (0, T). \)
For some fixed \( j \leq m \), we write \( y = y(x) \) for \( y = P_j(x) \) and \( x = x(y) \) for \( x = P_j^{-1}(y) \).

Note that \( Y^k(z) = [\Delta x^{\beta}_{kl}(z, v', v''_{x}) v_{x_{\beta}}] + [\Delta x^{\beta}_{kl}(z, v', v''_{x}) v_{x_{\beta}}] \equiv Y^k_I(z) + Y^k_{II}(z) \). It is sufficient to prove that
\[
(1.27) \quad \|Y_I\|_{H^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma_T)} \leq h(M)\|\hat{v}\|_{X_T} \left(T^{\alpha_0/2} + T^{(1-\alpha_0)/2}\right).
\]

An analogous estimate for \( Y_{II} \) is more easily derived in the same way.

In the local coordinates \( y \in B^+_1, y = (y', y_n), y' \in \sigma \), we have
\[
\hat{Y}^k_I(y', t) \equiv \Delta x^{\beta}_{kl} \left(x(y), t, v'(x(y), t), v''(x(y), t) \cdot \frac{\partial y}{\partial x}\right) \hat{v}_{y_{\gamma}}^l \cdot \frac{\partial y}{\partial x_{\beta}}.
\]
\[
(1.28) \quad \hat{v}_{y_{\gamma}}^l (x(y), t) \cdot \frac{\partial y}{\partial x_{\beta}} \bigg|_{y' \in \sigma, y_n = 0} \equiv \Delta x^{\gamma}_{kl} (y', t, v', v''_{y}) \hat{v}_{y_{\gamma}}^l + (\hat{x}_{kl}(y', t, v', v''_{y}) - \hat{x}_{kl}^*(y', t, v', v''_{y})) v''_{y_{\gamma}}^l,
\]
where in the last equality we set \( v = v(x(y'), 0, t) \).

We shall estimate \( \|\hat{Y}_I\|_{H^{1+\alpha_0,(1+\alpha_0)/2}(\Sigma_T)} \) according to definition (1.9), (1.10).

In the local coordinates
\[
\|\hat{v}\|_{\Sigma_T} \leq \|\hat{v}\|_{X_T} \cdot T; \quad \|\hat{y}_y\|_{\Sigma_T} \leq K_1 \|\hat{v}\|_{X_T} \cdot T^{\alpha_0/2};
\]
\[
(1.29) \quad [\hat{v}]^{(\alpha_0)}_{\Sigma_T} + [\hat{y}_y]^{(\alpha_0)}_{\Sigma_T} \leq K_2 \|\hat{v}\|_{X_T} \cdot T^{\alpha_0/2};
\]
\[
\|\hat{v}_y\|_{\Sigma_T} \leq K_3 \|\hat{v}\|_{X_T} \cdot T^{\alpha_0/2}; \quad [\hat{v}_y]^{(\alpha_0)}_{\Sigma_T} \leq K_4,
\]
where constants \( K_i \) depend on \( T_1 \) and \( C^{2+\alpha_0} \) characteristics of \( x(y) \) and \( y(x) \).

It is easy to see that \( l_1 = \|\hat{Y}_I\|_{\infty, \Sigma_T} \leq h(M)\|\hat{v}\|_{X_T} T^{\alpha_0/2} \).

For the next step we omit indexes of functions and write
\[
(1.30) \quad \hat{Y}_I (y') = \Delta \hat{z}_{y'} \hat{v}_y + \Delta \hat{z}(\hat{v}_y)_{y'} + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial v \partial y'} ds \hat{v}_y'' + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial v \partial y'_{y'}} ds \hat{v}_y'' + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial p \partial v} \hat{v}_y' ds \hat{v}_y''
\]
\[
+ \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial v} ds \hat{v}_y'' + \hat{v}(v''_{y})_{y'} + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial p \partial v} \hat{v}_y' ds \hat{v}_y''
\]
\[ + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial p \partial p} (\hat{v}_y)_{y'} ds \hat{v}_y v'' + \int_0^1 \frac{\partial^2 \hat{z}(\ldots)}{\partial p \partial y'} ds \hat{v}_y v'' \]

\[ + \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial p} ds ((\hat{v}_y)_{y'} v'' + \hat{v}_y (v''_{y'})_{y'}). \]

Now we apply definition (1.28), conditions A_3, estimates (1.29) and equality (1.30) to deduce that

\[ l_2 = \| (\hat{Y}_I)_{y'} \|_{\infty, \Sigma_T} \leq h(M) \| \hat{v} \|_{X^T} T^{\alpha_0/2}. \]

Further,

\[ l_3 = \left[ (\hat{Y}_I)_{y'} \right]_{\Sigma_T}^{(\alpha_0)} \leq \left\{ [\Delta \hat{z}]_{\Sigma_T}^{\alpha_0} \| \hat{v}_y \|_{\infty, \Sigma_T} + [\Delta \hat{z}_y]_{\Sigma_T}^{(\alpha_0)} \right\}

\[ + [\Delta \hat{z}]_{\Sigma_T}^{(\alpha_0)} \| (\hat{v}_y)_{y'} \|_{\infty, \Sigma_T} + [\Delta \hat{z}]_{\Sigma_T}^{(\alpha_0)} \| (\hat{v}_y)_{y} \|_{\Sigma_T} + \ldots \]

\[ \ldots + \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial p} ds (\hat{v}_y'')_{y'} \right\}_{\infty, \Sigma_T}^{[(\hat{v}_y)_{y'}]_{\Sigma_T}^{(\alpha_0)}}, \]

where there are twenty two terms in the braces. We have not enough place to write and calculate all of them. Note only that all the terms are estimated by \( h(M) \| \hat{v} \|_{X^T} T^{\alpha_0/2} \) by using A_3 and inequalities (1.29).

At last,

\[ l_4 = \left( \hat{Y}_I \right)_{t, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \leq \left( \Delta \hat{z}(y', t, v', v''_y) \hat{v}_y \right)_{t, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \]

\[ + \left( \int_0^1 \frac{\partial \hat{z}(y', t, v, v'_y)}{\partial v} ds v''_y \right)_{t, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \| \hat{v} \|_{\infty, \Sigma_T} \]

\[ + \left( \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial v} ds v''_y \right)_{\infty, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \| \hat{v}_y \|_{\infty, \Sigma_T} \]

\[ + \left( \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial p} ds v''_y \right)_{t, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \| \hat{v}_y \|_{\infty, \Sigma_T} \]

\[ + \left( \int_0^1 \frac{\partial \hat{z}(\ldots)}{\partial p} ds v''_y \right)_{\infty, \Sigma_T}^{\left(\frac{1+\alpha_0}{2}\right)} \| \hat{v}_y \|_{t, \Sigma_T} = j_1 + j_2 + j_3 + j_4 + j_5. \]
For example, we estimate \( j_1 \):
\[
 j_1 \leq \langle \Delta \hat{\nu} \rangle_{t, \Sigma T} \left( \frac{1+\alpha_0}{2} \right) \| \hat{\nu} \|_{\infty, \Sigma T} + \| \Delta \hat{\nu} \|_{\infty, \Sigma T} \langle \hat{\nu}_y \rangle_{t, \Sigma T} \left( \frac{1+\alpha_0}{2} \right) .
\]
Here \( \langle \Delta \hat{\nu} \rangle_{t, \Sigma T} = \langle \Delta \hat{\nu}(y', t, v'(y', t), v'_y(y', t)) \rangle_{t, \Sigma T} \leq h(M), \| \Delta \hat{\nu} \|_{\infty, \Sigma T} \leq h(M)T^{(1+\alpha_0)/2} \). Now \( j_1 \) is estimated by \( h(M)\| \hat{\nu} \|_{X_T}T^{\alpha_0/2} \) with the help of (1.29). All other \( j_k \)'s are estimated in the same way (about the arguments (…) see (1.28)). Summarizing, we have estimated \( l_1-l_4 \) and \( \| \hat{Y}_T \|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Sigma_T)} \) by \( h(M)\| \hat{\nu} \|_{X_T}T^{\alpha_0/2} \) with some \( h(M) > 0 \), hence (1.27) follows.

4) Estimation of \( \| Z \|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Gamma_T)} \).

We write \( Z_k \) in the form \( Z^k(z) = \int_0^1 \frac{\partial G^k(z, \hat{v}(z))}{\partial n^m} \, ds \, \hat{v}^m(z) \), \( \hat{v} = v'' + s\hat{v} \) and put \( \hat{Z}(y', t) = Z(x(y', 0), t), \ y' \in \Omega, \ t \in (0, T) \).

To deduce the estimate
\[
\| \hat{Z} \|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Sigma_T)} \leq h(M)\| \hat{\nu} \|_{X_T} \left( T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right)
\]
we make use of conditions \( \text{A}_4 \), inequalities (1.29) and argue in the same way as at the previous step. From (1.31) and definition (1.9), (1.10), the estimate of \( \| Z \|_{H^{1+\alpha_0, (1+\alpha_0)/2}(\Gamma_T)} \) follows.

Now we go back to estimate (1.25) and obtain that for some \( h(M) > 0 \)
\[
\| \hat{\nu} \|_{X_T} \leq c_0 h(M)\| \hat{\nu} \|_{X_T} \left( T^{\alpha_0/2} + T^{(1-\alpha_0)/2} \right).
\]
Here \( c_0 \) depends on the same data as in the statement of Theorem 1. By inequality (1.32) with \( \theta = c_0 h(M) \), (1.17) follows. Lemma 2 is proved.

2. Weak global in time solvability

Using M. Struwe’s idea [6], we shall construct a global solution of the Cauchy-Neumann problem to the class of parabolic systems studied in [9].

Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) and \( T > 0 \) is fixed arbitrarily, \( Q = \Omega \times (0, T) \). For some functions \( f: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}^1 \) and \( G: \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^1 \), \( N > 1 \), we consider a solution \( u: Q \to \mathbb{R}^N, \ u = (u^1, \ldots, u^N) \), of the problem
\[
\begin{align*}
  & u_t^k - \frac{d}{dx_\alpha} f_{p^k_\alpha}(x, u, u_x) + f_{u^k}(x, u, u_x) = 0, \quad z = (x, t) \in Q, \\
  & f_{p^k_\alpha}(x, u, u_x) \cos(n, x_\alpha) + g^k(x, u)|_{\Gamma} = 0, \quad \Gamma = \partial \Omega \times (0, T), \quad k \leq N, \\
  & u|_{t=0} = \varphi(x),
\end{align*}
\]
where \( g = \nabla uG, \varphi : \overline{\Omega} \to \mathbb{R}^N \) is a given function, \( \mathbf{n} = \mathbf{n}(x) \) is the outward to \( \Omega \) normal vector at a point \( x \in \partial \Omega \).

It is easy to see that the corresponding to (2.1) stationary problem describes stationary points of the functional

\[
E[u] = \int_{\Omega} f(x, u, u_x) \, dx + \int_{\partial \Omega} G(x, u) \, ds.
\]

Now we fix a number \( \alpha_0 \in (0, 1) \) and formulate all assumptions on \( \partial \Omega, \varphi, f, G \) and \( g \).

\( D_1. \) \( \partial \Omega \in C^{3+\alpha_0}, \varphi \in W^{1,2}_2(\Omega) \).

\( D_2. \) \( f \) is defined on the set \( \mathcal{M} = \partial \Omega \times \mathbb{R}^N \times \mathbb{R}^{2N} \) with the derivatives mentioned below and satisfies the following conditions:

1. \( \nu_0 |p|^2 \leq f \leq \mu_1 + \mu_0 |p|^2, \)
2. \( \left| f_u \right| + \left| f_{ux} \right| + \left| f_{uu} \right| \leq \mu_2 (1 + |p|^2), \quad \left| f_p \right| + \left| f_{px} \right| + \left| f_{pu} \right| \leq \mu_2 (1 + |p|), \)
3. \( \left| f_{pp} \right| + \left| f_{ppx} \right| \leq \mu_2, \quad \langle f_{pp}(x, u, p) \xi, \xi \rangle \geq \nu |\xi|^2, \forall \xi \in \mathbb{R}^{2N}, \)

with positive constants \( \nu_0, \mu_0, \mu_2, \nu \) and \( \mu_1 \geq 0 \).

2) Derivatives \( f_{px}, f_{ppx} \) are continuous on \( \mathcal{M} \) and are Hölder continuous in \( x \) with the exponent \( \alpha_0 \) on any compact set of \( \mathcal{M} \).

3) \( \psi(x, u, p) = f_{up}(x, u, p) \) is continuously differentiable in \( x, u, p \) on the set \( \mathcal{M} \).

4) On any compact subset of \( \mathcal{M} \), the function \( \Lambda(x, u, p) = f_{pp}(x, u, p) \) is twice continuously differentiable in all arguments and \( \Lambda_{xu}, \Lambda_{xp}, \Lambda_{uu}, \Lambda_{up}, \Lambda_{pp} \) are Hölder continuous in all arguments with the exponent \( \alpha_0 \).

\( \mathcal{D}_3. \) (1) \( G(x, u) \) is a continuous function on the set \( \mathcal{M}_0 = \overline{\Omega} \times \mathbb{R}^N \), it has continuous derivative \( G_x \) and satisfies

\[
G \geq h_0|u|^2 - h_1, \quad |G| + |G_x| \leq h_2(1 + |u|^2),
\]

\( h_0, h_1 = \text{const} \geq 0, \) \( h_2 = \text{const} > 0 \).

2) The function \( g(x, u) = \nabla uG(x, u) \) and its derivatives \( g, g_x, g_{xx}, g_u, g_{ux}, g_{uu} \) are continuous on \( \mathcal{M}_0 \) and

\[
|g| + |g_x| + |g_{xx}| \leq h_3(1 + |u|), \quad |g_u| + |g_{ux}| + |g_{uu}| \leq h_3,
\]

\( h_3 = \text{const} > 0 \).

3) On any compact subset of \( \mathcal{M}_0 \), \( g_x \) is Hölder continuous in \( x \) with the exponent \( \alpha_0 \) and \( g_{xu}, g_u \) are Hölder continuous in \( x, u \) with the exponent \( \alpha_0 \).
II. Local and global solvability results

It is evident that under assumptions $\mathbb{D}_1-\mathbb{D}_3$, the parabolic system (2.1) has non-diagonal main matrix and quadratic nonlinearity in the gradient. In general, the weak global solvability for such a type systems was not proved yet.

If we put $f(x,u,p) = \frac{1}{2} A_{k\alpha}^{\beta}(x,u)p^l_{\alpha} p^k_{\beta}$, $G(x,u) = \frac{1}{2} h(x)|u|^2 + (u, r(x))$, where $A_{k\alpha}^{\beta}$ are $C^{2+\alpha_0}$ smooth functions on $\Omega \times \mathbb{R}^N$ and $A_{k\alpha}^{\beta} = A^{\alpha\beta}_{l\mu}$,

$$A_{k\alpha}^{\beta}(x,u)\xi^k_{\alpha} \xi^l_{\beta} \geq \nu|\xi|^2, \quad \forall \xi \in \mathbb{R}^{2N}, \quad \nu = \text{const} > 0,$$

$h, r \in C^2(\Omega)$, $h(x) \geq 0$, then conditions $\mathbb{D}_2$, $\mathbb{D}_3$ hold. In this case we have the quasilinear problem (2.1) in the form

(2.6)

$$u^k_t - (A_{k\alpha}^{\beta}(x,u)u^l_{x,\beta})_{x,\alpha} + \frac{1}{2} (A_{m\alpha}^{\beta}(x,u))'_{uk} u^m_{x,\beta} u^l_{x,\alpha} = 0, \quad (x,t) \in Q,$$

$$\left. \left( \frac{\partial u}{\partial n_A} \right) \right|_\Gamma \equiv A_{k\alpha}^{\beta}(x,u)u^l_{x,\beta} \cos(n, \alpha) + h(x)u^k + r^k(x) \right|_\Gamma = 0, \quad k \leq N,$$

$$u|_{t=0} = \varphi.$$

We shall construct a weak global solution to (2.1) (and, in particular, to (2.6)) in five steps.

**Step 1.** First of all, we “smooth” the initial function $\varphi$.

**Proposition 1.** Under conditions $\mathbb{D}_1-\mathbb{D}_3$ there exists a sequence $\{\varphi_m(x)\}_{m \in \mathbb{N}}$, $\varphi_m \in C^{2+\alpha_0}(\Omega)$, with $\varphi_m \to \varphi$ in $W^1_2(\Omega)$ and such that every function $\varphi_m$ satisfies the compatibility condition:

(2.7)

$$l^{|k|} [\varphi_m] = f_{p^k_{\alpha}}(x, \varphi_m(x), (\varphi_m(x))_{x}) \cos(n, x_{\alpha}) + g^k(x, \varphi_m(x)) + \xi^k_{\alpha} \right|_{x \in \partial \Omega} = 0, \quad k \leq N.$$

As $\partial \Omega \in C^{3+\alpha_0}$, there exists a sequence $\{\psi_m\}$, $\psi_m \in C^{3+\alpha_0}(\Omega)$, $\psi_m \to \varphi$ in $W^1_2(\Omega)$. If some function $\psi_m$ does not satisfy (2.7) then we can “correct” it in a boundary layer with the help of the distance function. The function belongs to $C^{2+\alpha_0}(\Omega)$ and the new sequence tends to $\varphi$ in $W^1_2(\Omega)$. To save place we omit the proof of Proposition 1.

**Step 2.** Now we study problem (2.1) with the initial condition $u|_{t=0} = \varphi_m$, $\varphi_m$ satisfies (2.7). To apply Theorem 1 we introduce the problem in the non-divergence form, $\hat{u} = u - \varphi_m$:

$$\hat{u}^k_t - A_{k\alpha}^{\beta}(x,\hat{u}, \hat{u}_x)\hat{u}^l_{x,\beta,\alpha} + b^k(x, \hat{u}, \hat{u}_x) = 0, \quad (x,t) \in Q,$$

$$\hat{u}^k_{x,\beta,\alpha}(x, \hat{u}, \hat{u}_x)\hat{u}^l_{x,\beta} + \hat{g}^k(x, \hat{u}) \right|_\Gamma = 0, \quad k = 1, \ldots, N,$$

$$\hat{u}|_{t=0} = 0.$$
where
\[ A_{kl}^{\alpha\beta}(x, \hat{u}, \hat{p}) = f_{\hat{p}^\alpha} p_{\beta}^0(x, \hat{u} + \varphi_m(x), \hat{p} + (\varphi_m(x))x), \]
\[ b^k(x, \hat{u}, \hat{p}) = f_{u^k}(x, \hat{u} + \varphi_m(x), \hat{p} + (\varphi_m(x))x) - f_{p_{\alpha}^k} u^m(\ldots)(\hat{p}^m_{\alpha} + \ldots) \]
\[ + (\varphi_m(x))x), \]
\[ \text{Step 3.} \]

(by \ldots we denote the same arguments as function \( f_{u^k} \) has);
\[ \chi_{kl}^{\beta}(x, \hat{u}, \hat{p}) = \int_0^1 f_{p_{\alpha}^k} p_{\beta}^l(x, \hat{u} + \varphi_m(x), (\varphi_m(x))x + s\hat{p}) \, ds \cdot \cos(n, x_{\alpha}), \]
\[ g^k(x, \hat{u}) = g^k(x, \hat{u} + \varphi_m(x)) + f_{p_{\alpha}^k}(x, \hat{u} + \varphi_m(x), (\varphi_m(x))x) \cos(n, x_{\alpha}). \]

Conditions \( \mathbb{D}_1-\mathbb{D}_3 \) and (2.7) imply the validity of the assumptions of Theorem 1. Thus, for some \( T_m > 0 \) there exists a unique smooth solution \( \hat{u}_m \) to (2.8) in a cylinder \( \hat{Q}_m = \hat{\Omega} \times [0, T_m) \). \( \hat{u}_m \in \mathcal{H}^{2+\alpha_0, 1+\alpha_0/2}(\hat{Q}_m) \). It implies the existence of a solution \( u_m \) to problem (2.1). We suppose that \( T_m \) defines the maximal interval of the smooth solution.

Step 3. We put \( E[u(t)] = \|u_{x}(\cdot, t)\|_{2, \Omega}^2 + \|u(\cdot, t)\|_{2, \partial \Omega}^2 \)
\[ E[u(t); \Omega_r(x^0)] = \|u_{x}(\cdot, t)\|_{2, \Omega_r(x^0)}^2 + \|u(\cdot, t)\|_{2, \gamma_r(x^0)}^2, \]
\[ \gamma_r(x^0) = \partial \Omega \cap B_r(x^0). \]

The functions \( u_m, m \in \mathcal{N}, \) satisfy the following inequalities
\[ \|u_t^m\|_{2, \Omega \times (0, t)}^2 + E[u_m(t)] \leq c_1 + c_2 E[\varphi_m] \]
\[ \leq c_1 + \hat{c}_2 E[\varphi] \equiv e_0, \quad \forall t \in [0, T_m), \]
\[ E[u_m(t''); \Omega_R(x^0)] \leq c_3 (R + (t'' - t')) + c_4 E[u_m(t'); \Omega_2 R(x^0)] \]
\[ + \frac{c_5 (t'' - t') e_0}{R^2}, \quad \forall t' \leq t'' < T_m, \forall x^0 \in \overline{\Omega}, \quad R < \min\{1, \text{diam } \Omega/2\}. \]

Inequalities (2.9), (2.10) follow from (13), (14) \[ 9 \]. By Remark 13 \[ 9 \], the constants \( c_1, \ldots, c_5, \) do not depend on \( T_m \).

Now we fix \( R_0 > 0 \) such that
\[ E[\varphi; \Omega_{2R_0}(x^0)] < \frac{\varepsilon_0}{8c_4}, \quad \forall x^0 \in \overline{\Omega}, \text{ and } R_0 < \min \left\{ 1, \frac{\varepsilon_0}{8c_3} \right\}, \]
where \( \varepsilon_0 \) is as in Theorem 1 \[ 9 \]; it depends on the data from conditions \( \mathbb{D}_1-\mathbb{D}_3 \).
Thus, most a finite number (the proof of Theorem 3 [9] and using (2.10) one can easily derive that
Moreover, we put \( \hat{T} = \theta R_0^2 \), where \( \theta < \frac{\varepsilon_0}{4(c_3 + c_5 \varepsilon_0)} \) and derive from (2.10) (with \( t' = 0 \), \( t'' = t, R = R_0 \)) and (2.11) the inequality
\[
\sup_{0 \leq t \leq \min \{ \hat{T}, T_m \}} \sup_{x^0 \in \bar{\Omega}} E[u_m(t); \Omega_{R_0}(x)] < \varepsilon_0, \quad \forall m \geq m_0.
\]
If \( T_m < \hat{T} \) then all assumptions of Theorem 1 [9] are valid and it is possible to extend the solution \( u_m \) up to \( t = T_m \). This contradicts the definition of \( T_m \). Thus, \( T_m > \hat{T} > 0 \) and
\[
(2.12) \quad \sup_{[0, \hat{T}]} \sup_{x \in \Omega} E[u_m(t), \Omega_{R_0}(x)] < \varepsilon_0.
\]
All functions \( u_m(t), m \geq m_0 \), are smooth on \( \bar{\Omega} \times [0, \hat{T}] \). According to Lemma 2 and Remark 7 [9], (2.12) guarantees that
\[
(2.13) \quad \|u_m \|_{2, \hat{Q}}^2 \leq c + c_\varphi \left( 1 + \hat{T} + \frac{T^2}{R_0^2} \right), \quad \hat{Q} = \Omega \times (0, \hat{T}),
\]
where the constants \( c \) and \( c_\varphi \) are defined by parameters from conditions (2.3)–(2.5) and \( C^{1,1} \) characteristics of \( \partial \Omega \), \( c_\varphi \) also depends on \( \|\varphi\|_{W_2^1(\Omega)} \).

By (2.9), (2.13), it follows that
\[
(2.14) \quad \sup_{[0, \hat{T}]} \|u_m(\cdot, t)\|_{W_2^1(\Omega)} + \|u_m\|_{W_2^{2,1}(\hat{Q})} \leq c, \quad \forall m \geq m_0.
\]

Whence, \( u_m \rightharpoonup u \) weakly in \( W_2^{2,1}(\hat{Q}) \), \( (u_m)_x \rightharpoonup u_x \) in \( L^2(\hat{Q}) \) for some sequence of \( m \to +\infty \). The limit function \( u \) is a solution to (2.1), \( u \in Y(\hat{Q}) = W_2^{2,1}(\hat{Q}) \cap L^\infty((0, \hat{T}), W_2^1(\Omega)) \). From Theorem 2' [9] it follows that \( u \) is a unique solution in this class. Applying Theorem 2 [9], we find that \( u \in H^{2+1+\alpha_0/2}(\Omega \times (0, \hat{T})) \) and \( u_{xt} \in L^{2,2+2\alpha_0}(\Omega \times (\delta, \hat{T})), \forall \delta > 0 \).

Suppose that \( T_1 > \hat{T} \) defines the maximal interval of the existence of smooth solution \( u \). According to Theorem 3 and Remark 12 [9], \( u \) admits a smooth extension to the set \( \Omega \times (0, T_1) \setminus \Sigma T_1 \), where the singular set \( \Sigma T_1 \) consists of at most a finite number \( (M_1) \) points, \( \Sigma T_1 = \{(x_1, T_1) \cup \ldots \cup (x_1^{M_1}, T_1)\} \). Analyzing the proof of Theorem 3 [9] and using (2.10) one can easily derive that \( M_1 \leq \frac{4c_5 \varepsilon_0}{\varepsilon_0} \).

Moreover, \( u(\cdot, t) \rightharpoonup u(\cdot, T_1) \) weakly in \( W_2^1(\Omega) \) and in \( W_2^{1,loc}(\Omega \setminus \{x_1^{1} \cup \ldots \cup x_1^{M_1}\}) \).
If $h_0 > 0$ in (2.4) or $G = 0$ on $\partial \Omega$ then the dominating constant $M_1$ does not depend on $T_1$ (see Remark 13).

**Step 4.** We denote $\varphi^{(1)}(x) = u(x, T_1) \in W^1_2(\Omega)$ and deduce that

$$(2.15) \quad \mathcal{E}[\varphi^{(1)}] \leq \mathcal{E}[\varphi] - \frac{\nu_0 \varepsilon_0}{4c_4} M_1,$$

in the same way as in [6]. Now we consider problem (2.1) for $t > T_1$ with initial function $\varphi^{(1)}(x)$. We argue precisely as we did at the previous step. As a result, we deduce the existence of a smooth solution $u^{(1)}(x, t)$ on some interval $(T_1, T_2)$,

$\quad u^{(1)}(\cdot, t) \to u^{(1)}(\cdot, T_2)$ weakly in $W^1_2(\Omega)$, $t \to T_2$.

We construct a sequence of intervals $(T_m, T_{m+1}) \subset (0, T)$ and of solutions $u^{(m)}(\cdot, t)$, $m = 0, 1, 2, \ldots$, $(T_0 = 0, u(0) = u, \varphi(0) = \varphi)$. Taking in consideration (2.15), we deduce that

$$(2.16) \quad M = \sum_{j=1}^{m+1} M_j \leq \frac{\mathcal{E}[\varphi] \cdot 4c_4}{\nu_0 \varepsilon_0} = m_0.$$ 

In the case when $h_0 > 0$ in (2.4) or $G = 0$ on $\partial \Omega$, $m_0$ in (2.16) does not depend on $T$.

Joining all the functions $u^{(m)}$, we obtain a solution $u$ (2.1). The solution is smooth on $\hat{\Omega} \times (0, T)$, except of at most finitely many points. Further, $u_t \in L^2(Q)$, $\sup \mathcal{E}[u(t)] \leq \mathcal{E}[\varphi]$, $u(\cdot, t) \to \varphi$ weakly in $W^1_2(\Omega)$ (indeed, one can prove that $u(\cdot, t) \to \varphi$ in the norm of $W^1_2(\Omega)$). The uniqueness of $u$ with the mentioned properties follows from Theorem 2’ [9] when applying the result to each interval

$\quad [T_j, T_{j+1}) \cup \bigcup_{j=0}^{M} [T_j, T_{j+1}) = [0, T).$

We have proved the following result.

**Theorem 1.** Let conditions $\mathbb{D}_1-\mathbb{D}_3$ hold. Then for a fixed number $T > 0$ and any function $\varphi \in W^1_2(\Omega)$ there exists a global solution $u : \Omega \times (0, T) \to \mathbb{R}^N$ to the problem (2.1) such that $u$ is $C^{2+\alpha_0, 1+\alpha_0/2}$ smooth function in $\overline{\Omega} \times (0, T) \setminus \Sigma$. The singular set $\Sigma$ consists of at most finitely many points $\{(x^j, t^j)\}_{j=1}^{M}$. The number $M$ is estimated by the data from assumptions $\mathbb{D}_1-\mathbb{D}_3$ and $T$. If $h_0 \neq 0$ in (2.4) or $G = 0$ on $\partial \Omega$ then $M$ is estimated by the data from $\mathbb{D}_1-\mathbb{D}_3$ only.

Every point $(x^j, t^j) \in \Sigma$ is characterized by the condition

$\lim_{t \to t^j} \|u_x(\cdot, t)\|_{L^2_{\Omega_R(x^j)}}^2 \geq \varepsilon_0, \quad \forall R > 0,$
II. Local and global solvability results

\(\varepsilon_0 > 0\) is taken from Theorem 1 [9]. Furthermore,

1. \(u \in L^\infty((0, T); W^1_2(\Omega)), \ u_t \in L^2(Q), \ \sup_{[0,T]} E[u(t)] \leq E[\varphi];\)
2. \(u\) is a unique solution with the above properties;
3. \(u\) satisfies the integral identity

\[
\int_Q u^k h^k + f_{p_\alpha}^k(x, u, u_x)h_x^k + f_{u^k}(x, u, u_x)h^k \, dQ
\]
\[
+ \int_\Gamma g^k(x, u)h^k \, d\Gamma = 0, \quad \forall \ h \in L^2((0, T); W^1_2(\Omega)) \cap L^1((0, T); L^\infty(\Omega)),
\]
\[
u(\cdot, t) \to \varphi \quad \text{in} \ W^1_2(\Omega).
\]

On the behavior of the solution at infinity

Here we suppose that \(h_0 \neq 0\) in (2.4) or \(G = 0\) on \(\partial\Omega\). In this case, the number \(m_0\) in (2.16) does not depend on \(T\). As \(T > 0\) was fixed arbitrarily we may discuss the behavior of \(u(\cdot, t)\) when \(t \to +\infty\).

First, we assume that all singularities in \(\overline{\Omega}\) are developed in a finite time interval. Then for some \(T > 0\) and \(R > 0\) we have the inequality

\[
\sup_{t > T} \sup_{x \in \Omega} \|u_x(\cdot, t)\|_{L^2(\Omega_R(x))}^2 < \varepsilon_0.
\]

Whence, (see [7, Chapter III]) along a certain sequence of indices \(j \to +\infty\) the sequence \(u(\cdot, t_j)\) weakly converges in \(W^2_2(\Omega)\) to a function \(u^\infty \in W^2_2(\Omega)\), \(u_t(\cdot, t_j) \to 0\) in \(L^2(\Omega)\). By the imbedding theorem, \(u_x(\cdot, t_j) \to (u^\infty)_x\) in \(L^s(\Omega)\), \(s < \infty\). To justify these facts note that for any \(t > T\) the following estimates are valid:

\[
\int_t^{t+1} \|u_{xx}(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau \leq c + c_\varphi \left( 1 + \frac{1}{R^2} \right); \quad \int_t^{t+1} \|u_t(\cdot, \tau)\|_{L^2 \Omega}^2 \, d\tau \to 0.\]

Furthermore, \(u_{xx}(\cdot, t_j) \to u_{xx}^\infty\) in the \(L^2(\Omega)\) norm. To prove this assertion we treat the local setting of (2.1) (see (24) [9]). In such a case, the functions \(u(\cdot, t_j), u^\infty\) transform to \(v(\cdot, t_j), v^\infty\) in \(B^1_1\). From the integral identity for \(v(\cdot, t_j)\) and \(v^\infty\) we derive that \(||(v(\cdot, t_j) - v^\infty)_{yy}\|_{L^2(\Omega)} \to 0.\) Returning to the functions \(u(\cdot, t)\) and \(u^\infty\), we deduce that \(||u_{xx}(\cdot, t_j) - u_{xx}^\infty\|_{L^2(\Omega)} \to 0.\)

Known results on the smoothness of weak solutions of nonlinear elliptic systems guarantee that \(u^\infty \in C^{2+\alpha_0}(\overline{\Omega}), u^\infty\) is an extremal point of the functional \(E[u] = \int_\Omega f(x, u, u_x) \, dx + \int_{\partial\Omega} G(x, u) \, ds.\)
In particular, if $E[\varphi] < \varepsilon_0/\nu$ ($\varepsilon_0 > 0$ is the constant from (2.3)), then from the monotonicity of $E[u(t)]$ it follows that $\sup_{[0,\infty)} \|u_x(\cdot,t)\|_{2,\Omega}^2 < \varepsilon_0$. In this case Theorem 1 [9] yields that solution $u(\cdot,t)$ to (2.1) is a smooth transformation of $\varphi$ to an extremal point $u^\infty$ when $t \in (0,\infty]$.

Suppose now that there exist singular points at the infinity. In this case $u^\infty$ is a smooth in $\Omega \setminus \{x^1 \cup \ldots \cup x^M\}$ solution to the problem

$$-rac{d}{dx^\alpha} f_{p^k_{\alpha}}(x, u, u_x) + f_{u^k}(x, u, u_x) = 0, \quad x \in \Omega,$$

$$f_{p^k_{\alpha}}(x, u, u_x) \cos(n, x^\alpha) + g^k(x, u)|_{x \in \partial \Omega} = 0.$$

According to De Giorgi’s lemma [8, Chapter II, Lemma 3.1], $u^\infty$ satisfies the identity

$$\int_\Omega (f_{p^k_{\alpha}} \eta_{x^\alpha} + f_{u^k} \eta^k) \, dx + \int_{\partial \Omega} g^k \eta^k \, ds = 0, \quad \forall \eta \in W^1_2(\Omega) \cap L^\infty(\Omega).$$

Concluding, note that the boundedness of the solution constructed was not stated. The estimate $\sup_{[0,T]} \|u_x(\cdot,t)\|_{2,\Omega} \leq \text{const}$ guarantees only that $\sup_{[0,T]} \|u(\cdot,t)\|_{L^{2,n}(\Omega)} \leq \text{const}$, $n = 2$.

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